A NOTE ON THE LEFSCHETZ FIXED POINT
THEOREM FOR ADMISSIBLE SPACES

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ABSTRACT. The Lefschetz fixed point theorem is extended to compact continuous maps defined on an admissible subset of a Hausdorff topological space.

1. Introduction

In this paper we present new Lefschetz fixed point theorems for single valued continuous compact maps \( f : X \to X \) where \( X \) is an admissible (to be defined later) subset of a Hausdorff topological space. Our definition of admissibility will include \( \text{NES} \) (compact) spaces so our results improve those in the literature; see [2] and the references therein.

For the remainder of this section we present some definitions and known results which will be needed throughout this paper. Consider vector spaces over a field \( K \). Let \( E \) be a vector space and \( f : E \to E \) an endomorphism. Now let \( N(f) = \{ x \in E : f^{(n)}(x) = 0 \text{ for some } n \} \) where \( f^{(n)} \) is the \( n^{th} \) iterate of \( f \), and let \( \tilde{E} = E \setminus N(f) \). Since \( f(N(f)) \subseteq N(f) \) we have the induced endomorphism \( \tilde{f} : \tilde{E} \to \tilde{E} \). We call \( f \) admissible if \( \dim \tilde{E} < \infty \); for such \( f \) we define the generalized trace \( Tr(f) \) of \( f \) by putting \( Tr(f) = tr(\tilde{f}) \) where \( tr \) stands for the ordinary trace.

**Definition 1.1.** Let \( f = \{ f_q \} : E \to E \) be an endomorphism of degree zero of a graded vector space \( E = \{ E_q \} \). We call \( f \) the **Leray endomorphism** if (i) all \( f_q \) are admissible and (ii) almost all \( E_q \) are trivial. For such \( f \) we define the generalized Lefschetz number \( \Lambda(f) \) by

\[
\Lambda(f) = \sum_q (-1)^q Tr(f_q).
\]

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Let \( H \) be the singular homology functor (with coefficients in the field \( K \)) from the category of topological spaces and continuous maps to the category of graded vector spaces and linear maps of degree zero. Thus \( H(X) = \{H_q(X)\} \) is a graded vector space, \( H_q(X) \) being the \( q \)-dimensional singular homology group of \( X \). For a continuous map \( f : X \to Y, \, H(f) \) is the induced linear map \( f_* = \{f_q\} \), where \( f_q : H_q(X) \to H_q(Y) \).

**Definition 1.2.** A continuous map \( f : X \to X \) is called a Lefschetz map provided \( f_* : H(X) \to H(X) \) is a Leray endomorphism. For such \( f \) we define the Lefschetz number \( \Lambda(f) \) of \( f \) by putting \( \Lambda(f) = \Lambda(f_*) \). We know if \( f \) and \( g \) are homotopic \( (f \sim g) \) and if \( f \) is a Lefschetz map, then \( g \) is a Lefschetz map with \( \Lambda(g) = \Lambda(f) \).

**Definition 1.3.** A space \( X \) is said to be a Lefschetz space provided any continuous map \( f : X \to X \) is a Lefschetz map and \( \Lambda(f) \neq 0 \) implies \( f \) has a fixed point.

By a space we mean a Hausdorff topological space. Let \( Q \) be a class of topological spaces. A space \( Y \) is a neighborhood extension space for \( Q \) (written \( Y \in NES(Q) \)) if \( \forall X \in Q, \forall K \subseteq X \) closed in \( X \), and for any continuous function \( f_0 : K \to Y \), there exists a continuous extension \( f : U \to Y \) of \( f_0 \) over a neighborhood \( U \) of \( K \) in \( X \).

The following result was established in [2].

**Theorem 1.1.** Every \( NES(\text{compact}) \) is a Lefschetz space.

**2. Fixed point theory**

We begin this section with a simple extension of Theorem 1.1. Let \( X \) be a subset of a Hausdorff topological space. Then \( X \) is said to be Borsuk \( NES(\text{compact}) \) if \( X \) is dominated by a \( NES(\text{compact}) \) space \( Y \) i.e. there exists a subset \( Y \) of a Hausdorff topological space with \( Y \in NES(\text{compact}) \), and continuous maps \( r : Y \to X, \, s : X \to Y \) with \( rs = 1_X \).

**Theorem 2.1.** Let \( X \) be a subset of a Hausdorff topological space and assume \( X \) is Borsuk \( NES(\text{compact}) \). Then \( X \) is a Lefschetz space.

**Proof.** Let \( f : X \to X \) be a continuous compact map. We know there exists \( Y, \, r \) and \( s \) as described above. Notice \( sf r : Y \to Y \) is a continuous compact map. From Theorem 1.1 we know \( \Lambda(sfr) \) is well defined. Also [3, Lemma 3] (or [2, Example 3.4]) guarantees that \( \Lambda(f) \)
is well defined and $\Lambda(f) = \Lambda(sf r)$. Next assume $\Lambda(f) \neq 0$. Then
$\Lambda(sf r) \neq 0$ so Theorem 1.1 guarantees that there exists $x \in Y$ with
$x = sf r(x)$. Let $w = r(x)$ and notice $x = sf(w) \Rightarrow w = rs f(w)$. This
together with $rs = 1_X$ yields $w = f(w)$, so the proof is complete. 

For our next result we assume $X$ is a subset of a Hausdorff topological
vector space $E$. We say $X$ is NES admissible if for every compact subset
$K$ of $X$ and every neighborhood $V$ of zero there exists a continuous
function $h_V : K \rightarrow X$ such that

(i) $x - h_V(x) \in V$ for all $x \in K$;
(ii) $h_V(K)$ is contained in a subset $C$ of $X$ with $C \in NES(\text{compact})$;
(iii) $h_V$ and $i : K \hookrightarrow X$ are homotopic.

**Theorem 2.2.** Let $E$ be a Hausdorff topological vector space and let
$X \subseteq E$ be NES admissible. Then $X$ is a Lefschetz space.

**Proof.** Let $f : X \rightarrow X$ be a continuous compact map. Next let $N$ be
a fundamental system of neighborhoods of the origin $0$ in $E$ and $V \in N$.
Let $K = f(X)$. Now there exists a continuous function $h_V : K \rightarrow X
$ and a $C \subseteq X$ with $C \in NES(\text{compact})$, $x - h_V(x) \in V$ for all $x \in K$,
h_V(K) \subseteq C$ and $h_V \sim i$. Notice also that $h_V f : X \rightarrow X$ is a continuous
compact map with $h_V f \sim f$. Let $g_V = h_V f|_C$ so $g_V : C \rightarrow C$ is
a continuous compact map. From Theorem 1.1 we know that $g_V$ is a
Lefschetz map so in particular $\Lambda(g_V)$ is well defined. Also [2, Lemma
3.2 (see Example 3.3)] implies that $h_V f : X \rightarrow X$ is a Lefschetz map
with $\Lambda(h_V f) = \Lambda(g_V)$. Now since $h_V f \sim f$ we have that $f : X \rightarrow X$ is
a Lefschetz map with $\Lambda(f) = \Lambda(h_V f)$.

Next assume $\Lambda(f) \neq 0$. Then $\Lambda(h_V f) \neq 0$ so Theorem 1.1 guarantees
that there exists $x_V \in C$ with $x_V = h_V (f(x_V))$. Now since $y_V = f(x_V) \subseteq K$ then from (i) above we have $y_V - h_V(y_V) \in V$ so $y_V - x_V \in V$.
Now since $K = \overline{F(X)}$ is compact we may assume without loss of
generality that there exists $x$ with $y_V \rightarrow x$. Also since $y_V - x_V \in V$ we
have $x_V \rightarrow x$. This together with $y_V = f(x_V)$ and the continuity of $f$
implies $x = f(x)$, and the proof is complete.

**Remark 2.1.** A similar result could be obtained if $C \in NES (\text{com-
 pact})$ in (ii) above is replaced by $C$ is Borsuk NES(\text{compact}); we only
need replace Theorem 1.1 with Theorem 2.1 in the proof of Theorem 2.2.

Let $X$ be a subset of a Hausdorff topological space. Then $X$ is said to
be Borsuk NES admissible if $X$ is dominated by a NES admissible space
$Y$ i.e., there exists a Hausdorff topological vector space $E$, a $Y \subseteq E$
which is NES admissible, and continuous maps \( r : Y \to X \), \( s : X \to Y \) with \( rs = 1_X \).

Essentially the same reasoning as in Theorem 2.1 establishes the following result.

**THEOREM 2.3.** Let \( X \) be a subset of a Hausdorff topological space and assume \( X \) is Borsuk NES admissible. Then \( X \) is a Lefschetz space.

Let \( X \) be a subset of a Hausdorff topological vector space \( E \). Let \( V \) be a neighborhood of the origin \( 0 \) in \( E \). \( X \) is said to be NES admissible \( V \)-dominated if there exists a NES admissible space \( X_V \) and two continuous functions \( r_V : X_V \to X \), \( s_V : X \to X_V \) such that \( x - r_V s_V(x) \in V \) for all \( x \in X \) and also that \( r_V s_V \) and \( i : X \to X \) are homotopic. \( X \) is said to be almost NES admissible dominated if \( X \) is NES admissible \( V \)-dominated for every neighborhood \( V \) of the origin \( 0 \) in \( E \).

**THEOREM 2.4.** Let \( X \) be a subset of a Hausdorff topological vector space \( E \). Also assume \( X \) is almost NES admissible dominated. Then \( X \) is a Lefschetz space.

**Proof.** Let \( f : X \to X \) be a continuous compact map and let \( N \) be a fundamental system of neighborhoods of the origin \( 0 \) in \( E \) and \( V \in N \). Let \( K = f(X) \). Now there exists a NES admissible space \( X_V \) and two continuous functions \( r_V : X_V \to X \), \( s_V : X \to X_V \) such that \( x - r_V s_V(x) \in V \) for all \( x \in X \) and \( r_V s_V \sim i \). Notice \( s_V f r_V : X_V \to X_V \) is a continuous compact map and from Theorem 2.2 we know that \( \Lambda(s_V f r_V) \) is well defined. Also [2, Lemma 3.2] guarantees that \( \Lambda(f r_V s_V) \) is well defined and \( \Lambda(f r_V s_V) = \Lambda(s_V f r_V) \). Also since \( r_V s_V \sim i \) we have immediately that \( f r_V s_V \sim f \). Thus \( f \) is a Lefschetz map and \( \Lambda(f) = \Lambda(f r_V s_V) = \Lambda(s_V f r_V) \).

Now assume \( \Lambda(f) \neq 0 \). Then \( \Lambda(s_V f r_V) \neq 0 \) so Theorem 2.2 guarantees that there exists \( x_V \in X_V \) with \( x_V = s_V f r_V(x_V) \). Let \( y_V = r_V(x_V) \) and notice \( y_V = r_V s_V f(y_V) \). Since \( x - r_V s_V(x) \in V \) for all \( x \in X \) we have

\[
  f(y_V) - r_V s_V f(y_V) \in V.
\]

Thus \( f(y_V) - y_V \in V \). Let \( w_V = f(y_V) \in K \). Now since \( K = \overline{F(X)} \) is compact we may assume without loss of generality that there exists a \( x \) with \( w_V \to x \). Also since \( w_V - y_V \in V \) we have \( y_V \to x \). These together with \( w_V = f(y_V) \) and the continuity of \( f \) implies \( x = f(x) \). \(\square\)

Next we extend Theorem 2.2 to the case of Hausdorff topological spaces. First we gather together some well known preliminaries. For a subset \( K \) of a topological space \( X \), we denote by \( \text{Cov}_X(K) \) the set of all
coverings of $K$ by open sets of $X$ (usually we write $Cov(K) = Cov_X(K)$). Given a map $f : X \rightarrow X$ and $\alpha \in Cov(X)$, a point $x \in X$ is said to be an $\alpha$-fixed point of $f$ if there exists a member $U \in \alpha$ such that $x \in U$ and $f(x) \in U$. Given two maps $f, g : X \rightarrow Y$ and $\alpha \in Cov(Y)$, $f$ and $g$ are said to be $\alpha$-close if for any $x \in X$ there exists $U_x \in \alpha$, $f(x) \in U_x$ and $g(x) \in U_x$.

The following result can be found in [2, p.272].

**Theorem 2.5.** Let $X$ be a topological space and $f : X \rightarrow X$ a continuous map. Suppose there exists a cofinal family of coverings $\theta \subseteq Cov_X(f(X))$ such that $f$ has an $\alpha$-fixed point for every $\alpha \in \theta$. Then $f$ has a fixed point.

**Remark 2.2.** From Theorem 2.5 in proving the existence of fixed points in uniform spaces for continuous compact maps it suffices [1, p.298] to prove the existence of approximate fixed points (since open covers of a compact set $A$ admit refinements of the form $\{U[x] : x \in A\}$ where $U$ is a member of the uniformity [4, p.199] so such refinements form a cofinal family of open covers). For convenience in this paper we will apply Theorem 2.5 only when the space is uniform.

Let $X$ be a subset of a Hausdorff topological space and let $X$ be a uniform space. Then $X$ is said to be Schauder NES admissible if for every compact subset $K$ of $X$ and every open covering $\alpha \in Cov_X(K)$ there exists a continuous function $\pi_\alpha : K \rightarrow X$ and a subset $C$ of $X$ with $C \in NES(\text{compact})$ and

(i) $\pi_\alpha$ and $i : K \hookrightarrow X$ are $\alpha$-close;
(ii) $\pi_\alpha(K)$ is contained in $C$;
(iii) $\pi_\alpha$ and $i : K \hookrightarrow X$ are homotopic.

**Theorem 2.6.** Let $X$ be a subset of a Hausdorff topological space and let $X$ be a uniform space. Also suppose $X$ is Schauder NES admissible. Then $X$ is a Lefschetz space.

**Proof.** Let $f : X \rightarrow X$ be a continuous compact map, $K = \overline{f(X)}$ and $\alpha \in Cov_X(K)$. Then there exists a continuous function $\pi_\alpha : K \rightarrow X$, a subset $C$ of $X$ with $C \in NES(\text{compact})$, $\pi_\alpha(K) \subseteq C$, $\pi_\alpha$ and $i : K \hookrightarrow X$ are $\alpha$-close and $\pi_\alpha \sim i$. Let $f_\alpha = \pi_\alpha f$ and notice $f_\alpha : X \rightarrow X$ is a continuous compact map with $f_\alpha \sim f$. Let $g_\alpha = f_\alpha|_C$ and note $g_\alpha : C \rightarrow C$ is a continuous compact map. From Theorem 1.1 we know that $g_\alpha$ is a Lefschetz map and also from [2, Lemma 3.2] we have that $f_\alpha : X \rightarrow X$ is a Lefschetz map with $\Lambda(f_\alpha) = \Lambda(g_\alpha)$. Next since $f_\alpha \sim f$ we have that $f : X \rightarrow X$ is a Lefschetz map with $\Lambda(f) = \Lambda(f_\alpha)$.
Next assume $\Lambda(f) \neq 0$. Then $\Lambda(f_\alpha) \neq 0$ so Theorem 1.1 guarantees that there exists $x \in C$ with $x = \pi_\alpha f(x)$. Since $\pi_\alpha$ and $i$ are $\alpha$-close there exists $U \in \alpha$ with $\pi_\alpha f(x) = x \in U$ and $f(x) \in U$. Thus $f$ has an $\alpha$-fixed point. The result now follows from Theorem 2.5 (with Remark 2.2). \hfill \Box

**Remark 2.3.** As in Remark 2.1 it is possible to replace $C \in NES$ (compact) in (ii) above with $C$ Borsuk NES(compact).

Let $X$ be a subset of a Hausdorff topological space. Then $X$ is said to be Borsuk Schauder NES admissible if $X$ is dominated by a uniform space $Y$ which is Schauder NES admissible i.e. there exists a uniform space $Y$ which is Schauder NES admissible, and continuous maps $r : Y \to X$, $s : X \to Y$ with $rs = 1_X$.

Essentially the same reasoning as in Theorem 2.1 establishes the following result.

**Theorem 2.7.** Let $X$ be a subset of a Hausdorff topological space and assume $X$ is Borsuk Schauder NES admissible. Then $X$ is a Lefschetz space.

Let $X$ be a Hausdorff topological space and let $\alpha \in Cov(X)$. $X$ is said to be Schauder NES admissible $\alpha$-dominated if there exists a Schauder NES admissible space $X_\alpha$ and two continuous functions $r_\alpha : X_\alpha \to X$, $s_\alpha : X \to X_\alpha$ such that $r_\alpha s_\alpha : X \to X$ and $i : X \to X$ are $\alpha$-close and also that $r_\alpha s_\alpha \sim i$. $X$ is said to be almost Schauder NES admissible dominated if $X$ is Schauder NES admissible $\alpha$-dominated for every $\alpha \in Cov(X)$.

Our next result was motivated by ideas in [2].

**Theorem 2.8.** Let $X$ be a uniform space and let $X$ be almost Schauder NES admissible dominated. Then $X$ is a Lefschetz space.

*Proof. Let $f : X \to X$ be a continuous compact map, $K = f(X)$, and $\alpha \in Cov_X(K)$. Now there exists a Schauder NES admissible space $X_\alpha$ and two continuous functions $r_\alpha : X_\alpha \to X$, $s_\alpha : X \to X_\alpha$ such that $r_\alpha s_\alpha : X \to X$ and $i : X \to X$ are $\alpha$-close and also that $r_\alpha s_\alpha \sim i$. Notice $s_\alpha f r_\alpha : X_\alpha \to X_\alpha$ is a continuous compact map and from Theorem 2.6 we know that $\Lambda(s_\alpha f r_\alpha)$ is well defined. Also [2, Lemma 3.2] guarantees that $\Lambda(f r_\alpha s_\alpha)$ is well defined and $\Lambda(f r_\alpha s_\alpha) = \Lambda(s_\alpha f r_\alpha)$. Since $r_\alpha s_\alpha \sim i$ we have immediately that $f r_\alpha s_\alpha \sim f$. Thus $f$ is a Lefschetz map and $\Lambda(f) = \Lambda(f r_\alpha s_\alpha) = \Lambda(s_\alpha f r_\alpha)$.

Now assume $\Lambda(f) \neq 0$. Then $\Lambda(s_\alpha f r_\alpha) \neq 0$ so Theorem 2.6 guarantees that there exists $x_\alpha \in X_\alpha$ with $x_\alpha = s_\alpha f r_\alpha(x_\alpha)$. Let $y_\alpha = r_\alpha(x_\alpha)$
and notice \( y_\alpha = r_\alpha s_\alpha f(y_\alpha) \). Now since \( i \) and \( r_\alpha s_\alpha \) are \( \alpha \)-close there exists \( U_\alpha \in \alpha \) with \( f(y_\alpha) \in U_\alpha \) and \( r_\alpha s_\alpha f(y_\alpha) \in U_\alpha \) i.e. \( f(y_\alpha) \in U_\alpha \) and \( y_\alpha \in U_\alpha \). In particular \( f \) has an \( \alpha \)-fixed point. The result now follows from Theorem 2.5 (with Remark 2.2).

\[ \square \]

References


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