UNIFORM TOPOLOGY ON DIFFERENCE ALGEBRAS

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Abstract. In this paper, we consider a collection of ideals of a difference algebra \( X \). We use the concept of congruence relation with respect to ideals to construct a uniformity that induces a topology on \( X \) which makes this to a topological difference algebras. We study the properties of this topology regarding different ideals.

1. Introduction

In [2], Hausdorff introduced the order group, which is a general algebraic system combining a partial ordered set and a group. In [7], Meng introduced the concept of difference algebra as result of combining a partial order and set difference operation. In [8], E. Roh et. al. study some algebraic property of this algebraic structure. In this note we consider a collection of ideals and use congruence relation with respect to ideals to define a uniformity and make the difference algebra into a uniform topological space. Then we obtain some related results which have been mentioned in the abstract.

2. Preliminaries

Definition 2.1. [7] A difference algebra is an algebra \( (X, *, \leq, 0) \) with binary operation \(*\) and a binary relation \( \leq \) on \( X \) and constant \( 0 \in X \) such that:

\begin{align*}
(D1) \quad & (X, \leq) \text{ is a poset,} \\
(D2) \quad & x \leq y \text{ implies } x * z \leq y * z, \\
(D3) \quad & (x * y) * z \leq (x * z) * y, \\
(D4) \quad & 0 \leq x * x, \\
(D5) \quad & x \leq y \text{ if and only if } x * y \leq 0.
\end{align*}

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We shall write the binary relation "≤" by putting \( x \leq y \) if and only if \((x, y) \in \leq\), for convenience.

**Lemma 2.2.** [7] In each difference algebra \( X \), the following relations hold for all \( x, y, z \in X \):

1. \((x * y) * z = (x * z) * y,\)
2. \(x * x = 0,\)
3. \(x * y \leq z\) implies that \(x * z \leq y,\)
4. \((x * (x * y)) * y = 0,\)
5. \(x \leq y\) implies that \(z * y \leq z * x,\)
6. \(x * (x * (x * y)) = x * y,\)
7. \(x * 0 = x,\)
8. \(0 * (x * y) = (0 * x) * (0 * y),\)
9. \((x * y) * (x * z) \leq z * y.\)

**Definition 2.3.** [8] A weak ideal of a difference algebra \( X \) is a nonempty subset \( I \) of \( X \) such that for all \( x, y \in X \), we have

1. \(0 \in I,\)
2. \(x * y \in I\) and \(y \in I\) imply \(x \in I.\)

**Definition 2.4.** [8] A weak ideal \( I \) of a difference algebra \( X \) is called ideal if it satisfies

1. \(x \leq y\) and \(y \in I\) imply \(x \in I.\)

**Definition 2.5.** A congruence relation on a difference algebra \( X \) is an equivalence relation \( R \) on \( X \) moreover if \(xRy\) and \(uRu\), then we have

1. \((x * u)R(y * v),\)
2. \((x * u)R(y * v)\) and \((u * x)R(v * y).\)
3. \((x \wedge u)R(y \wedge v)\) and \((x \vee u)R(y \vee v).\)

**Theorem 2.6.** [8] Let \( I \) be an ideal of a difference algebra \( X \). Define:

\[ x \equiv_I y \text{ if and only if } x * y \in I \text{ and } y * x \in I. \]

Then \( \equiv_I \) is a congruence relation on \( X. \)

**3. Uniformity in difference algebra**

From now on \((X, *, \leq, 0)\) (briefly, \( X \)) is a difference algebra, unless otherwise is stated.

Let \( X \) be a nonempty set and \( U, V \) be any subset of \( X \times X \). Define:
$U \circ V = \{(x, y) \in X \times X \mid (z, y) \in U \text{ and } (x, z) \in V, \text{ for some } z \in X\},$

$U^{-1} = \{(x, y) \in X \times X \mid (y, x) \in U\},$

$\Delta = \{(x, x) \in X \times X \mid x \in X\}.$

**Definition 3.1.** [5] By a uniformity on $X$ we shall mean a nonempty collection $\mathcal{K}$ of subsets of $X \times X$ which satisfies the following conditions:

$(U_1)$ $\Delta \subseteq U$ for any $U \in \mathcal{K},$

$(U_2)$ if $U \in \mathcal{K}$, then $U^{-1} \in \mathcal{K},$

$(U_3)$ if $U \in \mathcal{K}$, then there exist a $V \in \mathcal{K}$, such that $V \circ V \subseteq U,$

$(U_4)$ if $U, V \in \mathcal{K}$, then $U \cap V \in \mathcal{K},$

$(U_5)$ if $U \in \mathcal{K}$, and $U \subseteq V \subseteq X \times X$ then $V \in \mathcal{K}.$

The pair $(X, \mathcal{K})$ is called a uniform structure (uniform space).

**Theorem 3.2.** Let $\Lambda$ be an arbitrary family of ideals of $X$ which is closed under intersection. If

$$U_I = \{(x, y) \in X \times X \mid x \equiv_I y\}$$

and

$$\mathcal{K}^* = \{U_I \mid I \in \Lambda\},$$

then $\mathcal{K}^*$ satisfies the conditions $(U_1)$-$$(U_4).$

**Proof.** $(U_1):$ Since $I$ is an ideal of $X$ then we have $x \equiv_I x$ for any $x \in X,$ hence $\Delta \subseteq U_I,$ for all $U_I \in \mathcal{K}^*.$

$(U_2):$ For any $U_I \in \mathcal{K}^*,$ we have

$$(x, y) \in (U_I)^{-1} \Leftrightarrow (y, x) \in U_I \Leftrightarrow y \equiv_I x \Leftrightarrow x \equiv_I y \Leftrightarrow (x, y) \in U_I.$$

$(U_3):$ For any $U_I \in \mathcal{K}^*,$ the transitivity of $\equiv_I$ implies that $U_I \circ U_I \subseteq U_I.$

$(U_4):$ For any $U_I, U_J \in \mathcal{K}^*,$ we claim that $U_I \cap U_J = U_{I \cap J}.$ Let $(x, y) \in U_I \cap U_J.$ Then $x \equiv_I y$ and $x \equiv_J y.$ Hence $x \ast y \in I,$ $y \ast x \in I$ and $x \ast y \in J,$ $y \ast x \in J.$ Then $x \equiv_{I \cap J} y$ and hence $(x, y) \in U_{I \cap J}.$

Conversely, let $(x, y) \in U_{I \cap J}.$ Then $x \equiv_{I \cap J} y,$ hence $x \ast y \in I \cap J$ and $y \ast x \in I \cap J.$ Then $x \ast y \in I,$ $y \ast x \in I,$ $x \ast y \in J$ and $y \ast x \in J.$ Therefore $x \equiv_I y$ and $x \equiv_J y.$ Then $(x, y) \in U_I \cap U_J.$ So $U_I \cap U_J = U_{I \cap J}.$ Since $I, J \in \Lambda,$ then $I \cap J \in \Lambda,$ $U_I \cap U_J \in \mathcal{K}^*.$ $\square$

**Theorem 3.3.** Let $\mathcal{K} = \{U \subseteq X \times X \mid U_I \subseteq U \text{ for some } U_I \in \mathcal{K}^*\}.$ Then $\mathcal{K}$ satisfies a uniformity on $X$ and the pair $(X, \mathcal{K})$ is a uniform structure.
Proof. By Theorem 3.2, the collection $\mathcal{K}$ satisfies the conditions $(U_1)$–$(U_4)$. It suffices to show that $\mathcal{K}$ satisfies $(U_5)$. Let $U \in \mathcal{K}$ and $U \subseteq V \subseteq X \times X$. Then there exists a $U_I \subseteq U \subseteq V$, which means that $V \in \mathcal{K}$. This proves the theorem. □

Let $x \in X$ and $U \in \mathcal{K}$. Define

$$U[x] := \{y \in X \mid (x,y) \in U\}.$$

Theorem 3.4. Given a difference algebra $X$, then

$$T = \{G \subseteq X \mid \forall x \in G, \exists U \in \mathcal{K}, U[x] \subseteq G\}$$

is a topology on $X$.

Proof. It is clear that $\emptyset$ and the set $X$ belong to $T$. Also from the definition, it is clear that $T$ is closed under arbitrary union. Finally to show that $T$ is closed under finite intersection, let $G, H \in T$ and suppose $x \in G \cap H$. Then there exist $U$ and $V \in \mathcal{K}$ such that $U[x] \subseteq G$ and $V[x] \subseteq H$. Let $W = U \cap V$, then $W \in \mathcal{K}$. Also $W[x] \subseteq U[x] \cap V[x]$ and so $W[x] \subseteq G \cap H$ therefore $G \cap H \in T$. Thus $T$ is topology on $X$. □

Note that for any $x$ in $X$, $U[x]$ is an open neighborhood of $x$.

Definition 3.5. Let $(X, \mathcal{K})$ be a uniform structure. Then the topology $T$ is called the uniform topology on $X$ induced by $\mathcal{K}$.

Proposition 3.6. Topological space $(X, T)$ is completely regular.

Proof. See Theorem 14.2.9, [5]. □

4. Topological property of space $(X, T)$

Let $X$ be a difference algebra and $C, D$ subsets of $X$. Then we define $C \ast D$ as follows:

$$C \ast D = \{x \ast y \mid x \in C, y \in D\}$$

Let $X$ be a difference algebra and $T$ a topology defined on the set $X$. Then we say that the pair $(X, T)$ is a topological difference algebra if the operation $\ast$ is continuous with respect to $T$. The continuity of the operations $\ast$ is equivalent to having the following property satisfied:

(C): Let $O$ be an open set and $a, b \in X$ such that $a \ast b \in O$. Then there are open sets $O_1$ and $O_2$ such that $a \in O_1$, $b \in O_2$ and $O_1 \ast O_2 \subseteq O$.

Theorem 4.1. The pair $(X, T)$ is a topological difference algebra.
Proof. Let us first prove (C). Indeed assume that $x \ast y \in G$, with $x, y \in X$ and $G$ an open subset of $X$. Then there exist $U \in \mathcal{K}$, $U[x \ast y] \subseteq G$ and an ideal $I$ such that $U_I \subseteq U$. We claim that the following relation holds:

$$U_I[x] \ast U_I[y] \subseteq U[x \ast y]$$

Indeed for $h \in U_I[x]$ and $k \in U_I[y]$ we get that $x \equiv_I h$ and $y \equiv_I k$. Hence $x \ast y \equiv_I h \ast k$. From that $(x \ast y, h \ast k) \in U_I \subseteq U$. Hence $h \ast k \in U_I[x \ast y] \subseteq U[x \ast y]$. Then $h \ast k \in G$. Thus the condition (C) is verified.

THEOREM 4.2. [5] Let $X$ be a set and $S \subset \mathcal{P} \,(X \times X)$ be a family such that for every $U \in S$ the following conditions hold:

(a) $\Delta \subseteq U$,
(b) $U^{-1}$ contains a member of $S$, and
(c) there exists a $V \in S$, such that $V \circ V \subseteq U$.

Then there exists a unique uniformity $\mathcal{U}$, for which $S$ is a subbase.

THEOREM 4.3. If we let $B = \{U_I \mid I$ is an ideal of $X\}$, then $B$ is a subbase for a uniformity of $X$. We denote this topology by $S$.

Proof. Since $\equiv_I$ is an equivalence relation, then it is clear that $B$ satisfies the axioms of Theorem 4.2.

We say that topology $\sigma$ is finer than $\tau$ if $\tau \subseteq \sigma$ as subsets of the power set. Then we have:

COROLLARY 4.4. Topology $S$ is finer than $T$.

THEOREM 4.5. Any ideal in the collection $\Lambda$ is a clopen subset of $X$.

Proof. Let $I$ be an ideal of $X$ in $\Lambda$ and $y \in I^c$. Then $y \in U_I[y]$ and we get that $I^c \subseteq \bigcup\{U_I[y] \mid y \in I^c\}$. We claim that, $U_I[y] \subseteq I^c$, for all $y \in I^c$. Let $z \in U_I[y]$, then $z \equiv_I y$. Hence $y \ast z \in I$. If $z \in I$ then $y \in I$, that is a contradiction. So $z \in I^c$ and we get $\bigcup\{U_I[y] \mid y \in I^c\} \subseteq I^c$. Hence $I^c = \bigcup\{U_I[y] \mid y \in I^c\}$ and since $U_I[y]$ is open for all $y \in X$, $I$ is a closed subset. We show that $I = \bigcup\{U_I[y] \mid y \in I\}$. If $y \in I$ then $y \in U_I[y]$ and we get $I \subseteq \bigcup\{U_I[y] \mid y \in I\}$. Let $y \in I$, if $z \in U_I[y]$ then $z \equiv_I y$ and so $z \ast y \in I$. Since $y \in I$ hence $z \in I$ and we get that $\bigcup\{U_I[y] \mid y \in I\} \subseteq I$. So $I$ is also an open subset of $X$.

THEOREM 4.6. For any $x \in X$ and $I \in \Lambda$, $U_I[x]$ is a clopen subset of $X$.

Proof. We show that $(U_I[x])^c$ is open. Let $y \in (U_I[x])^c$, then $x \ast y \in I^c$ or $y \ast x \in I^c$. Let $y \ast x \in I^c$. Hence by Theorems 4.1 and 4.2, $(U_I[y] \ast U_I[x]) \subseteq U_I[y \ast x] \subseteq I^c$. We claim that: $U_I[y] \subseteq (U_I[x])^c$. Let
$z \in U_I[y]$, then $z \ast x \in (U_I[z] \ast U_I[x])$. So $z \ast x \in I^c$ then we get $z \in (U_I[x])^c$. Hence $U_I[x]$ is closed. It is clear that $U_I[x]$ is open. So $U_I[x]$ is clopen subset of $X$. □

A topological space $X$ is connected if and only if has only $X$ and $\emptyset$ as clopen subsets. Therefore we have

**Corollary 4.7.** The space $(X, T)$ is not a connected space.

We denote the uniform topology obtained by an arbitrary family $\Lambda$, by $T_\Lambda$ and if $\Lambda = \{I\}$, we denote it by $T_I$.

**Theorem 4.8.** $T_\Lambda = T_J$, where $J = \bigcap \{J \mid I \in \Lambda\}$.

**Proof.** Let $\mathcal{K}$ and $\mathcal{K}^*$ be as in Theorems 3.2 and 3.3. Now consider $\Lambda_0 = \{J\}$, define:

$$(\mathcal{K}_0)^* = \{U_J\}$$

and

$$\mathcal{K}_0 = \{U \mid U_J \subseteq U\}.$$ 

Let $G \in T_\Lambda$. So for all $x \in G$, there exist $U \in \mathcal{K}$ such that $U[x] \subseteq G$. From $J \subseteq I$, we get that $U_J \subseteq U_I$, for all ideals $I$ of $X$. Since $U \in \mathcal{K}$, there exist $I \in \Lambda$ such that $U_I \subseteq U$. Hence $U_J[x] \subseteq U_I[x] \subseteq G$. Since $U_J \in \mathcal{K}_0$, $G \in T_J$. So $T_\Lambda \subseteq T_J$.

Conversely, let $H \in T_J$ then for all $x \in H$, there exist $U \in \mathcal{K}_0$ such that $U[x] \subseteq H$. So $U_J[x] \subseteq H$ and since $\Lambda$ is closed under intersection, $J \in \Lambda$. Then we get $U_J \in \mathcal{K}$ and so $H \in T_\Lambda$. Thus $T_J \subseteq T_\Lambda$. □

**Corollary 4.9.** Let $I$ and $J$ be ideals of $X$ and $I \subseteq J$. Then $J$ is clopen in topological space $(X, T_I)$.

**Proof.** Consider $\Lambda^* = \{I, J\}$. Then by Theorem 4.8, $T_\Lambda = T_I$ and therefore $J$ is clopen in topological space $(X, T_I)$. □

**Theorem 4.10.** Let $I$ and $J$ be ideals of $X$. Then $T_I \subseteq T_J$ if $J \subseteq I$.

**Proof.** Let $J \subseteq I$. Consider:

$\Lambda_1 = \{I\}$, $\mathcal{K}_1^* = \{U_I\}$, $\mathcal{K}_1 = \{U \mid U_I \subseteq U\}$ and $\Lambda_2 = \{J\}$, $\mathcal{K}_2^* = \{U_J\}$, $\mathcal{K}_2 = \{U \mid U_J \subseteq U\}$.

Let $G \in T_I$. Then for all $x \in G$, there exist $U \in \mathcal{K}_1$ such that $U[x] \subseteq G$. Since $J \subseteq I$, then $U_J \subseteq U_I$ and since $U_I[x] \subseteq G$, we get $U_J[x] \subseteq G$. $U_J \in \mathcal{K}_2$ and so $G \in T_J$. □

Recall that a uniform space $(X, \mathcal{K})$ is totally bounded if for each $U \in \mathcal{K}$, there exists $x_1, \ldots, x_n \in X$ such that $X = \bigcup_{i=1}^n U[x_i]$ and $X$ is compact if any open cover of $X$ has a finite subcover.
Theorem 4.11. Let $I$ be an ideal of $X$. Then the following conditions are equivalent:

1. Topological space $(X, T_I)$ is compact,
2. Topological space $(X, T_I)$ is totally bounded,
3. There exists $P = \{x_1, x_2, \ldots, x_n\} \subseteq X$ such that for all $a \in X$ there exists $x_i \in P$ where $a \cdot x_i \in I$ and $x_i \cdot a \in I$.

Proof. $(1) \rightarrow (2)$: It is clear by Theorem 14.3.8 of [5].

$(2) \rightarrow (3)$: Let $U_I \in \mathcal{K}$ since $(X, T_I)$ is totally bounded, then there exists $x_1, x_2, \ldots, x_n \in I$ such that $X = \bigcup_{i=1}^{n} U_I[x_i]$. Now let $a \in X$ then there exist $x_i$ such that $a \in U_I[x_i]$, therefore $a \cdot x_i \in I$ and $x_i \cdot a \in I$.

$(3) \rightarrow (1)$: For all $a \in X$ by hypothesis there exists $x_i \in P$ where $a \cdot x_i \in I$ and $x_i \cdot a \in I$. Hence $a \in U_I[x_i]$, thus $X = \bigcup_{i=1}^{n} U_I[x_i]$. Now let $X = \bigcup_{\alpha \in \Omega} O_\alpha$, where each $O_\alpha$ is an open set of $X$, then for any $x_i \in X$ there exists $\alpha_i \in \Omega$ such that $x_i \in O_{\alpha_i}$, since $O_{\alpha_i}$ is an open set then $U_I[x_i] \subseteq O_{\alpha_i}$, so $X = \bigcup_{i=1}^{n} U_I[x_i] \subseteq \bigcup_{i=1}^{n} O_{\alpha_i}$, therefore $X = \bigcup_{i=1}^{n} O_{\alpha_i}$ which means that $(X, T_I)$ is compact.

Theorem 4.12. If $I$ is an ideal of $X$, then $U_I[x]$ is a compact set in topological space $(X, T_I)$, for all $x \in X$.

Proof. Let $U_I[x] \subseteq \bigcup_{\alpha \in \Omega} O_\alpha$, where each $O_\alpha$ is an open set of $X$. Since $x \in U_I[x]$, then there exists $\alpha \in \Omega$ such that $x \in O_\alpha$. Then $U_I[x] \subseteq O_\alpha$. Hence $U_I[x]$ is compact.

References


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