SKewed POWER SERIES EXTENSIONS OF $\alpha$-RIGID P.P.-RINGS

EBRAHIM HASHEBIM AND AHMAD MOUSSAVI

ABSTRACT. We investigate skew power series of $\alpha$-rigid p.p.-rings, where $\alpha$ is an endomorphism of a ring $R$ which is not assumed to be surjective. For an $\alpha$-rigid ring $R$, $R[[x; \alpha]]$ is right p.p., if and only if $R[[x, x^{-1}; \alpha]]$ is right p.p., if and only if $R$ is right p.p. and any countable family of idempotents in $R$ has a join in $I(R)$.

1. Introduction

Throughout this paper $R$ denotes an associative ring with identity and $\alpha : R \to R$ is an endomorphism. We denote $C(R)$ the center of $R$ and $S = R[[x; \alpha]]$ the skew power series ring, whose elements are power series of the form $\sum_{i=0}^{\infty} r_i x^i$ with coefficients $r_i \in R$, where the addition is defined as usual and the multiplication subject to the condition $xb = \alpha(b)x$, for any $b \in R$. The set $\{x^i\}_{i \geq 0}$ is an Ore subset of $R[[x; \alpha]]$, so that one can localize $R[[x; \alpha]]$ and form the skew Laurent series ring $R[[x, x^{-1}; \alpha]]$. Elements of $R[[x, x^{-1}; \alpha]]$ are formal combinations of elements of the form $x^{-j} r x^i$, where $r \in R$ and $i, j$ are nonnegative integers.

Recall that $R$ is (quasi-)Baer if the right annihilator of every (right ideal) non-empty subset of $R$ is generated (as a right ideal) by an idempotent of $R$. These definitions are left-right symmetric. The study of Baer rings has its roots in functional analysis. In [19] Rickart studied $C^*$-algebras with the property that every right annihilator of any element is generated by a projection (i.e., $p$ is a projection if $p = p^2 = p^*$, where $*$ is the involution on the algebra). Using Rickart's work, Kaplansky [13] defined an AW*-algebra as a C*-algebra with the stronger property that the right annihilator of the nonempty subset is generated by a projection. A ring satisfying a generalization of Rickart's condition

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(i.e., every right annihilator of any element is generated (as a right ideal)
by an idempotent) has a homological characterization as a right p.p.-
ring. A ring \( R \) is called a right (resp. left) p.p.-ring if every principal
right (resp. left) ideal is projective (equivalently, if the right (resp. left)
annihilator of an element of \( R \) is generated (as a right (resp. left) ideal)
by an idempotent of \( R \)). \( R \) is called a p.p.-ring if it is both right and
left) \textit{principally quasi-Baer} (or simply right (resp. left) \textit{p.q.-Baer}) if
the right annihilator of a principal right (resp. left) ideal of \( R \) is generated
by an idempotent. A ring is called \textit{p.q.-Baer} if it is both right and left
\textit{p.q.-Baer}. Observe that every biregular ring and every quasi-Baer ring
is \textit{p.q.-Baer}. Note that in a \textit{reduced} ring \( R \) (i.e. it has no nonzero nilpo-
tent elements), \( R \) is \textit{p.q.-Baer} if and only if \( R \) is p.p. For more details
and examples of right \textit{p.q.-Baer} rings, see [4].

In [5], Birkenmeier et al. showed that the quasi-Baer condition is pre-
served by many polynomial extensions including \( R[[x; \alpha]] \) and \( R[[x, x^{-1};
\alpha]] \). Following Krempa [15], a ring \( R \) is said to be \( \alpha \)-\textit{rigid} if for each
\( a \in R, a\alpha(a) = 0 \) implies that \( a = 0 \). Note that \( \alpha \)-rigid rings are re-
duced, and hence \textit{abelian} (i.e. every idempotent is central). In [9] Hong
et al. showed that, an \( \alpha \)-rigid ring \( R \) is quasi-Baer if and only if \( R[[x; \alpha]] \)
is quasi-Baer. Following [18], a ring \( R \) is called \textit{Armendariz} if whenever
two polynomials \( f(x) = \sum_{i=0}^{n} a_i x^i, \ g(x) = \sum_{j=0}^{n} b_j x^j \in R[x] \) satisfy
\( f(x)g(x) = 0 \) we have \( a_i b_j = 0 \) for every \( i, j \). By [2, Theorem 10], for an
Armendariz ring \( R, R \) is left p.p. if and only if \( R[x] \) is left p.p. Fraser
and Nicholson in [7] showed that \( R[[x]] \) is reduced p.p. if and only if
\( R \) is reduced p.p. and any countable family of idempotents of \( R \) has
a least upper bound in \( I(R) \), the set of all idempotents. Z. Liu in [16,
Theorem 3], showed that: If \( R \) is a ring such that all left semicentral
idempotents are central, then \( R[[x]] \) is right p.q.-Baer if and only if \( R \)
is right p.q.-Baer and any countable family of idempotents in \( R \) has a
generalized join in \( I(R) \).

In this paper we show that for an \( \alpha \)-rigid ring \( R, R[[x; \alpha]] \) is right
p.p. if and only if \( R[[x, x^{-1}; \alpha]] \) is right p.p. if and only if \( R \) is right p.p.
and any countable family of idempotents in \( R \) has a join in \( I(R) \). As a
consequence, for a reduced ring \( R, R[[x, x^{-1}]] \) is right p.p. if and only
if \( R[[x]] \) is right p.p. if and only if \( R \) is right p.p. and any countable
family of idempotents in \( R \) has a join in \( I(R) \). This extends the main
result of Fraser and Nicholson [7].
2. Skew power series extensions of \( \alpha \)-rigid p.p.-rings

In this section, we give a necessary and sufficient condition for some rings under which the ring \( R[[x; \alpha]] \) is right p.p.

For a nonempty subset \( X \) of \( R \), \( r_R(X) \) and \( \ell_R(X) \) denote the right and left annihilators of \( X \) in \( R \) respectively. We put \( r\text{Ann}_R(2^R) = \{ r_R(V) \mid V \subseteq R \} \) and \( \ell\text{Ann}_R(2^R) = \{ \ell_R(V) \mid V \subseteq R \} \).

Motivated by results in Armendariz [2], Anderson and Camillo [1], Kim and Lee [14], Hong et al. [9] and [10], we introduce conditions (SA1) and (SA2) which are skew power series versions of the Armendariz rings:

**Definition 2.1.** For a ring \( R \) and a monomorphism \( \alpha : R \rightarrow R \), we say \( R \) satisfies the (SA1) condition if for each \( f(x) = \sum_{i=0}^{\infty} a_i x^i \) and \( g(x) = \sum_{j=0}^{\infty} b_j x^j \in S = R[[x; \alpha]] \), \( f(x)g(x) = 0 \), implies that \( a_i b_j = 0 \) for all \( i, j \).

**Lemma 2.2.** [9, Lemma 4]. Let \( R \) be \( \alpha \)-rigid. Then we have the following:

(i) If \( ab = 0 \), then \( \alpha^n(b) = \alpha^n(a)b = 0 \) for each positive integer \( n \).

(ii) If \( \alpha^k(b) = 0 \) for some positive integer \( k \), then \( ab = 0 \).

**Proposition 2.3.** Let \( R \) be \( \alpha \)-rigid and \( S \) the skew power series ring \( R[[x; \alpha]] \). Then we have the following:

(i) \( R \) satisfies condition (SA1);

(ii) \( \varphi : r\text{Ann}_R(2^R) \rightarrow r\text{Ann}_S(2^S) ; A \rightarrow AS \) is bijective;

(iii) \( \psi : \ell\text{Ann}_R(2^R) \rightarrow \ell\text{Ann}_S(2^S) ; B \rightarrow SB \) is bijective.

**Proof.** (i) It follows from [10, Proposition 17]. (ii) It is clear that \( \varphi \) is a well defined map. Let \( J \) be an element of \( r\text{Ann}_S(2^S) \). There exists a nonempty subset \( Y \) of \( S \) such that \( r_S(Y) = J \). Suppose that \( X \) is the set of coefficients of elements of \( Y \). We show that \( r_S(Y) = r_R(X)S \). Since \( R \) is \( \alpha \)-rigid, \( r_R(X) \subseteq r_S(Y) \) and hence \( r_R(X)S \subseteq r_S(Y) \). Let \( f(x) = a_0 + a_1 x + \cdots \in r_S(Y) \). Since \( R \) satisfies condition (SA1), \( Xa_i = 0 \) for \( i = 0, 1, \cdots \). Hence \( f(x) \in r_R(X)S \), thus \( r_S(Y) = r_R(X)S \).

Similarly we can prove (iii). \( \square \)

**Definition 2.4.** (Z. Liu, [16]). Let \( \{ e_0, e_1, \cdots \} \) be a countable family of idempotents of \( R \). We say \( \{ e_0, e_1, \cdots \} \) has a join in \( I(R) \) if there exists an idempotent \( e \in I(R) \) such that

1. \( e_i(1 - e) = 0 \), and

2. If \( f \in I(R) \) is such that \( e_i(1 - f) = 0 \), then \( e(1 - f) = 0 \).

**Theorem 2.5.** Let \( R \) be \( \alpha \)-rigid. Then the following conditions are equivalent:
1. \(S = R[[x;\alpha]]\) is right p.p.
2. \(R\) is right p.p. and any countable family of idempotents in \(R\) has a join in \(I(R)\).

\textbf{Proof.} 1\(\Rightarrow\)2. Let \(a \in R\). There exists an idempotent \(e(x) = e_0 + e_1x + \cdots \in S\) such that \(r_S(a) = e(x)S\). By [9, Corollary 7], \(e(x) = e_0\) and thus \(r_S(a) = e_0S\). Therefore \(r_R(a) = e_0R\). Suppose that \(\{e_0, e_1, \cdots\}\) is a countable family of idempotents in \(R\). Set \(\phi(x) = e_0 + e_1x + e_2x^2 + \cdots \in S\). Since \(S\) is right p.p., there exists an idempotent \(e(x) = f_0 + f_1x + \cdots \in S\), such that \(r_S(\phi(x)) = e(x)S\). By a similar argument we have, \(r_S(\phi(x)) = f_0S\). Hence, by Lemma 2.2, \(e_i f_0 = 0\) for \(i = 0, 1, \cdots\). Let \(g = 1 - f_0\). Then \(e_i(1-g) = 0\) for each \(i\). Suppose that \(\phi\) is an idempotent of \(R\) such that \(e_i(1-h) = 0\) for each \(i\). Then by Lemma 2.2, \((1-h) \in r_S(\phi(x))\). Thus \((1-h) = f_0(1-h)\) and \(g(1-h) = (1-f_0)(1-h) = 0\). Hence \(g\) is a join of the set \(\{e_0, e_1, \cdots\}\).

2\(\Rightarrow\)1. Let \(f(x) = a_0 + a_1x + \cdots \in S\). Then there exist idempotents \(e_i\), with \(i = 0, 1, \cdots\), such that \(r_R(a_i) = e_iR\). Suppose that \(h\) is a join of the set \(\{1- e_i\mid i = 0, 1, \cdots\}\). Thus \((1-e_i)(1-h) = 0\) and hence \((1-h) = e_i(1-h)\). Thus, \(a_i(1-h) = a_i e_i(1-h) = 0\) for \(i = 0, 1, \cdots\). Hence \((1-h) \in r_S(f(x))\), by Lemma 2.2, which implies that \((1-h)S \subseteq r_S(f(x))\). Suppose that \(g(x) = b_0 + b_1x + \cdots \in r_S(f(x))\). Since \(R\) satisfies condition (SA1), \(a_ib_j = 0\) for all \(i,j\). Then \(b_j = e_ib_j\) for all \(i,j\). Now \(b_j(1-e_i) = 0\) because \(e_i \in C(R)\) for all \(i,j\). Since \(R\) is right p.p., \(r_R(b_j) = f_jR\) for idempotents \(f_j \in R\). Thus \((1-e_i)(1-h) = 0\), so \((1-e_i) = f_j(1-e_i)\) for all \(i,j\). Hence from \((1-e_i) \in C(R)\), we have \((1-e_i)(1-f_j) = 0\). Since \(h\) is a join of \(\{1-e_i\mid i = 0, 1, \cdots\}\), \(h(1-f_j) = 0\) for all \(j\). Hence \(b_j = b_j - b_j f_j = (1-f_j)b_j = (1-h)(1-f_j)b_j \in (1-h)R\) for all \(j\). So \(g(x) \in (1-h)S\). Therefore \(r_S(f(x)) = (1-h)S\), and hence \(S\) is right p.p.

\textbf{Corollary 2.6.} (Fraser and Nicholson [7, Theorem 3]). Let \(R\) be a reduced ring. Then the following conditions are equivalent:
1. \(R[[x]]\) is right p.p.
2. \(R\) is right p.p. and any countable family of idempotents in \(R\) has a join in \(I(R)\).

3. Skew Laurent power series extensions of \(\alpha\)-rigid p.p.-rings

In this section, we give a necessary and sufficient condition for some rings under which the ring \(R[[x; x^{-1}; \alpha]]\) is right p.p.
Now consider D.A. Jordan's construction of the ring $A(R, \alpha)$ (See [12], for more details). Let $A(R, \alpha)$ or $A$ be the subset $\{x^{-i}rx^i \mid r \in R, i \geq 0\}$ of the skew power series ring $R[[x, x^{-1}; \alpha]]$. For each $j \geq 0$, $x^{-i}rx^i = x^{-(i+j)}\alpha^j(r)x^{(i+j)}$. It follows that the set of all such elements forms a subring of $R[[x, x^{-1}; \alpha]]$ with $x^{-i}rx^i + x^{-j}rx^j = x^{-(i+j)}(\alpha^j(r) + \alpha^i(s))x^{(i+j)}$ and $(x^{-i}rx^i)(x^{-j}sx^j) = x^{-(i+j)}\alpha^j(r)\alpha^i(s)x^{(i+j)}$ for $r, s \in R$ and $i, j \geq 0$. Note that $\alpha$ is actually an automorphism of $A(R, \alpha)$. We have $R[[x, x^{-1}; \alpha]] \simeq A[[x, x^{-1}; \alpha]]$, by way of an isomorphism which maps $x^{-i}rx^i$ to $\alpha^{-i}(r)x^{j-i}$. Also for an automorphism $\alpha$ of $R$ we have $R = A(R, \alpha)$.

**Definition 3.1.** For a ring $R$ and a monomorphism $\alpha : R \to R$, we say $R$ satisfies the (SA2) condition if for each $f(x) = \sum_{i=m}^{\infty} u_i^{(i)} x^i$ and $g(x) = \sum_{j=n}^{\infty} v_j x^j \in T = A[[x, x^{-1}; \alpha]]$, $f(x)g(x) = 0$, implies that $u_iv_j = 0$ for all $i, j$.

**Proposition 3.2.** Let $\alpha$ be an automorphism of $R$. Let $R$ be $\alpha$-rigid and $T$ the skew Laurent power series ring $R[[x, x^{-1}; \alpha]]$. Then we have the following:

(i) $R$ satisfies condition (SA2);
(ii) $\varphi : r\text{Ann}_R(2^R) \to r\text{Ann}_T(2^T); A \to AT$ is bijective;
(iii) $\psi : \ell\text{Ann}_R(2^R) \to \ell\text{Ann}_T(2^T); B \to TB$ is bijective.

**Proof.** (i) Let $f(x) = \sum_{i=m}^{\infty} u_i^{(i)} x^i$, $g(x) = \sum_{j=n}^{\infty} v_j x^j \in T = A[[x, x^{-1}; \alpha]]$ and $f(x)g(x) = 0$ with $m, n \in \mathbb{Z}$. Put $f_1(x) = x^{-m}f(x)$ and $g_1(x) = g(x)x^{-n}$, hence $f_1(x)g_1(x) = (\sum_{i=m}^{\infty} \alpha^m(u_i)x^{i-m})(\sum_{j=n}^{\infty} v_j x^j) = 0$. By [9, Proposition 17], $\alpha^m(u_i)v_j = 0$ for all $i, j$. Hence $u_iv_j = 0$ for all $i, j$, by Lemma 2.2.

In a similar way as in the proof of Propositions 2.2, we can prove (ii) and (iii).

**Lemma 3.3.** A ring $R$ is $\alpha$-rigid if and only if $A(R, \alpha)$ is $\alpha$-rigid.

**Proof.** It is clear that any subring of an $\alpha$-rigid ring is also $\alpha$-rigid. Suppose that $R$ is $\alpha$-rigid and $(x^{-i}rx^i)\alpha(x^{-i}rx^i) = 0$, where $i \geq 0$ and $r \in R$. Hence $r\alpha(r) = 0$, and so $r = 0$.

**Lemma 3.4.** Let $R$ be $\alpha$-rigid. Then each countable family of idempotents in $R$ has a join in $I(R)$ if and only if each countable family of idempotents in $A(R, \alpha)$ has a join in $I(A(R, \alpha))$.

**Proof.** Let $\{e_i^i \mid i = 0, 1, \ldots\}$ be a countable family of idempotents in $A$. For each $e_i$ there exists an idempotent $e_i \in R$ and nonnegative integer $j_i$ such that $e_i' = x^{-j_i}e_ix^{j_i}$. Then $\{e_i \mid i = 0, 1, \ldots\}$ has a join $e$ in
We show that $e$ is a join of $\{e_i' \mid i = 0, 1, \cdots\}$. Since $e_i(1-e) = 0$, $e_i'(1-e) = 0$ for all $i$, by Lemma 2.2. Suppose that $f' \in I(A)$ is such that $e_i(1-f') = 0$ for all $i$. Then there exist an idempotent $f \in R$ and nonnegative integer $n$ such that $f' = x^{-n}fx^n$. Then $1-f' = x^{-n}(1-f)x^n$. Since $e_i'(1-f') = 0$, $e_i(1-f) = 0$ for all $i$ by [9, Proposition 5], because $\alpha(e) = e$. Since $e$ is a join of $\{e_i \mid i = 0, 1, \cdots\}$, $e(1-f) = 0$. By Lemma 2.2, $e(1-f') = 0$, and hence $e$ is a join of $\{e_i' \mid i = 0, 1, \cdots\}$. Conversely, suppose that $\{e_i \mid i = 0, 1, \cdots\}$ is a countable family of idempotents in $R$. Then $\{e_i \mid i = 0, 1, \cdots\}$ has a join $e'$ in $I(A)$. There exist an idempotent $e \in R$ and nonnegative integer $n$ such that $e' = x^{-n}ex^n$. By a similar argument one can show that $e$ is a join of $\{e_i \mid i = 0, 1, \cdots\}$. □

**Lemma 3.5.** Let $R$ be $\alpha$-rigid. Then $R$ is right p.p. if and only if $A(R, \alpha)$ is right p.p.

**Proof.** Assume that $R$ is right p.p. Let $a = x^{-i}tx^i$ be an element of $A$ and $x^{-j}bx^j \in r_A(a)$. By Lemma 2.2, $b \in r_R(t)$. Since $R$ is right p.p., $r_R(t) = eR$ for an idempotent $e \in R$. Thus $eb = b$, so by Lemma 2.2, $\alpha^n(e)b = b$ for each positive integer $n$. Hence $e(x^{-j}bx^j) = x^{-j}bx^j$, thus $r_A(a) \subseteq eA$. Since $R$ is $\alpha$-rigid, $eA \subseteq r_A(a)$. Hence $r_A(a) = eA$, thus $A$ is right p.p. Conversely, suppose that $A$ is right p.p. Let $t \in R$. Since $R$ is $\alpha$-rigid and $A$ is p.p., $r_A(t) = (x^{-j}ex^j)A$, where $e$ is an idempotent of $R$ and $j$ is a nonnegative integer. By Lemma 2.2, $eR \subseteq r_R(t)$. Now let $b \in r_R(t)$. By Lemma 2.2, $b \in r_A(t) = (x^{-j}ex^j)A$, hence $b = (x^{-j}ex^j)b$. Therefore $b = eb$ and so $r_R(t) \subseteq eR$, which implies that $R$ is right p.p. □

**Theorem 3.6.** Let $R$ be $\alpha$-rigid. Then the following conditions are equivalent:

1. $R[[x, x^{-1}; \alpha]]$ is right p.p.
2. $R$ is right p.p. and any countable family of idempotents in $R$ has a join in $I(R)$.

**Proof.** We have $R[[x, x^{-1}; \alpha]] \simeq A[[x, x^{-1}; \alpha]]$ where $\alpha$ is an automorphism of $A$. By Lemma 3.3, $R$ is $\alpha$-rigid if and only if $A$ is $\alpha$-rigid. By Lemma 3.4, any countable family of idempotents in $R$ has a join in $I(R)$ if and only if any countable family of idempotents in $A$ has a join in $I(A)$. By Lemma 3.5, $R$ is right p.p. if and only if $A$ is right p.p. The rest of the proof is similar to the proof of Theorem 2.5. □

**Lemma 3.7.** Every $\alpha$-rigid ring satisfies condition (SA2).

**Proof.** We observe that $\alpha$ is an automorphism of $A(R, \alpha)$ and by Lemma 3.4, $A$ is $\alpha$-rigid. Now the proof follows from Proposition 3.2. □
The following result is a generalization of Fraser and Nicholson [7]:

**COROLLARY 3.8.** For an $\alpha$-rigid ring $R$, the following conditions are equivalent:
1. $R[[x; \alpha]]$ is right p.p.
2. $R[[x, x^{-1}; \alpha]]$ is right p.p.
3. $R$ is right p.p. and any countable family of idempotents in $R$ has a join in $I(R)$.

**Proof.** It follows from Theorems 2.5 and 3.6. □

**COROLLARY 3.9.** For a reduced ring $R$, the following conditions are equivalent:
1. $R[[x]]$ is right p.p.
2. $R[[x, x^{-1}]]$ is right p.p.
3. $R$ is right p.p. and any countable family of idempotents in $R$ has a join in $I(R)$.

The following example [6, Example 3.6] shows that condition "any countable family of idempotents in $R$ has a join in $I(R)$" is not superfluous.

**EXAMPLE 3.10.** There is a reduced right p.p.-ring $R$ such that $R[[x; \alpha]]$ is not a right p.p.-ring. For a given field $F$, let

$$R = \{(a_n)_{n=1}^{\infty} \in \Pi_{n=1}^{\infty} F_n \mid a_n \text{ is eventually constant } \}$$

which is a subring of $\Pi_{n=1}^{\infty} F_n$, where $F_n = F$ for $n = 1, 2, \ldots$. Then the ring $R$ is a commutative von Neumann regular ring and hence it is right p.p. Let $\alpha$ be the identity map on $R$. Then $R$ is $\alpha$-rigid, but $R[[x; \alpha]]$ is not right p.p.

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DEPARTMENT OF MATHEMATICS, UNIVERSITY OF TARBIAT MODARRES, P.O. BOX 14115-170, TEHRAN, IRAN
E-mail: eb.hashemi@yahoo.com
moussava@modares.ac.ir