SOME DUALITY OF WEIGHTED BERGMAN SPACES OF THE HALF-PLANE

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Abstract. In the setting of the half-plane of the complex plane, we introduce a modified reproducing kernel and we show that for \( r > -1/2 \), \( B^{1,r} \)-cancellation property holds and the Bloch space is the dual space of \( B^{1,r} \).

1. Introduction

It is well-known that the duality plays an important role in mathematics. We will discuss the Bloch space and weighted Bergman spaces.

Let \( H = \{x + iy : y > 0\} \) denote the half-plane of the complex plane \( \mathbb{C} \) and \( dA \) the area measure on \( H \). Put \( K(z, w) = -1/\pi(z - \overline{w})^2 \). In fact, \( K(z, \cdot) \) is the reproducing kernel for \( B^2 \), that is, for any \( f \in B^2 \), \( f(z) = \int_H f(w)K(z, w)dA(w) \) for all \( z \in H \). For \( r > -1/2 \), the weighted Bergman Spaces \( B^{1,r} \) of the half-plane is the space of analytic functions in \( L^1(H, dA_r) \), where \( dA_r(z) = (2r + 1)K(z, z)^{-r}dA \).

The main result of this paper is to show that for \( r > -1/2 \), the dual space of \( B^{1,r} \) is the Bloch space.

In Section 2, using a modified reproducing kernel

\[ M(\cdot, w)^{1+r} = K(\cdot, w)^{1+r} - K(\cdot, i)^{1+r}, \]

we get bounded linear operators \( Q_r \) and we introduce a dense subspace of \( B^{1,r} \). Moreover, we prove that \( B^{1,r} \)-cancellation property holds. Section 3 is devoted to the duality of \( B^{1,r} \). To do so, for each element \( g \)
of $\mathcal{B}^{0}(i)$, $\Lambda_g$ is in $(B^{1,r})^*$, where for $r > -1/2$,
$$
\mathcal{B}^{r}(i) = \{ f \in A(H) : f(i) = 0 \text{ and } \| f \|_{\mathcal{B}^{r}} = \sup_{z=x+iy \in H} y^{1+2r}|f'(z)| < \infty \}
$$
and $\Lambda_g : B^{1,r} \to \mathbb{C}$ is defined by $\Lambda_g(f) = \int_H f(z)\overline{g(z)} \, dA_r(z)$ for all $f \in B^{1,r}$. Using the bounded linear operator $Q_r$ of $L^\infty$ into $\mathcal{B}^{0}(i)$, we show that $\mathcal{B}^{0}(i)$ can be identified with the dual space of $B^{1,r}$. Throughout this paper, we use the symbol $A \lesssim B(A \approx B$, respectively) for nonnegative constants $A$ and $B$ to indicate that $A$ is dominated by $B$ times some positive constant ($A \lesssim B$ and $B \lesssim A$, respectively).

2. Weighted Bergman spaces

In this section, we deal with bounded linear operators $Q_r$ and dense subspaces of $B^{1,r}$.

**Proposition 2.1.** Suppose $r > -1/2$ and $f \in B^{1,r}$. Then for each $z=x+iy \in H$,
$$
|f(z)| \leq \frac{4}{(2r+1)(\pi y^2)^{1+r}}\| f \|_{1,r}.
$$

**Proof.** Since $z=x+iy$ and $f$ is analytic on $H$, the mean-value property implies that $|f(z)| \leq (1/|B(z,y/2)|) \int_{B(z,y/2)} |f(w)| \, dA(w)$. Since for any $w \in B(z,y/2)$, $\text{Im} \, w > y/2$, \( K(w,w) = 1/4\pi(\text{Im} \, w)^2 \leq 1/\pi y^2 \) and hence $|f(z)| \leq (1/(2r+1)(\pi y^2)^{1+r})\| f \|_{1,r}$. \qed

**Proposition 2.2.** For $r > -1/2$, $B^{1,r}$ is a closed subspace of $L^1(H,dA_r)$.

**Proof.** It can be easily derived from Proposition 2.1. \qed

Let
$$
\mathcal{B}^{r}(i) = \{ f \in A(H) : \| f \|_{\mathcal{B}^{r}} = \sup_{z=x+iy \in H} y^{1+2r}|f'(z)| < \infty \text{ and } f(i) = 0 \}.
$$

Since each element of $\mathcal{B}^{r}(i)$ vanishes at $i$, $\| \cdot \|_{\mathcal{B}^{r}}$ is the norm on $\mathcal{B}^{r}(i)$, in fact, for $r > -1/2$, is also a Banach space (see [4]). We note that the weighted Bergman kernel $K(\cdot, w)^{1+r} \not\in L^1(H,dA_r)$ (see [5]) and hence we need a modified reproducing kernel $M(\cdot, w)^{1+r} = K(\cdot, w)^{1+r} - K(\cdot, i)^{1+r}$. For $r > -1/2$, we define $Q_r : L^\infty \to \mathcal{B}^{0}(i)$ by
$$
Q_r(b)(w) = (2r+1) \int_H b(z)M(z,w)^{1+r}K(z,z)^{-r} \, dA(z).
$$

Then we have the following:
THEOREM 2.3. (1) $M(\cdot, w)^{1+r}$ is in $L^1(H, dA_r)$ and hence in $B^{1,r}$.
(2) For $r > -1/2$, $Q_r$ is a bounded linear operator.

Proof. (1) Suppose $2 + 2r = q/p$ for some $p, q \in \mathbb{Z}$. Since

$$
|M(z, w)^{1+r}| 
= \frac{1}{\pi^{1+r}} \left| \frac{(z+i)^{2+2r} - (z-w)^{2+2r}}{(z-w)^2 + 2r} \times (z+i)^{2+2r} \right|
= \frac{1}{\pi^{1+r}} \left| \frac{(z+i)^{\frac{q}{p}} - (z-w)^{\frac{q}{p}}}{(z-w)^\frac{q}{p} \times (z+i)^{\frac{q}{p}}} \right|
= \frac{1}{\pi^{1+r}} \frac{|(z+i)^{\frac{q-1}{p}} + (z+i)^{\frac{q-2}{p}} (z-w)^{\frac{1}{p}} + \cdots + (z-w)^{\frac{q-1}{p}}|}{|z - w|^{\frac{q}{p}} |z + i|^{\frac{q}{p}}} \times |(z+i)^{\frac{1}{p}} - (z-w)^{\frac{1}{p}}|
= \frac{1}{\pi^{1+r}} \frac{|(z+ i)^{\frac{p-1}{p}} + (z + i)^{\frac{p-2}{p}} (z-w)^{\frac{1}{p}} + \cdots + (z-w)^{\frac{p-1}{p}}|}{|(z+i)^{\frac{p-1}{p}} + (z+i)^{\frac{p-2}{p}} (z-w)^{\frac{1}{p}} + \cdots + (z-w)^{\frac{p-1}{p}}|} \times \frac{|z + i - z + \bar{w}|}{|z - w|^{\frac{q}{p}} |z + i|^{\frac{q}{p}}},
$$

$$
\int_H |M(z, w)^{1+r}|K(z, z)^{-r} dA(z)
\leq \frac{4^r}{\pi} (1 + |w|) \int_H \frac{|(z+i)^{\frac{q-1}{p}} + \cdots + (z-w)^{\frac{q-1}{p}}| \text{Im} z^{\frac{q-2}{p}} dA(z)}{|z - w|^{\frac{q}{p}} |z + i|^{\frac{q}{p}} (z+i)^{\frac{p-1}{p}} + \cdots + (z-w)^{\frac{p-1}{p}}|}.
$$

Since $\frac{q}{p} + \frac{q}{p} + \frac{p-1}{p} - \frac{q-1}{p} - \frac{q}{p} = 2$, $M(\cdot, w)^{1+r} \in L^{1,r}$.

(2) Clearly, each $Q_r$ is linear and $Q_r(b)(i) = 0$. Take any closed contour $C$ in $H$. By Fubini’s Theorem,

$$
\int_C Q_r(b)(w) dA(w)
= (2r + 1) \int_H b(z) \int_C M(z, w) dA(w) K(z, z)^{-r} dA(z) = 0
$$

and hence $Q_r(b)$ is analytic on $H$. Let $z = x + iy$ and $w = s + it$ be in $H$. Since
\[
\frac{d}{dz} M(w, z)^{1+r} = \left(-\frac{1}{\pi}\right)^{1+r} \frac{2 + 2r}{(w - z)^{3+2r}} y \left| \frac{d}{dz} Q_r(b) (z) \right|
\]
\[
= (2r + 1)y \left| \int_H b(w) \left(-\frac{1}{\pi}\right)^{1+r} \frac{2 + 2r}{(w - z)^{3+2r}} K(w, w)^{-r} dA(w) \right|
\]
\[
\leq \frac{(2r + 1)(2r + 2) ||b||_{\infty}}{\pi^{1+r}} y \int_0^\infty \int_0^\infty \frac{(4\pi)^r t^{2r}(y + t)}{(s - x)^2 + (y + t)^2} (y + t)^{2+2r} ds dt
\]
\[
\leq 4^r (2r + 1)(2r + 2) ||b||_{\infty} y \int_0^\infty \frac{1}{(y + t)^2} dt
\]
\[
\approx ||b||_{\infty}.
\]

This implies that \( ||Q_r(b)||_{B^0} \lesssim ||b||_{\infty} \), that is, \( Q_r \) is bounded. \( \square \)

**Lemma 2.4.** \( B^{1,r} \cap H^\infty \) is dense in \( B^{1,r} \).

**Proof.** Take any \( h \) in \( B^{1,r} \) and any \( w \) in \( H \). For any \( \delta > 0 \), let \( h_\delta(z) = h(z + i\delta) \) for all \( z \in H \). Since

\[
|h_\delta(w)| = |h(w + i\delta)|
\]

\[
= \left| \frac{1}{|B(w + i\delta, \delta/2)|} \int_{B(w + i\delta, \delta/2)} h dA(z) \right|
\]
\[
\lesssim \int_{B(w + i\delta, \delta/2)} |h(z)| K(z, z)^{-r} dA(z) \lesssim ||h||_{1,r},
\]

\( h_\delta \) is in \( H^\infty \). On the other hand,

\[
||h_\delta||_{1,r} = (2 + 1) \int_H |h_\delta(z)| K(z, z)^{-r} dA(z)
\]
\[
\lesssim \int_H |h(z)| K(z, z)^{-r} dA(z) \approx ||h||_{1,r}
\]

and hence \( h_\delta \in B^{1,r} \cap H^\infty \). Since \( C_C(H) \) is dense in \( L^{1,r} \), for any \( f \) in \( B^{1,r} \) and any \( \epsilon > 0 \), there is \( g \in C_C(H) \) such that \( ||g - f||_{1,r} < \epsilon \). Since \( ||f_\delta - f||_{1,r} \leq ||f_\delta - g_\delta||_{1,r} + ||g_\delta - g||_{1,r} + ||g - f||_{1,r} \), \( B^{1,r} \cap H^\infty \) is dense in \( B^{1,r} \). \( \square \)
In order to prove that for any $f \in B^{1,r}$ and any $g \in B^0(i)$, $f \cdot g \in B^{1,r}$, we define the set

$$D_r = \{ f \in B^{1,r} : \text{there is a constant } C_f \text{ such that} \quad |f(z)| \leq \frac{C_f}{(1 + |z|)^{3+2r}} \text{ for all } z \in H \}.$$ 

The set $D_r$ satisfies the following property.

**Proposition 2.5.** $D_r$ is dense in $B^{1,r}$.

**Proof.** For each $n \in \mathbb{N}$, let $\varphi_n(z) = (ni)^{3+2r}/(ni + z)^{3+2r}$ for all $z \in H$. Then for $z = x + iy$ in $H$,

$$|\varphi_n(z)| = \frac{n^{3+2r}}{(x^2 + (y + n)^2)^{3+2r}} \leq \left( \frac{\sqrt{2}n}{1 + |z|} \right)^{3+2r}$$

and hence $\varphi_n \in D_r$. Take any $f$ in $B^{1,r}$ and any $\epsilon > 0$. By Lemma 2.4, there is $g \in B^{1,r} \cap H^\infty$ such that $\|g - f\|_{1,r} < \epsilon$. Since $g\varphi_n \in D_r$ and $|g(z)\varphi_n(z) - g(z)| \leq |g(z)|(1 + |\varphi_n(z)|) \leq 2|g(z)|$, Lebesgue dominated convergence theorem implies that $\lim_{n \to \infty} \int_H |g(z)\varphi_n - g(z)| \, dA_r(z) = 0$ and hence $\lim_{n \to \infty} \|g\varphi_n - g\|_{1,r} = 0$. Since $\|g\varphi_n - f\|_{1,r} \leq \|g\varphi_n - g\|_{1,r} + \|g - f\|_{1,r}$, $D_r$ is dense in $B^{1,r}$. \hfill \Box

**Lemma 2.6.** For any $g \in B^0(i)$ and any $z = x + iy \in H$,

$$|g(z)| \leq \|g\|_{B^0}(1 + |\log y| + 2 \log(1 + |x|)).$$

**Proof.** Suppose $z = x + iy \in H$. Since $g(i) = 0$,

$$|g(z)| \leq |g(x + iy) - g(x + i(1 + |x|))| + |g(x + i(1 + |x|)) - g((1 + |x|)i)| + |g((1 + |x|)i) - g(i)|

= \left| \int_0^1 \frac{g'(x + ((y - |x| - 1)t + 1 + |x|)i)((y - 1 - |x|)t + 1 + |x|)}{(y - 1 - |x|)t + 1 + |x|} \right| \times (y - 1 - |x|) \, dt

+ \left| \int_0^1 \frac{g'(tx + (1 + |x|)i)((1 + |x|)x + 1)}{1 + |x|} \, dx \right| \times \left| \int_0^1 \frac{g'(tx)}{i |x| + 1} |x| \, dt \right|

\leq \|g\|_{B^0}(1 + 2 \log(1 + |x|)). \hfill \Box$$

**Proposition 2.7.** Suppose $f \in D_r$ and $g \in B^0(i)$. Then $f \cdot g \in B^{1,r}$. 
Proof. Since $f \in \mathcal{D}_r$, there is a constant $C_f$ such that $|f(z)| \leq C_f/(1 + |z|)^{3+2r}$ for all $z \in H$. Put $z = x + iy$. By Lemma 2.6,

\[
\int_H |f(z)g(z)| \, dA_r(z) = (2r + 1) \int_H |f(z)g(z)|K(z, z)^{-r} \, dA(z) \\
\leq C_f \|g\|_{B^0} \int_H \frac{1 + |\log y| + 2 \log(1 + |x|)}{(1 + |z|)^{3+2r}} (4\pi)^r (\text{Im} \, z)^{2r} \, dA(z) \\
\lesssim \|g\|_{B^0} \int_H \frac{1 + |\log y| + 2 \log(1 + |x|)}{(1 + |z|)^3} \, dA(z).
\]

Since there is a compact subset $K$ of $H$ such that for $z \notin K$,

\[
\log(1 + |z|) < (1 + |z|)^{\frac{1}{2}}, \quad f \cdot g \in B^{1,r}.
\]

3. The dual space of $B^{1,r}$

The main goal of this section is to prove that the dual space of weighted Bergman spaces $B^{1,r}$ can be identified with $B^0(i)$. To do so, we define the following linear functional $\Lambda_g$ on $\mathcal{D}_r$ as for $g$ in $B^0(i)$,

\[
\Lambda_g(f) = (2r + 1) \int_H f(z)g(z)K(z, z)^{-r} \, dA(z).
\]

By Proposition 2.7, $\Lambda_g$ is well-defined and linear. Let $h(z) = \frac{1+z}{1-z}i$. The $h$ is a Riemann map from $\mathbb{D}$ onto $H$. Let $A^{1,r}(\mathbb{D})$ denote the space of analytic functions in $L^p(\mathbb{D}, dA_r)$, where $dA_r(z) = (2r + 1)K_D(z, z)^{-r} \, dA(z)$. Then $B^{1,r}$ can be identified with $A^{1,r}(\mathbb{D})$ via $\Psi(f)(w) = 2^{2+r}f(h(w))/(1 - w)^{4+4r}$ (see [3]). Let $B(\mathbb{D})$ be the Bloch space of $\mathbb{D}$. Then we have the following:

**Lemma 3.1.** (1) For any $f \in B^{1,r}$, $f(h(z))/(1 - z)^{4+4r}$ is in $A^{1,r}(\mathbb{D})$.

(2) For any $g \in B^0(i)$, $\|g \circ h\|_{B(\mathbb{D})} = 2\|g\|_{B^0}$.

**Proof.** (1) Since

\[
\int_\mathbb{D} \frac{|f(h(z))|}{|1 - z|^{4+4r}} K_D(z, z)^{-r} \, dA(z)
\]

\[
= \int_H \frac{|f(h^{-1}(z))|}{|1 - h^{-1}(z)|^{4+4r}} K_D(h^{-1}(z), h^{-1}(z))^{-r} |(h^{-1})'(z)|^2 \, dA(z)
\]

\[
= \pi^r \int_H \frac{|f(z)|}{|1 - \frac{z-i}{z+i}|^{4+4r}} \left(1 - \left|\frac{z-i}{z+i}\right|^2\right)^{2r} \left|\frac{2i}{(z+i)^2}\right|^2 \, dA(z)
\]
\[ = \frac{\pi^r}{4^{1+r}} \int_H |f(z)|(\text{Im } z)^{2r} \, dA(z) \approx \|f\|_{1,r} < \infty, \]

\(f(h(z))/|1 - z|^{4 + 4r}\) is in \(A^{1,r}(\mathbb{D})\).

(2) Suppose \(g \in \mathcal{B}^0(i)\). Then
\[
\|g \circ h\|_{\mathcal{B}(\mathbb{D})} = \sup_{z = x + iy \in \mathbb{D}} (1 - |z|^2)|g \circ h)'(z)| + |g \circ h(0)|
= \sup_{z \in \mathbb{D}} (1 - |z|^2)|g'(h(z))||h'(z)|
= \sup_{w = s + it \in H} \frac{|w + i|^2 - |w - i|^2}{|w + i|^2} |g'(w)| \frac{|w + i|^2}{2}
= \sup_{w = s + it \in H} 2t |g'(w)| = 2\|g\|_{\mathcal{B}^0}. \quad \square
\]

**Proposition 3.2.** For \(g \in \mathcal{B}^0(i)\), the linear functional \(\Lambda_g : \mathcal{D}_r \to \mathbb{C}\) defined by \(\Lambda_g(f) = (2r + 1) \int_H f(z)g(z)K(z,z)^{-r} \, dA(z)\) is bounded.

**Proof.** Since
\[
\Lambda_g(f) = (2r + 1) \int_H f(z)\overline{g(z)}K(z,z)^{-r} \, dA(z)
= (2r + 1) \int_\mathbb{D} f(h(z))\overline{g(h(z))}K(h(z),h(z))^{-r}|h'(z)|^2 \, dA(z)
\]
and \((A^{1,r})^* = \mathcal{B}(\mathbb{D})\), \(|\Lambda_g(f)| \lesssim \|g \circ h\|_{\mathcal{B}(\mathbb{D})}\|f\|_{1,r} \lesssim \|g\|_{\mathcal{B}^0}\|f\|_{1,r}\) and hence \(\|\Lambda_g\| \lesssim \|g\|_{\mathcal{B}^0}\), that is, \(\Lambda_g\) is bounded. \(\square\)

**Proposition 3.3.** For \(g \in \mathcal{B}^0(i)\), we define \(\Lambda_g : B^{1,r} \to \mathbb{C}\) by
\[
\Lambda_g(f) = (2r + 1) \int_H f(z)\overline{g(z)}K(z,z)^{-r} \, dA(z).
\]
Then \(\Lambda_g\) is bounded.

**Proof.** Take any \(f \in B^{1,r}\). Since \(\mathcal{D}_r\) is dense in \(B^{1,r}\), there is a sequence \(\{f_n\}\) in \(\mathcal{D}_r\) such that \(\lim_{n \to \infty} \|f_n - f\|_{1,r} = 0\). Since \(\{\Lambda_g(f_n)\}\) is a Cauchy sequence in \(\mathbb{C}\), \(\lim_{n \to \infty} \Lambda_g(f_n)\) exists. Suppose that \(\{h_n\}\) is a sequence in \(\mathcal{D}_r\) such that \(\lim_{n \to \infty} \|h_n - f\|_{1,r} = 0\). Since \(|\Lambda_g(f_n - h_n)| \leq \|\Lambda_g\|\|f_n - h_n\|_{1,r} = \|\Lambda_g\|\|f - f\|_{1,r} + \|f - h_n\|_{1,r}\), we define \(\Lambda_g(f) = \lim_{n \to \infty} \Lambda_g(f_n)\). Since
\[
|\Lambda_g(f)| = \lim_{n \to \infty} |\Lambda_g(f_n)| \leq \lim_{n \to \infty} \|\Lambda_g\|\|f_n\|_{1,r} = \|\Lambda_g\|\|f\|_{1,r},
\]
\(\Lambda_g\) is a bounded linear functional on \(B^{1,r}\). \(\square\)

**Theorem 3.4.** We define \(\Phi : \mathcal{B}^0(i) \to (B^{1,r})^*\) by \(\Phi(g) = \Lambda_g\) for all \(g \in \mathcal{B}^0(i)\). Then \(\Phi\) is a bounded linear functional.
Proof. It is immediate from Proposition 3.2 and Proposition 3.3. □

Main result of this paper is to show that \((B^{1,r})^*\) can be identified with \(B^0(i)\). To do so, we need the \(B^{1,r}\)-cancellation property.

**Lemma 3.5.** For any \(f \in B^{1,r}\), \(\int_H f(w)K(w,w)^{-r}\,dA(w) = 0\).

**Proof.** Take any \(f\) in \(B^{1,r}\). Then

\[
\int_H f(w)K(w,w)^{-r}\,dA(w)
= \int_D f(h(z))K(h(z),h(z))^{-r}|h'(z)|^2\,dA(z)
= 4^{1+r}\pi\int_D \frac{f(g(z))}{|1-z|^{4+4r}}(1-|z|^2)^{2r}\,dA(z)
= 4^{1+r}\pi\int_D \frac{f(g(z))}{(1-z)^{4+4r}}(1-|z|^2)^{2r}\,dA(z)
= 4^{1+r}\pi\lim_{t\to 1^-}\int_D \frac{f(g(z))}{(1-z)^{4+4r}}(1-tz)^{2+2r}\,dA(z) \quad (\text{Lemma 3.1})
= \frac{4^{1+r}\pi^{1+r}}{2r+1}\lim_{t\to 1^-} \int_D \left(\frac{f(g(z))(1-tz)^{2+2r}}{(1-z)^{4+4r}}(2r+1)\right)^{1+r}
\times \pi^r(1-|z|^2)^{2r}\,dA(z)
= \frac{4^{1+r}\pi^{1+r}}{2r+1}\lim_{t\to 1^-} \frac{f(g(t))}{(1-t)^{4+4r}}(1-t^2)^{2+2r}.
\]

Since

\[
\lim_{t\to 1^-} \frac{|f(g(t))|}{(1-t)^{4+4r}}
= \lim_{t\to 1^-} \frac{1}{|B(t,(\frac{1-t^2}{2})^{1+r})|} \left| \int_{B(t,(\frac{1-t^2}{2})^{1+r})} \frac{f(g(z))}{(1-z)^{4+4r}}\,dA(z) \right|
\leq \lim_{t\to 1^-} \frac{4^{1+r}}{\pi(1-t^2)^{2+2r}} \left| \int_{B(t,(\frac{1-t^2}{2})^{1+r})} \frac{f(g(z))}{(1-z)^{4+4r}}\,dA(z) \right|
\leq \lim_{t\to 1^-} \frac{4^{1+r}}{\pi} \int_{B(t,(\frac{1-t^2}{2})^{1+r})} \left| \frac{f(g(z))}{(1-z)^{4+4r}} \right|\,dA = 0
\]

and hence \(\int_H f(w)K(w,w)^{-r}\,dA(w) = 0\). □
Proposition 3.6. Let \( \Phi \) be a map defined in Theorem 3.4. Then \( \Phi \) is 1-1 and onto.

Proof. Take any \( g \) in \( \ker \Phi \). Let \( \varphi_n(z) = (ni)^{3+2r}/(ni + z)^{3+2r} \) for all \( z \in H \) and for each \( n \in \mathbb{N} \). Then \( g \in \mathcal{B}^0(i) \) and \( \varphi_n(z) \in \mathcal{D}_r \) and hence \( g \cdot \varphi_n \in \mathcal{B}^{1,r} \). Since \( \varphi_n \) is bounded, for any \( f \in \mathcal{B}^{1,r} \),

\[
\lim_{n \to \infty} \int_H f(z) \varphi_n(z) g(z) K(z, z)^{-r} \, dA(z) \quad = \quad \int_H f(z) g(z) K(z, z)^{-r} \, dA(z) = 0.
\]

Since \( M(\cdot, w)^{1+r} \in \mathcal{B}^{1,r} \),

\[
0 = \lim_{n \to \infty} \int_H \varphi_n(z) g(z) M(z, w)^{1+r} K(z, z)^{-r} \, dA(z) \\
= \lim_{n \to \infty} \left( \int_H \varphi_n(z) g(z) K(z, w)^{1+r} K(z, z)^{-r} \, dA(z) \\
- \int_H \varphi_n(z) g(z) K(z, i)^{1+r} K(z, z)^{-r} \, dA(z) \right) \\
= \frac{1}{2r+1} \lim_{n \to \infty} (\varphi_n(w) g(w) - \varphi_n(i) g(i)) \\
= \frac{1}{2r+1} g(w)
\]

and hence \( g = 0 \), that is, \( \ker \Phi = \{0\} \). This implies that \( \Phi \) is one-to-one. Take any \( \Psi \) in \( (\mathcal{B}^{1,r})^* \). By Hahn-Banach Extension Theorem, there is a unique \( \tilde{\Psi} \) in \( (L^{1,r})^* \) such that \( \tilde{\Psi} = \Psi \) on \( \mathcal{B}^{1,r} \) and \( \|\tilde{\Psi}\| = \|\Psi\| \). The Riesz Representation Theorem implies that there is a unique \( b \) in \( L^\infty \) such that \( \tilde{\Psi}(b) = \int_H f(w) \tilde{b}(z) K(w, w)^{-r} \, dA(w) \) for all \( f \in \mathcal{B}^{1,r} \). Put \( g = Q_r(b) \). Then \( g \in \mathcal{B}^0(i) \) and for any \( f \in \mathcal{B}^{1,r} \),

\[
\Phi(g)(f) = \Lambda_g(f) \\
= \int_H f(w) g(w) K(w, w)^{-r} \, dA(w) \\
= \int_H f(w) \overline{Q_r(b)}(w) K(w, w)^{-r} \, dA(w)
\]
\[
(2r + 1) \int_H f(w) \int_H b(z) \overline{M(z, w)^{1+r}} K(z, z)^{-r} dA(z) \\
\times K(w, w)^{-r} dA(w)
\]
\[
= (2r + 1) \int_H f(w) \int_H b(z) (K(z, w)^{1+r} - K(z, i)^{1+r}) K(z, z)^{-r} dA(z) \\
\times K(w, w)^{-r} dA(w)
\]
\[
= \int_H \overline{b(z)}(2r + 1) \int_H f(w) (K(w, z)^{1+r} - K(i, z)^{1+r}) K(w, w)^{-r} dA(w) \\
\times K(z, z)^{-r} dA(z)
\]
\[
= \int_H \overline{b(z)} f(z) K(z, z)^{-r} dA(z) = \Psi(f),
\]
that is, \( \Phi(g) = \Psi \). This completes the proof.

**Theorem 3.7.** For \( r > -1/2 \), \( B^0(i) \) is the dual space of \( B^{1,r} \).

**Proof.** The open mapping theorem implies that \( \Phi \) is an isomorphism.

**References**


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