\textbf{h-STABILITY FOR NONLINEAR PERTURBED DIFFERENCE SYSTEMS}

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\textbf{ABSTRACT.} We show that two concepts of h-stability and h-stability in variation for nonlinear difference systems are equivalent by using the concept of $n_\infty$-summable similarity of their associated variational systems. Also, we study h-stability for perturbed nonlinear system $y(n + 1) = f(n, y(n)) + g(n, y(n), Sy(n))$ of nonlinear difference system $x(n + 1) = f(n, x(n))$ using the comparison principle and extended discrete Bihari-type inequality.

1. Introduction

Let $\mathbb{N}(n_0) = \{n_0, n_0 + 1, \ldots, n_0 + k, \ldots\}$, where $n_0$ is a nonnegative integer and $\mathbb{R}^m$ the $m$-dimensional real Euclidean space. We consider the nonlinear difference system

\begin{equation}
\tag{1.1} x(n + 1) = f(n, x(n)), \quad x(n_0) = x_0
\end{equation}

and its perturbed system

\begin{equation}
\tag{1.2} y(n + 1) = f(n, y(n)) + g(n, y(n), Sy(n)), \quad y(n_0) = y_0
\end{equation}

where $f : \mathbb{N}(n_0) \times \mathbb{R}^m \to \mathbb{R}^m$ and $g : \mathbb{N}(n_0) \times \mathbb{R}^m \times F(\mathbb{N}(n_0), \mathbb{R}^m) \to \mathbb{R}^m$, and $S : F(\mathbb{N}(n_0), \mathbb{R}^m) \to F(\mathbb{N}(n_0), \mathbb{R}^m)$ is an operator on $F(\mathbb{N}(n_0), \mathbb{R}^m)$ = \{y|y : \mathbb{N}(n_0) \to \mathbb{R}^m is a sequence\}, and $f(n, 0) = 0 = g(n, 0, 0)$.

We assume that $f_x = \frac{\partial f}{\partial x}$ exists and is continuous and invertible on $\mathbb{N}(n_0) \times \mathbb{R}^m$. Let $x(n) = x(n, n_0, x_0)$ be the unique solution of (1.1). Also, we consider its associated variational systems

\begin{equation}
\tag{1.3} v(n + 1) = f_x(n, 0)v(n)
\end{equation}

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and
\[ z(n + 1) = f_x(n, x(n, n_0, x_0))z(n). \]
The fundamental matrix solution \( \Phi(n, n_0, 0) \) of (1.3) is given by
\[ \Phi(n, n_0, 0) = \frac{\partial x(n, n_0, 0)}{\partial x_0} \]
and the fundamental matrix solution \( \Phi(n, n_0, x_0) \) of (1.4) is given by
\[ \Phi(n, n_0, x_0) = \frac{\partial x(n, n_0, x_0)}{\partial x_0}. \]
in [10].

Equation (1.2) represents several interesting equations, namely, difference equation of Volterra type as
\[ y(n + 1) = A(n)y(n) + \sum_{l=n_0}^{n} B(n, l)y(l) \]
where \( f(n, y(n)) = A(n)y(n) \) and \( Sy(n) = \sum_{l=n_0}^{n} B(n, l)y(l) \), \( A(n) \) and \( B(n, l) \) are \( m \times m \) matrix functions on \( \mathbb{N}(n_0) \) and \( \mathbb{N}(n_0) \times \mathbb{N}(n_0) \) respectively, and delay difference equation as
\[ y(n + 1) = f(n, y(n)) + g(n, y(n), y(n - \tau)) \]
etc..

The symbol \( |\cdot| \) will be used to denote any convenient vector norm in \( \mathbb{R}^m \).

When we study the asymptotic stability it is not easy to work with non-exponential types of stability. Medina and Pinto [11–14] extended the study of exponential stability to a variety of reasonable systems called \( h \)-systems. They introduced the notion of \( h \)-stability for difference systems as well as for differential systems. To study the various stability notions of nonlinear difference systems, the comparison principle [10] and variation of constants formula by Agarwal [1] play a fundamental role.

Now, we recall some definitions of stability notions in [12].

**Definition 1.1.** System (1.1) is called an \( h \)-system around the null solution, or more briefly an \( h \)-system, if there exist a positive function \( h : \mathbb{N}(n_0) \to \mathbb{R} \) and constant \( c \geq 1 \), such that
\[ |x(n, n_0, x_0)| \leq c|x_0|h(n)h^{-1}(n_0), \quad n \geq n_0 \]
for \( |x_0| \) small enough (here \( h^{-1}(n) = \frac{1}{h(n)} \)).

If \( h \) is a bounded function, then an \( h \)-system permits the following types of stability:
**DEFINITION 1.2.** The zero solution of system (1.1), or more briefly system (1.1), is said to be

(hS) **h-stable** if \( c \geq 1, \delta \) exist as well as a positive bounded function \( h : \mathbb{N}(n_0) \to \mathbb{R} \) such that

\[
|x(n, n_0, x_0)| \leq c|x_0|h(n)h^{-1}(n_0), \quad n \geq n_0
\]

for \( |x_0| \leq \delta \).

(GhS) **globally h-stable** if system (1.1) is hS for every \( x_0 \in D \), where \( D \subset \mathbb{R}^m \) is a region which includes the origin,

(hSV) **h-stable in variation** if the zero solution of system (1.4) is hS,

(GhSV) **globally h-stable in variation** if the zero solution of system (1.4) is GhS.

The various notions about h-stability given by Definition 1.2 include several types of known stability properties such as uniform stability, uniform Lipschitz stability and exponential asymptotic stability. See [4–9,11–14].

**DEFINITION 1.3.** A function \( w : \mathbb{R}^+ \to \mathbb{R}^+ \) is said to be of the class \( F \) if

(F1) \( w(u) \) is nondecreasing and continuous for \( u \geq 0 \) and positive for \( u > 0 \),

(F2) there exists a nonnegative function \( r(\text{multiplier function}) \) defined on \( (0, \infty) \) such that

\[
w(\alpha u) \leq r(\alpha) w(u), \quad \text{for} \quad \alpha > 0, u \geq 0,
\]

(F3) \( \lim_{\alpha \to 0^+} \frac{r(\alpha)}{\alpha} \) exists.

Conti [3] defined two \( m \times m \) matrix functions \( A \) and \( B \) on \( \mathbb{R}^+ \) to be \( t_\infty \)-**similarity** if there is an \( m \times m \) matrix function \( S \) defined on \( \mathbb{R}^+ \) such that \( S'(t) \) is continuous, \( S(t) \) and \( S^{-1}(t) \) are bounded on \( \mathbb{R}^+ \), and

\[
\int_0^\infty |S' + SB - AS| dt < \infty.
\]

Now, we introduce the notion of \( n_\infty \)-summable similarity which is the corresponding \( t_\infty \)-similarity for the discrete case.

Let \( \mathcal{M} \) denote the set of all \( m \times m \) invertible matrices \( A(n) \) defined on \( \mathbb{N}(n_0) \) and \( \mathcal{S} \) be the subset of \( \mathcal{M} \) consisting of those nonsingular bounded matrices \( S(n) \) such that \( S^{-1}(n) \) is also bounded.

**DEFINITION 1.4.** A matrix \( A(n) \in \mathcal{M} \) is \( n_\infty \)-**summable similarity** to a matrix \( B(n) \in \mathcal{M} \) if there exists an \( m \times m \) matrix \( F(n) \) absolutely
summable over $\mathbb{N}(n_0)$, i.e.

$$\sum_{l=n_0}^{\infty} |F(l)| < \infty$$

such that

$$(1.5) \quad S(n + 1)B(n) - A(n)S(n) = F(n)$$

for some $S(n) \in \mathcal{S}$.

For the example of $n_\infty$-summable similarity, see [8, Example 2.6].

Medina and Pinto studied the important properties about hS for the various differential systems and the nonlinear difference systems [11–14]. We also investigated hS for nonlinear differential or difference systems [4–7].

In this paper, we show that two concepts of h-stability and h-stability in variation for nonlinear difference systems are equivalent by using the $n_\infty$-summable similarity of their associated variational systems. Also, we study h-stability for perturbed nonlinear system of (1.1) using the comparison principle and extended discrete Bihari-type inequality.

**Remark 1.5.** We can easily show that the $n_\infty$-summable similarity is an equivalence relation by the same method of Trench in [15]. Also, if two $m \times m$ matrices $A$ and $B$ are $n_\infty$-summable similar with $F(n) = 0$, then we say that they are *kinematically similar*.

**2. h-stability in variation for nonlinear difference systems**

For the linear difference systems, Medina and Pinto [12] showed that

$$\text{GhSV} \iff \text{GhS} \iff \text{hS} \iff \text{hSV}.$$  

Also, the associated variational system inherits the property of hS from the original nonlinear system. i.e., the zero solution $v = 0$ of (1.3) is hS when the zero solution $x = 0$ of (1.1) is hS in Theorem 2 [12]. Our purpose is to characterize the h-stability in variation via $n_\infty$-summable similarity. To do this, we need the following lemma.

**Lemma 2.1.** [8, Lemma 3.3] Assume that $f_x(n, 0)$ is $n_\infty$-summable similar to $f_x(n, x(n, n_0, x_0))$ for $n \geq n_0 \geq 0$ and $|x_0| \leq \delta$ for some constant $\delta > 0$ and $\sum_{n=n_0}^{\infty} \frac{h(n)}{h(n+1)} |F(n)| < \infty$ with the positive function $h(n)$ defined on $\mathbb{N}(n_0)$. Then the solution $v = 0$ of (1.3) is an h-system if and only if the solution $z = 0$ of (1.4) is an h-system.
Letting $h(n)$ be bounded on $\mathbb{N}(n_0)$, we obtain the following result [13, Theorem 3.5] as a corollary of Lemma 2.1.

**Corollary 2.2.** Assume that

$$
\sum_{n=n_0}^{\infty} \frac{h(n)}{h(n+1)} |f_x(n, x(n, n_0, x_0)) - f_x(n, 0)| < \infty, \quad n_0 \geq 0
$$

holds for $|x_0| \leq \delta$ with some $\delta > 0$. Then the solution $v = 0$ of (1.3) is hS if and only if the solution $z = 0$ of (1.4) is hS.

**Proof.** Setting $F(n) = f_x(n, x(n, n_0, x_0)) - f_x(n, 0)$ and $S(n) = I$ for $n \geq n_0 \geq 0$, we can easily see that $f_x(n, x(n, n_0, x_0))$ and $f_x(n, 0)$ are $n_0\infty$-summable similar. Thus all conditions of Lemma 2.1 are satisfied, and hence the solution $z = 0$ of (1.4) is hS. \qed

Medina and Pinto showed that hSV implies hS [12, Theorem 3] by using the formula

$$
x(n, n_0, x_0) = [\int_0^1 \Phi(n, n_0, sx_0)ds]x_0.
$$

Also, they proved the converse when the condition (2.1) holds [12, Theorem 14]. In the following theorem, we can improve Theorem 14 in [12] by assuming that $f_x(n, 0)$ is $n_0\infty$-summable similar to $f_x(n, x(n, n_0, x_0))$, instead of the above condition (2.1).

**Theorem 2.3.** Under the same conditions of Lemma 2.1, the solution $x = 0$ of (1.1) is hS if and only if the solution $x = 0$ of (1.1) is hSV.

**Proof.** If $z = 0$ of (1.4) is hS, then $x = 0$ of (1.1) is also hS by Theorem 2 [12]. Also, if $x = 0$ of (1.1) is hS, then $v = 0$ of (1.3) is also hS by Theorem 3.3 [13]. Thus $x = 0$ of (1.1) is hSV by Lemma 2.1. \qed

**Remark 2.4.** If $h(n)$ is a positive bounded function on $\mathbb{N}(n_0)$, then $\frac{h(n)}{h(n+1)}$ is bounded in general.

For example, letting $h(n) = \exp(-\sum_{s=n_0}^{n-1} s)$, $h(n)$ is a positive bounded function on $\mathbb{N}(n_0)$ but $\lim_{n\to\infty} \frac{h(n)}{h(n+1)} = \lim_{n\to\infty} \exp(n) = \infty$. Thus if $\frac{h(n)}{h(n+1)}$ is bounded, then the condition $\frac{h(n)}{h(n+1)} |F(n)| \in l_1(\mathbb{N}(n_0))$ in Lemma 2.1 can be replaced by $|F(n)| \in l_1(\mathbb{N}(n_0))$.

**Example 2.5.** To illustrate Theorem 2.3, we consider the scalar difference equation with the initial value

$$
x(n + 1) = f(n, x(n)) = \frac{e^{\lambda(n)}x(n)}{\sqrt{1 + 2x^2(n)}}, \quad x(n_0) = x_0, \quad n \geq n_0
$$
where \( \lambda(n) \) is a function defined on \( \mathbb{N}(n_0) \) with \( 0 < c < -\lambda(n) \) for some constant \( c \). Then the two concepts of hS and hSV of (2.2) are equivalent.

**Proof.** The solution \( x(n) \) of Eqn (2.2) with the initial value \( x(n_0) = x_0 \) is given by

\[
x(n, n_0, x_0) = \frac{\exp(\sum_{l=n_0}^{n-1} \lambda(l)) x_0}{\sqrt{1 + 2x_0^2 [1 + \sum_{l=n_0}^{n-2} \exp(2 \sum_{j=n_0}^{l} \lambda(j))]}}
\]

where \( \sum_{l=0}^{n-2} \exp(2 \sum_{j=0}^{2} \lambda(j)) = -1 \) and \( \sum_{l=0}^{n-1} \exp(2 \sum_{j=0}^{2} \lambda(j)) = 0 \). Then we obtain

\[
|x(n, n_0, x_0)| \leq |x_0| \exp(\sum_{l=n_0}^{n-1} \lambda(l)) = c_1|x_0|h(n)h^{-1}(n_0), \quad n \geq n_0,
\]

where \( h(n) = \exp(\sum_{l=0}^{n-1} \lambda(l)) \) and \( c_1 = 1 \). Thus the zero solution \( x = 0 \) of (2.2) is hS for a function \( \lambda(n) \) with \( 0 < c < -\lambda(n) \) for some constant \( c \). The fundamental matrix solutions of (1.4) and (1.3) are given by, respectively,

\[
\Phi(n, n_0, x_0) = \frac{\partial x(n, n_0, x_0)}{\partial x_0} = \frac{\exp(\sum_{l=n_0}^{n-1} \lambda(l))}{\{1 + 2x_0^2 [1 + \sum_{l=n_0}^{n-2} \exp(2 \sum_{j=n_0}^{l} \lambda(j))]\}^{\frac{3}{2}}}
\]

and

\[
\Phi(n, n_0, 0) = \frac{\partial x(n, n_0, 0)}{\partial x_0} = \exp(\sum_{l=n_0}^{n-1} \lambda(l)).
\]

Then we have

\[
|\Phi(n, n_0, x_0)| \leq |\Phi(n, n_0, 0)| \leq \exp(\sum_{l=n_0}^{n-1} \lambda(l)) \leq h(n)h^{-1}(n_0), \quad n \geq n_0.
\]

Thus \( x = 0 \) of (2.2) is hSV. Also, we easily can see that \( f_x(n, 0) \) and \( f_x(n, x(n, 0, x_0)) \) are \( n_\infty \)-summableley similar with some bounded invertible matrix

\[
S(n) = \frac{1 + 2x_0^2 [1 + \sum_{l=0}^{n-2} \exp(2 \sum_{j=0}^{l} \lambda(j))]}{1 + 2x_0^2 [1 + \sum_{l=0}^{n-3} \exp(2 \sum_{j=0}^{l} \lambda(j))]}.
\]
whose inverse $S^{-1}(n)$ is bounded. In fact, there exists $F(n)$ absolutely summable over $\mathbb{N}(0)$, i.e.,

$$\sum_{n=0}^{\infty} |F(n)| < \infty$$

such that

$$S(n+1)f_x(n, x(n, 0, x_0)) - f_x(n, 0)S(n)$$

$$= \left( \frac{1 + 2x_0^2 [1 + \sum_{j=0}^{n-1} \exp(2 \sum_{j=0}^{l} \lambda(j))]}{1 + 2x_0^2 [1 + \sum_{j=0}^{n-1} \exp(2 \sum_{j=0}^{l} \lambda(j))]} \right)^{\frac{1}{2}} e^{\lambda(n)}$$

$$- e^{\lambda(n)} \left( \frac{1 + 2x_0^2 [1 + \sum_{j=0}^{n-2} \exp(2 \sum_{j=0}^{l} \lambda(j))]}{1 + 2x_0^2 [1 + \sum_{j=0}^{n-3} \exp(2 \sum_{j=0}^{l} \lambda(j))]} \right)$$

$$= F(n),$$

since

$$\sum_{n=0}^{\infty} |F(n)| \leq e^{-c} \sum_{n=0}^{\infty} |2x_0^2 \exp\left(2 \sum_{l=0}^{n-2} \lambda(l)\right)|$$

$$\leq 2x_0^2 e^{-c} \sum_{n=0}^{\infty} e^{-2c(n-1)}$$

$$\leq 2x_0^2 e^c \left(1 + \frac{1}{e^{2c} - 1}\right) < \infty, \ |x_0| < \infty.$$

Since all conditions of Lemma 2.1 are satisfied, the solution $v = 0$ of (1.3) is hS if and only if the solution $z = 0$ of (1.4) is also hS. Hence the two notions hS and hSV of (1.1) are equivalent by Theorem 2.3. □

3. $h$-stability for perturbed difference systems

In this section, we examine the property of hS for the perturbed difference system of nonlinear difference system (1.1) using the comparison principle and Bihari-type inequalities. In our subsequent discussion we assume that for any two sequences $y(n)$ and $z(n) \in F(\mathbb{N}(n_0), \mathbb{R}^m)$, the operator $S$ satisfies the following property: $|y(n)| \leq |z(n)|$ implies $|Sy(n)| \leq |Sz(n)|$ and $|Sy(n)| \leq |S||y(n)|$ for each finite interval $n_0 \leq n \leq l$ of $\mathbb{N}(n_0)$ and $|S| : F(\mathbb{N}(n_0), \mathbb{R}^+) \to F(\mathbb{N}(n_0), \mathbb{R}^+)$ is a non-decreasing operator.
THEOREM 3.1. Assume that \( x = 0 \) of (1.1) is hSV with the nonincreasing function \( h(n) \) and

\[
|g(n, y, Sy)| \leq \iota(n, |y|, |Sy|)
\]

where \( \iota : \mathbb{N}(n_0) \times \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}^+ \) is strictly increasing in \( r \) and \( u \) for each fixed \( n \in \mathbb{N}(n_0) \) with \( \iota(n, 0, 0) = 0 \). Consider the scalar difference equation

\[
(3.1) \quad u(n + 1) = u(n) + \alpha(n, u(n), |S|u(n)), \quad u(n_0) = u_0, \quad c > 1.
\]

If the zero solution \( u = 0 \) of (3.1) is hS, then the zero solution \( y = 0 \) of (1.2) is also hS whenever \( u_0 = c|y_0| \).

Proof. Using the discrete analogue of Alekseev's formula in [1], the solutions of (1.1) and (1.2) with the same initial values are related by

\[
y(n, n_0, y_0) = x(n, n_0, y_0) + \sum_{l=n_0}^{n-1} \int_{l}^{1} \Phi(n, l + 1, \mu(y(l), \tau))d\tau \cdot g(l, y(l), Sy(l)),
\]

where \( \mu(y(n), \tau) = f(n, y(n)) + \tau g(n, y(n), Sy(n)) \), \( \tau \in [0, 1] \) and \( \Phi(n, n_0, x_0) \) is the fundamental matrix of (1.4). Then we have

\[
|y(n, n_0, y_0)| \leq |x(n, n_0, y_0)| + \sum_{l=n_0}^{n-1} \int_{l}^{1} |\Phi(n, l + 1, \mu(y(l), \tau))|d\tau |g(l, y(l), Sy(l))|
\]

\[
\leq c|y_0|h(n)h^{-1}(n_0) + c \sum_{l=n_0}^{n-1} h(n)h^{-1}(l + 1)\iota(l, |y(l)|, |Sy(l)|)
\]

\[
\leq c|y_0| + c \sum_{l=n_0}^{n-1} \iota(l, |y(l)|, |S| |y(l)|),
\]

since \( h(n) \) is nonincreasing. Thus we obtain

\[
|y(n)| - c \sum_{l=n_0}^{n-1} \iota(l, |y(l)|, |Sy(l)|) \leq c|y_0| = u_0
\]

\[
= u(n) - c \sum_{l=n_0}^{n-1} \iota(l, u(l), |S| u(l)).
\]
By the comparison principle in [10], we have $|y(n)| < u(n)$ for all $n \geq n_0 \geq 0$. Also we see that

$$|y(n)| < u(n) \leq c_1 u_0 h(n) h^{-1}(n_0) = d |y_0| h(n) h^{-1}(n_0), \quad d = c_1 c > 1,$$

since $u = 0$ of (3.1) is hS. This completes the proof.

\[ \square \]

**Remark 3.2.** Letting $g(n, y, Sy) = g(n, y)$ and $\iota(n, u, w) = \iota(n, u)$ in Theorem 3.1, we obtain the same result as that of Theorem 10 in [6].

We need the following lemma related to the discrete Bihari-type inequality to obtain the property of hS for difference system (1.2).

**Lemma 3.3.** Suppose that all conditions of Lemma 2.8 in [9] hold and let $w(u) = u^p$, where $0 < p < \infty$. Then

(i) $u(n) \leq d \exp[\sum_{l=n_0}^{n-1} (a(l) + b(l) \sum_{k=n_0}^{l-1} c(k))]$ for all $n \geq n_0$, where $p = 1$,

(ii) $u(n) \leq [d^p + q \sum_{l=n_0}^{n-1} (a(l) + b(l) \sum_{k=n_0}^{l-1} c(k))]^{1/2}$ for $n \leq m$, where $p \neq 1, q = 1 - p$ and

$$m = \sup\{n \in \mathbb{N}(n_0) | W(d) + \sum_{l=n_0}^{n-1} (a(l) + b(l) \sum_{k=n_0}^{l-1} c(k)) \in \text{Dom } W^{-1}\}$$

$$= \sup\{n \in \mathbb{N}(n_0) | W(d) + \sum_{l=n_0}^{n-1} a(l) + b(l) \sum_{k=n_0}^{l-1} c(k) \geq -\frac{u_0^q}{q}\}.$$

**Proof.** Let

$$v(n) = d + \sum_{l=n_0}^{n-1} a(l) u^p(l) + \sum_{l=n_0}^{n-1} b(l) \sum_{k=n_0}^{l-1} c(k) u^p(k), \quad v(n_0) = d.$$
Then we have
\[ \Delta v(n) = a(n)u^p(n) + b(n) \sum_{l=n_0}^{n-1} c(l)u^p(l) \leq a(n)v^p(n) + b(n) \sum_{l=n_0}^{n-1} c(l)v^p(l) \leq [a(n) + b(n) \sum_{l=n_0}^{n-1} c(l)]v^p(n). \]

If \( p = 1 \) then \( W(u) = \ln \left( \frac{u}{u_0} \right) \) and \( W^{-1}(u) = u_0 \exp u \). Thus we obtain
\[ v(n) \leq u_0 \exp \left[ \ln \left( \frac{d}{u_0} \right) + \sum_{l=n_0}^{n-1} (a(l) + b(l) \sum_{k=n_0}^{l-1} c(k)) \right] \]
\[ = d \exp \left[ \sum_{l=n_0}^{n-1} (a(l) + b(l) \sum_{k=n_0}^{l-1} c(k)) \right] \]
for all \( n \geq n_0 \).

If \( p \neq 1 \) and let \( d(n) = 0, k(n) = a(n) + b(n) \sum_{l=n_0}^{n-1} c(l) \) in Lemma 15.5 in [2], then we have
\[ v(n) \leq [d^q + q \sum_{l=n_0}^{n-1} (a(l) + b(l) \sum_{k=n_0}^{l-1} c(k))^{\frac{1}{q}}] \]
for \( n_0 \leq n \leq m \).

**Theorem 3.4.** Suppose that the zero solution \( z = 0 \) of (1.4) is hS with the positive function \( h(n) \) and for any \( n \geq n_0 \)
\[ |g(n, y, Sy)| \leq a(n)|y| + b(n) \sum_{l=n_0}^{n-1} c(l)|y(l)| \]
where \( a, b, c \in F(\mathbb{N}(n_0), \mathbb{R}^+) \) and
\[ M(n) = \exp c_1 \left[ \sum_{l=n_0}^{n-1} \frac{h(l)}{h(l+1)} a(l) + \frac{b(l)}{h(l+1)} \sum_{k=n_0}^{l-1} h(k)c(k) \right] < \infty. \]

Then the zero solution \( y = 0 \) of (1.2) is hS.
Proof. In view of the formula

\[ x(n, n_0, y_0) = \left[ \int_{n_0}^{n} \Phi(n, n_0, s y_0) ds \right] y_0 \]

and assumption we have

\[
|y(n)| \leq c_1 h(n) h^{-1}(n_0)|y_0| + \sum_{l=n_0}^{n-1} c_1 h(n) h^{-1}(l+1)|g(l, y(l), S y(l))| \\
\leq c_1 h(n) h^{-1}(n_0)|y_0| + c_1 \sum_{l=n_0}^{n-1} h(n) h^{-1}(l+1)|a(l)|y(l)| \\
+ b(l) \sum_{k=n_0}^{l-1} c(k)|y(k)|. 
\]

Putting \( u(n) = |y(n)| h^{-1}(n) \), then we obtain the following inequality from Lemma 3.3 (i):

\[
u(n) \leq c_1 u(n_0) \\
+ c_1 \sum_{l=n_0}^{n-1} \left[ \frac{h(l)}{h(l+1)} a(l) u(l) + \frac{b(l)}{h(l+1)} \sum_{k=n_0}^{l-1} h(k)c(k)u(k) \right] \\
\leq c_1 u(n_0) \exp \left[ c_1 \sum_{l=n_0}^{n-1} \left[ \frac{h(l)}{h(l+1)} a(l) + \frac{b(l)}{h(l+1)} \sum_{k=n_0}^{l-1} h(k)c(k) \right] \right] \\
\leq c_1 u(n_0) M(\infty).
\]

Hence we obtain

\[
|y(n)| \leq M|y_0|h(n) h^{-1}(n_0), \quad M = c_1 M(\infty) \geq 1,
\]

for all \( n \geq n_0 \), and the proof is complete.

\[
\square
\]

Corollary 3.5. Suppose that the zero solution \( z = 0 \) of (1.4) is hS with the positive increasing function \( h(n) \) and for any \( n \geq n_0 \)

\[
|g(n, y, S y)| \leq a(n)|y| + b(n) \sum_{l=n_0}^{n-1} c(l)|y(l)|
\]

where \( a, b, c \in l_1(\mathbb{N}(n_0)) \). Then the zero solution \( y = 0 \) of (1.2) is also hS.
Proof. From the increasing function of \( h(n) \), we obtain

\[
M(n) = \exp c_1 \left[ \sum_{l=n_0}^{n-1} \left( \frac{h(l)}{h(l+1)} a(l) + \frac{b(l)}{h(l+1)} \sum_{k=n_0}^{l-1} c(k) \right) \right] \\
\leq \exp c_1 \left( \sum_{l=n_0}^{n-1} [a(l) + b(l) \sum_{k=n_0}^{l-1} c(k)] \right) < \infty.
\]

Hence the zero solution \( y = 0 \) of (1.2) is also hS by Theorem 3.4.

\( \square \)

Example 3.6. To illustrate Theorem 3.4, we consider the scalar difference equation (2.2)

\[
x(n+1) = f(n, x(n)) = \frac{e^{\lambda(n)}x(n)}{\sqrt{1 + 2x^2(n)}}, \quad x(n_0) = y_0
\]

and its perturbation equation with same initial value

\[
y(n+1) = f(n, y(n)) + g(n, y(n), Sy(n)) = \frac{e^{\lambda(n)}y(n)}{\sqrt{1 + 2y^2(n)}} + \frac{e^{-n}y^7(n)}{1 + 4y^6(n)} \sin(ny(n)) + e^{-2n}Sy(n), \quad y(n_0) = y_0,
\]

where an operator \( S : F(\mathbb{N}(n_0), \mathbb{R}) \to F(\mathbb{N}(n_0), \mathbb{R}) \) is given by

\[
Sy(n) = \sum_{l=n_0}^{n-1} e^{-3l} \frac{y^7(l)}{1 + 4y^6(l)} \sin(ly(l)), \quad n \geq l \geq n_0.
\]

Then the zero solution \( y = 0 \) of (3.2) is hS with a positive function \( h(n) \), provided that \( \lambda(n) \) is a function defined on \( \mathbb{N}(n_0) \) and for all \( n \geq n_0 \geq 0, \)

\[
M(n) = \exp \left[ \sum_{l=n_0}^{n-1} (e^{-l} + e^{-2l} \sum_{k=n_0}^{l-1} e^{-3k}) \right] \\
\leq \exp \left[ e^{-(n_0-1)} + \frac{e^{-5(n_0-1)}}{(e^3 - 1)(e^5 - 1)} - \frac{e^{-2(n_0-4)}}{(e^3 - 1)(e^5 - 1)} \right].
\]

Proof. It follows from Example 2.5 that \( x = 0 \) of (2.2) is hSV with the positive function \( h(n) = e^{\sum_{l=0}^{n-1} \lambda(l)} \). On the other, the perturbation \( g \) of the nonlinear equation (3.2) satisfies the following conditions
of Theorem 3.4 for any $n \geq n_0$,

$$|g(n, y(n), Sy(n))| \leq e^{-n}|y(n)| + e^{-2n} \sum_{l=n_0}^{n-1} e^{-3l}|y(l)|$$

and

$$M(n) \leq \exp\left[\frac{e^{-n_0} - e^{-(n-n_0)}}{1 - e^{-1}} \right. \\
+ \sum_{l=n_0}^{n-1} \left(\frac{e^{-3n_0}e^{-2l}}{1 - e^{-3}} - \frac{e^{3n_0}}{1 - e^{-3}} \sum_{k=n_0}^{l-1} e^{-5k}\right) \\
\leq \exp\left[\frac{e^{-2(n_0-1)}}{e - 1} + \frac{e^{-5(n_0-1)}}{(e^3 - 1)(e^5 - 1)} - \frac{e^{-2(n_0-4)}}{(e^3 - 1)(e^5 - 1)}\right] \\
= M(\infty) < \infty.$$ 

Hence the zero solution $y = 0$ of (3.2) is $h$S by Theorem 3.4.

**Theorem 3.7.** Suppose that $x = 0$ of (1.1) is $h$S and

$$|g(n, y, Sy)| \leq a(n)w(|y|) + b(n)|Sy|,$$

$$|Sy| \leq \sum_{l=n_0}^{n-1} c(l)w(|y|), \, n \geq n_0$$

where $w \in \hat{F}$ with corresponding multiplier function $r$ and

$$\hat{a}(n) = \frac{h(n_0)a(n)}{h(n_0+1)|y_0|} r\left(\frac{h(n)|y_0|}{h(n_0)}\right),$$

$$\hat{b}(n) = \frac{h(n_0)b(n)}{h(n_0+1)|y_0|}, \, \hat{c}(n) = r\left(\frac{h(n)|y_0|}{h(n_0)}\right)c(n),$$

$$K = \max_{n \in \mathbb{N}(n_0)} W^{-1}[W(d) + \sum_{l=n_0}^{n-1} (\hat{a}(l) + \hat{b}(l) \sum_{k=n_0}^{l-1} \hat{c}(k))] < \infty.$$

Then for $y_0$ sufficiently small, every solution $y(n) = y(n, n_0, y_0)$ of (1.2) satisfies

$$|y(n)| \leq Kh(n)h^{-1}(n_0)|y_0|.$$
Proof. Using the assumptions of the theorem, we have the following inequality:

\[ |y(n)| \leq dh(n)h^{-1}(n_0)|y_0| + d \sum_{l=n_0}^{n-1} h(n)h^{-1}(l+1)[a(l)w(|y(l)|)] + b(l) \sum_{k=n_0}^{l-1} c(k)w(|y(k)|) \]

for all \( n \geq n_0 \). Let \( u(n) = \frac{|y(n)|h(n_0)}{h(n)|y_0|} \). Then we have

\[ u(n) \leq d + \sum_{l=n_0}^{n-1} \hat{a}(l)w(u(l)) + \sum_{l=n_0}^{n-1} \hat{b}(l) \sum_{k=n_0}^{l-1} \hat{c}(k)w(u(k)), \]

where \( \hat{a}(n) = \frac{h(n_0)a(n)}{h(n+1)|y_0|} \) and \( \hat{b}(n) = \frac{h(n_0)b(n)}{h(n+1)|y_0|} \), and \( \hat{c}(n) = r\left( \frac{h(n)|y_0|}{h(n_0)} \right)c(n) \). By Lemma 2.8 in [9] we have

\[ |y(n)| \leq h(n)h^{-1}(n_0)|y_0|W^{-1}[W(d) + \sum_{l=n_0}^{n-1} (\hat{a}(l) + \hat{b}(l) \sum_{k=n_0}^{l-1} \hat{c}(k))] \]

for \( n \leq m \), where

\[ m = \sup\{n \in \mathbb{N}(n_0)|W(d) + \sum_{l=n_0}^{n-1} (\hat{a}(l) + \hat{b}(l) \sum_{k=n_0}^{l-1} \hat{c}(k)) \in \text{Dom}W^{-1} \}. \]

\[ \square \]

**Corollary 3.8.** Suppose that \( x = 0 \) of (1.1) is hS with a nondecreasing \( h(n) \) and

\[ |\Phi(n, l+1, \mu(y(l), \tau)g(n, y, Sy)| \]

\[ \leq h(n)h^{-1}(l)[a(l)w(|y(l)|) + b(l) \sum_{k=n_0}^{l-1} c(k)w(|y(k)|)], \quad n \geq n_0 \]

where \( w \in \hat{F} \) with corresponding multiplier function \( r(\alpha) = \alpha \) for all \( \alpha > 0 \). Then

\[ |y(n)| \leq h(n)h^{-1}(n_0)|y_0|W^{-1}[W(d) + \sum_{l=n_0}^{n-1} (a(l) + b(l) \sum_{k=n_0}^{l-1} c(k))] \]

for all \( n \geq n_0 \).
Proof. Put \( u(n) = \frac{|y(n)| h(n_0)}{h(n)|y_0|} \), then we have

\[
    u(n) \leq d + \sum_{l=n_0}^{n-1} [a(l)w(u(l)) + b(l) \sum_{k=n_0}^{l-1} c(k)w(u(k))]
\]

for all \( n \geq n_0 \). Thus theorem is proved by Lemma 2.8 in [9]. \( \Box \)

References

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