SCALAR EXTENSION OF SCHUR ALGEBRAS

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Abstract. Let $K$ be an algebraic number field. If $k$ is the maximal cyclotomic subextension in $K$ then the Schur $K$-group $S(K)$ is obtained from the Schur $k$-group $S(k)$ by scalar extension. In the paper we study projective Schur group $PS(K)$ which is a generalization of Schur group, and prove that a projective Schur $K$-algebra is obtained by scalar extension of a projective Schur $k$-algebra where $k$ is the maximal radical extension in $K$ with mild condition.

1. Introduction

Let $K$ be a field. A finite dimensional central simple $K$-algebra is a Brauer algebra. A Brauer $K$-algebra $A$ is a projective Schur algebra if there is a finite group $G$ and a 2-cocycle $\alpha \in Z^2(G, K^*)$ such that $A$ is a homomorphic image of the twisted group algebra $KG^\alpha$, where $K^* = K \setminus \{0\}$ is regarded as a $G$-module with respect to the trivial $G$-action. The similarity class containing a Brauer $K$-algebra $A$ is denoted by $[A]$, and they form a Brauer group $B(K)$. The projective Schur group $PS(K)$ is a subgroup of $B(K)$ consisting of similarity classes which are represented by projective Schur $K$-algebras (refer to [1, 4]). When $\alpha = 1$, the projective Schur $K$-algebra $A$ is called a Schur $K$-algebra, and the set of $[A]$'s forms the Schur group $S(K)$. If characteristic of $K$ is positive then $S(K)$ is trivial.

Assume that $K$ is an algebraic number field (i.e., a finite extension of the rational field $\mathbb{Q}$) with algebraic closure $E$. Let $\mathbb{Q}(\mu)$ denote the maximal cyclotomic extension of $\mathbb{Q}$ contained in $E$, where $\mu$ is the group of all roots of unity in $E$. Let $k = \mathbb{Q}(\mu) \cap K$ and $K \otimes_k S(k)$ be the subgroup of $B(K)$ obtained from $S(k)$ by extension of scalars. Then

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$K \otimes_k S(k)$ is contained in $S(K)$, and it was proved in [9, (4.6)] that
\[ S(K) = K \otimes_k S(k) = \{ [K \otimes_k A] \mid [A] \in S(k) \}. \]

We may refer to [5, Theorem 3.4], and [7] over ring $K$ of algebraic integers.

The purpose of this paper is to study situations that a projective Schur $K$-algebra is obtained from a projective Schur $k$-algebra by scalar extensions where $k$ is a subfield of $K$. Upon using projective characters, we recast the definition of projective Schur algebra in Theorem 7, and prove that if $K$ is an algebraic number field and $k$ is a maximal radical extension field of $Q$ contained in $K$ with mild conditions then a projective Schur $K$-algebra $A$ can be written as $A \cong K \otimes_k A'$ for a projective Schur $k$-algebra $A'$ in Theorem 8. Moreover we construct a subgroup $PS_F(K)$ of $PS(K)$ with $F \subset K$ such that $PS_F(K) = K \otimes_k PS_F(k)$ for some subfield $k$ of $K$ in Theorem 12. As an application to a special class of algebras, radical algebra was discussed in Theorem 13.

In what follows, we mean that $K^*$ is the multiplicative subgroup of a field $K$, $\varepsilon_n$ is a primitive $n$-th root of unity for $n > 0$ and $\mu$ is the set of roots of unity. For a finite group $G$, $|G|$ is the order of $G$ and $o(g)$ is the order of $g \in G$. We denote by $Q$ the rational number field, and by $Z(A)$ the center algebra of an algebra $A$.

2. Projective group character

We always assume that $K$ is a field of characteristic 0 with an algebraic closure $E$, and $G$ is a finite group of exponent $n$. Let $\rho$ be an irreducible $E$-representation of $G$, and $\chi$ be the $E$-character of $G$ afforded by $\rho$. Then
\[
e(\chi) = \frac{\chi(1)}{|G|} \sum_{g \in G} \chi(g^{-1})g
\]
is a block idempotent of the group algebra $EG$ (see [9, (1.1)] or [3, Vol.1(19.2.7)]). Moreover $e(\chi)$ is a block idempotent of $K(\varepsilon_n)G$, because all values of $\chi$ are contained in $K(\varepsilon_n)$. And the Galois group $\text{Gal}(K(\varepsilon_n)/K)$ acts on the set of idempotents
\[
\{ e(\chi) \mid \chi \text{ irreducible } E\text{-characters of } G \}
\]
by $\tau \cdot e(\chi) = e(\chi^\tau)$, where $\chi^\tau(g) = \tau(\chi(g))$ for all $\tau \in \text{Gal}(K(\varepsilon_n)/K)$ and $g \in G$. 
Let \( K(\chi) \) denote the subfield of \( E \) generated by \( K \) and the character values \( \chi(g) \) for all \( g \in G \). Since \( \chi(g) \) is a sum of \( o(g) \)-th roots of unity over \( K \), \( K(\chi) = K(\varepsilon_d) \) where \( d \mid n \). And the block idempotent \( v(\chi) \) of \( KG \) such that \( e(\chi) \) is a summand of \( v(\chi) \) forms (see [3, Vol.3(14.1.14)])

\[
(2) \quad v(\chi) = \sum_{\tau \in \text{Gal}(K(\chi)/K)} e(\chi^\tau) \quad (\text{the } e(\chi^\tau) \text{ are all distinct}).
\]

We shall denote the simple component \( A \) of \( KG \) corresponding to \( \chi \) by \( KGv(\chi) \). Then \( A \) is central over \( K \) if and only if \( K = K(\chi) \) ([9, (1.4)]). Hence in this case, \( \text{Gal}(K(\chi)/K) = 1 \), \( e(\chi) = v(\chi) \) and \( A \) is isomorphic to \( KG(\chi) \). Thus the definition of Schur algebra can be stated as follow.

**Definition 1.** [3, Vol.3, p.819] Let \( K \) be a field of characteristic 0 and \( E \) be an algebraic closure of \( K \). Then a central simple algebra \( A \) is a Schur \( K \)-algebra if and only if there is a finite group \( G \) and an irreducible \( E \)-character \( \chi \) of \( G \) such that \( K = K(\chi) \) and \( A \cong KG(\chi) \) where \( e(\chi) \) is as in (1).

In [9, (4.6)], the equality \( S(K) = K \otimes_k S(k) \) was proved by employing Brauer-Witt theorem which states that every Schur \( K \)-algebra is similar to a cyclotomic algebra. Though there is a radical algebra which is an analog of cyclotomic algebra in projective Schur algebra theory, only projective Schur division algebra is a radical algebra ([1]). Hence in next theorem we will prove the identity \( S(K) = K \otimes_k S(k) \) by making use of group characters, so that the similar method will be extended to projective Schur algebra case.

**Theorem 2.** Let \( K \) be an algebraic number field and \( k \) be the maximal cyclotomic extension field contained in \( K \). Then \( S(K) = K \otimes_k S(k) \).

**Proof.** Let \( [S] \in S(K) \). Then there is a Schur algebra \( A \in [S] \) such that

\[
A \cong KG(\chi) \quad \text{(as } K \text{-algebras)} \quad \text{and } K = K(\chi),
\]

for a finite group \( G \), an irreducible \( E \)-character \( \chi \) of \( G \), and the block idempotent \( e(\chi) \) of \( EG \) as in (1).

Let \( A' \) be the simple component of \( kG \) corresponding to \( \chi \), and \( k(\chi) \) be the extension field adjoining all values of \( \chi \) to \( k \). Then \( k \subseteq k(\chi) \subseteq E \). Since \( e(\chi) \) belongs to \( k(\chi)G \), \( v'(\chi) = \sum_{\tau \in \text{Gal}(k(\chi)/k)} e(\chi^\tau) \) (the \( e(\chi^\tau) \) are all distinct) is a block idempotent of \( kG \) (see (2)). Thus \( A' \cong kGv'(\chi) \).

All \( \chi(g) \) \( (g \in G) \) are contained in \( K(\chi) \cap \mathbb{Q}(\mu) = K \cap \mathbb{Q}(\mu) \), where \( \mu \) is the set of primitive roots of unity in \( E \). But since \( K \cap \mathbb{Q}(\mu) \) is a
cyclotomic extension of \( \mathbb{Q} \) in \( K \), we have \( K \cap \mathbb{Q}(\mu) \subseteq k \) and \( \chi(g) \in k \) for all \( g \in G \). Thus \( k(\chi) = k, \ v'(\chi) = e(\chi) \) and \( A' \) is a central simple \( k \)-algebra such that \( A' \cong kGe(\chi) \). Hence we have \([S] = [A] = [KGe(\chi)] = [K \otimes_k kGe(\chi)] = K \otimes_k [A'] \subseteq K \otimes_k S(k)\).

Throughout the paper we always assume that \( K \) is a field of characteristic 0 and \( E \) is an algebraic closure of \( K \). Let \( \alpha \) be a 2-cocycle in \( Z^2(G, K^*) \) with \( \alpha(x, 1) = \alpha(1, x) = 1 \) for all \( x, y \in G \), and \( \{a_x | x \in G\} \) be a basis of the twisted group algebra \( KG^\alpha \) satisfying \( a_xa_y = \alpha(x, y)a_{xy} \). We denote by \( \rho \) an irreducible projective \( \alpha \)-representation of \( G \) over \( E \) and by \( \chi_\alpha \) the \( \alpha \)-character afforded by \( \rho \).

**Theorem 3.** Let \( K, E \) and \( \chi_\alpha \) be defined as above. Then there is a finite Galois radical extension \( F \) over \( K \) in \( E \) containing \( K(\chi_\alpha) \). That is, \( F = K(\Omega) \), where \( \Omega \) is a Gal(d\( F/K \))-invariant subgroup of \( F^* \) such that \( \Omega K^* / K^* \) is finite, and \( K(\chi_\alpha) \subseteq F \).

**Proof.** For any \( g \in G \), let

\[
\lambda_g = \prod_{i=1}^{o(g)} \alpha(g^i, g) \in K
\]

and let \( \delta_g \) in \( E \) be any \( o(g) \)-th root of \( \lambda_g \). Let \( \Omega_\alpha \) be the subset

\[
\Omega_\alpha = \langle \mu, \{\delta_g | g \in G\} \rangle \subseteq E^*,
\]

where \( \mu \) is the set of \( |G| \)-th root of unity in \( E \). Then \( K \subseteq K(\Omega_\alpha) \subseteq E \), and \( \Omega_\alpha K^* \) is torsion over \( K^* \). And since \( \delta_g \) is a root of the polynomial \( X^{o(g)} - \lambda_g \in K[X] \), any automorphism on \( K(\Omega_\alpha) \) maps \( \delta_g \) to another root of \( X^{o(g)} - \lambda_g \) that belongs to \( \Omega_\alpha \). Thus \( \Omega_\alpha \) is Gal\( (K(\Omega_\alpha)/K) \)-invariant, and \( K(\Omega_\alpha) \) is a finite Galois radical extension field of \( K \). Moreover since \( \chi_\alpha(g) \) is a sum of \( \delta_g \) ([3, Vol.3(1.2.6)]), it follows that \( \chi_\alpha(g) \) belongs to \( K(\Omega_\alpha) \), hence \( K(\chi_\alpha) \) is a subfield of \( K(\Omega_\alpha) \).

Maintaining the above notations, we get next corollary.

**Corollary 4.** Let \( \alpha \in Z^2(G, K^*) \) be of finite order \( o(\alpha) \). Then \( K(\chi_\alpha) \) is a subfield of a cyclotomic extension field over \( K \) in \( E \).

**Proof.** For \( g \in G \), we use the same notations \( \lambda_g \in K \) and \( \delta_g \in E \) as in Theorem 3. Let \( \rho \) be the irreducible \( \alpha \)-representation of \( G \) over \( E \) affording \( \chi_\alpha \).

Consider any positive multiple \( n = o(g)s \) with some \( s > 0 \). Let \( \lambda'_{g} = \prod_{i=1}^{n} \alpha(g^i, g) \) and let \( \delta'_{g} \) be an \( n \)-th root of \( \lambda'_{g} \) in \( E \). Then \( \rho(g)^n = \prod_{i=1}^{n} \alpha(g^i, g) \).
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\[ \prod_{i=1}^{n} \alpha(g^i, g) \rho(g^n) = \lambda'_g I, \text{ where } I \text{ is the identity matrix, and} \]

\[ (\delta_g')^n = \lambda'_g = \prod_{i=1}^{n} \alpha(g^i, g) = \left( \prod_{i=1}^{o(g)} \alpha(g^i, g) \right)^s = (\lambda_g)^s = (\delta_g)^s = (\delta_g)^n. \]

Since \( o(\alpha) \) is finite, if we consider \( n = o(g)o(\alpha) \) then

\[ (\delta_g)^n = \left( \prod_{i=1}^{o(g)} \alpha(g^i, g) \right)^{o(\alpha)} = \prod_{i=1}^{o(g)} \alpha^{o(\alpha)}(g^i, g) = 1, \]

and we may choose \( \delta_g \) as an \( n \)-root of unity. Thus \( K(\chi_\alpha) \) is contained in a cyclotomic subfield of \( E \).

Since the algebraic closure \( E \) is a splitting field for \( EG^\alpha \),

\begin{equation}
(4)
\end{equation}

\[ e(\chi_\alpha) = \frac{\chi_\alpha(1)}{|G|} \sum_{g \in G} \alpha^{-1}(g, g^{-1}) \chi_\alpha(g^{-1}) a_g \]

is a block idempotent of \( EG^\alpha \) associated with \( \chi_\alpha \) ([3, Vol.3(1.11.1)]). If \( \Gamma \) is a finite dimensional \( K \)-algebra and \( V \) is a \( \Gamma \)-module then \( \Gamma^E = E \otimes_k \Gamma \) and \( V^E = E \otimes_k V \) are \( E \)-algebra and \( \Gamma^E \)-module respectively. And any block idempotent \( v \) of \( \Gamma \) is a central idempotent of \( \Gamma^E \). Thus \( v \) can be written uniquely as a sum of distinct block idempotents \( e \) of \( \Gamma^E \) ([3, Vol.3(7.1.1)]).

**Theorem 5.** For \( \alpha \in Z^2(G, K^*) \), let \( \chi_\alpha \) be the irreducible \( \alpha \)-character of \( G \) over \( E \) afforded by an irreducible \( \alpha \)-representation \( \rho \) of \( G \). Let \( e(\chi_\alpha) \) be the block idempotent of \( EG^\alpha \) as in (4), and \( v(\chi_\alpha) \) be the block idempotent of \( KG^\alpha \) such that \( e(\chi_\alpha) \) is a summand of \( v(\chi_\alpha) \). Then, as \( K \)-algebras,

\[ KG^\alpha v(\chi_\alpha) \cong \rho(KG^\alpha) \text{ and } K(\chi_\alpha) \cong Z(\rho(KG^\alpha)). \]

**Proof.** When \( e(\chi_\alpha) \) is a summand of \( v(\chi_\alpha) \), we shall write \( e(\chi_\alpha) \subset v(\chi_\alpha) \). Let \( U \) be a simple \( EG^\alpha \)-module corresponding to \( \chi_\alpha \) and \( V \) be a simple \( KG^\alpha \)-module such that \( U \) is an irreducible constituent of \( V^E \). Let \( \Omega_\alpha \) be the subset of \( E^* \) consisting of the set \( \mu \) of \(|G|\)-th roots of unity in \( E \) and the set \( \{ \delta_g | g \in G \} \), where \( \delta_g \) is an \( o(g) \)-th root of \( \prod_{i=1}^{o(g)} \alpha(g^i, g) \) (see (3)). Then \( K(\Omega_\alpha) \) is a finite Galois radical extension field of \( K \) containing \( K(\chi_\alpha) \) (Theorem 3).
Clearly the block idempotent

\[ e(\chi_\alpha) = \frac{\chi_\alpha(1)}{|G|} \sum_{g \in G} \alpha^{-1}(g, g^{-1})\chi_\alpha(g^{-1})a_g \]

of \( EG^\alpha \) belongs to \( K(\chi_\alpha)G^\alpha \). And the Galois group \( \text{Gal}(K(\Omega_\alpha)/K) = G \) acts on the twisted group algebra \( K(\Omega_\alpha)G^\alpha \) by \( \tau \cdot \sum_{g \in G} x_g a_g = \sum_{g \in G} \tau(x_g)a_g \) for \( \tau \in G \) and \( x_g \in K(\Omega_\alpha) \). Thus if we consider \( \chi_\alpha^\tau = \tau \chi_\alpha \) then \( \chi_\alpha^\tau \) is an \( \alpha \)-character of \( G \) over \( E \) corresponding to the \( EG^\alpha \)-module \( U^\tau \). Since \( \tau e(\chi_\alpha) = \frac{\chi_\alpha(1)}{|G|} \sum_{g \in G} \alpha^{-1}(g, g^{-1})\chi_\alpha^\tau(g^{-1})a_g = e(\chi_\alpha^\tau) \), we have

\[ \sigma \left( \sum_{\tau \in G} e(\chi_\alpha^\tau) \right) = \sum_{\tau \in G} \sigma e(\chi_\alpha^\tau) = \sum_{\tau \in G} e(\chi_\alpha^\sigma \tau) = \sum_{\tau \in G} e(\chi_\alpha^\tau) \]

for all \( \sigma \in G \), thus \( \sum_{\tau \in G} e(\chi_\alpha^\tau) \) is contained in \( KG^\alpha \).

We notice however that \( \sum_{\tau \in G} e(\chi_\alpha^\tau) \) may not be a block idempotent in \( KG^\alpha \), because some of idempotents \( e(\chi_\alpha^\tau) \) might appear more than once in the summation. We now let

\[ (5) \quad v(\chi_\alpha) = \sum e(\chi_\alpha^\tau), \]

where the sum runs over \( \tau \in G \) such that \( e(\chi_\alpha^\tau) \) are all distinct. And we may generously assume that \( e(\chi_\alpha) \subset v(\chi_\alpha) \). Let \( \sigma \) be any element in \( G \).

Since \( \tau \) runs over \( G \) where \( e(\chi_\alpha^\tau) \) are all distinct in the summation \( v(\chi_\alpha) \), so does \( \sigma \tau \) and \( e(\chi_\alpha^{\sigma \tau}) \) are all distinct. Hence \( \sigma (v(\chi_\alpha)) = \sum e(\chi_\alpha^{\sigma \tau}) \) is the sum of all distinct idempotents of \( EG^\alpha \), so is equal to \( v(\chi_\alpha) \) for all \( \sigma \in G \). Thus \( v(\chi_\alpha) \) is a block idempotent in \( KG^\alpha \) associated with \( \chi_\alpha \). We note that \( e(\chi_\alpha) \subset v(\chi_\alpha) \subset e(\chi_\alpha) + \sum_{\tau \in \mathbb{G} - G_0} e(\chi_\alpha^\tau) \) where \( G_0 = \text{Gal}(K(\Omega_\alpha)/K(\chi_\alpha)) \).

For the \( \alpha \)-representation \( \rho \) on \( G \), the mapping on \( EG^\alpha \) defined by \( \sum x_g a_g \mapsto \sum x_g \rho(g) \ (x_g \in E) \) is a homomorphism of \( E \)-algebras. We shall use the same notation \( \rho \) for the homomorphism on \( EG^\alpha \). Since \( U \) is a simple \( EG^\alpha \)-module corresponding to \( \chi_\alpha \), \( e(\chi_\alpha) \) acts as identity and the other block idempotents must annihilate \( U \). Thus

\[ \rho(v(\chi_\alpha)) = 1 \quad \text{and} \quad \rho(KG^\alpha v(\chi_\alpha)) = \rho(KG^\alpha). \]

Hence \( \rho \) induces a surjective homomorphism of \( KG^\alpha v(\chi_\alpha) \) onto \( \rho(KG^\alpha) \). But since \( KG^\alpha v(\chi_\alpha) \) is simple, \( \rho \) is one to one and \( KG^\alpha v(\chi_\alpha) \cong \rho(KG^\alpha) \).

And the second statement \( K(\chi_\alpha) \cong Z(\rho(KG^\alpha)) \) follows immediately from Theorem 7.3.8 (iii) in [3, Vol.3]. \( \square \)
COROLLARY 6. Let the context be the same as in Theorem 5. Let $A$ be a simple component of $KG^\alpha$ corresponding to $\chi_\alpha$. Then $A$ is central over $K$ if and only if $K = K(\chi_\alpha)$.

Proof. The simple component $A$ of $KG^\alpha$ is isomorphic to $KG^\alpha v(\chi_\alpha)$ with a block idempotent $v(\chi_\alpha)$ of $KG^\alpha$. Thus $A$ is central over $K$ if and only if $K = Z(A) \cong Z(KG^\alpha v(\chi_\alpha)) \cong K(\chi_\alpha)$ by Theorem 5.

We are now able to recast the definition of projective Schur algebra in the following form.

THEOREM 7. An algebra $A$ is a projective Schur $K$-algebra if and only if there exists a finite group $G$, a $2$-cocycle $\alpha \in Z^2(G, K^*)$ and an irreducible $\alpha$-character $\chi_\alpha$ of $G$ over $E$ such that $K = K(\chi_\alpha)$ and $A \cong KG^\alpha e(\chi_\alpha)$, where $e(\chi_\alpha)$ is as in (4).

Proof. Let $A$ be a projective Schur $K$-algebra. Then $A$ is a central simple $K$-algebra that is a homomorphic image of $KG^\alpha$ for a finite group $G$ and a $2$-cocycle $\alpha \in Z^2(G, K^*)$.

Let $v$ be a block idempotent of $KG^\alpha$ such that $A = KG^\alpha v$. Since $v$ is a sum of distinct block idempotents of $EG^\alpha$, we may let $e$ be a block idempotent of $EG^\alpha$ which is a summand of $v$. Let $\chi_\alpha$ be the irreducible $\alpha$-character of $G$ over $E$ associated with $e$. Then we can write $e = e(\chi_\alpha) = (\chi_\alpha(1)/|G|) \sum_{g \in G} \alpha^{-1}(g, g^{-1}) \chi_\alpha(g^{-1}) a_g$. By considering the fields $K \subset K(\chi_\alpha) \subset K(\Omega_\alpha) \subset E$ as in Theorem 3 and by letting $G = \text{Gal}(K(\Omega_\alpha)/K)$, without loss of generality we may write $v = v(\chi_\alpha) = \sum e(\chi_\alpha^\tau)$ which is the sum of distinct $e(\chi_\alpha^\tau)$ for $\tau \in G$ as in (5).

Since $A = KG^\alpha v(\chi_\alpha)$ is central, we have $K = K(\chi_\alpha)$ due to Corollary 6. And since $e(\chi_\alpha) \subset v(\chi_\alpha) \subset e(\chi_\alpha) + \sum e(\chi_\alpha^\tau)$ where the sum ranges over $\tau \in \text{Gal}(K(\Omega_\alpha)/K) - \text{Gal}(K(\Omega_\alpha)/K(\chi_\alpha))$ by the proof of Theorem 5, it follows that $e(\chi_\alpha) = v(\chi_\alpha)$, so $A = KG^\alpha v(\chi_\alpha)$ is isomorphic to $KG^\alpha e(\chi_\alpha)$. The other direction is easy to see.

3. Projective Schur algebra over a field

A $K$-algebra $\Gamma$ is said to be definable over a subfield $L$ of $K$ if $\Gamma$ is isomorphic to $K \otimes_L \Gamma' = \Gamma'^K$ for some $L$-algebra $\Gamma'$. It is known that $KG$ is definable over $Q$ if $\text{char}K = 0$, and $KG^\alpha$ is definable over a subfield $L$ of $K$ if $L$ contains the values of $\alpha \in Z^2(G, K^*)$ [3, Vol.3(7.1.1)]. For a simple $\Gamma$-module $V$, $V^E$ need not be a semisimple $\Gamma^E$-module. However if $\Gamma$ is definable over a perfect subfield of $K$ (or if $K$ itself is perfect) then $V^E$ is semisimple ([3, Vol.3(7.1.3)]).
THEOREM 8. Let $K$ be an algebraic number field and $A$ be a projective Schur $K$-algebra which is an image of $KG^\alpha$ for a finite group $G$ and $\alpha \in Z^2(G, K^*)$. Let $M_\alpha$ be a subfield of $K$ containing the values of $\alpha$. Then for the maximal radical extension field $k$ of $M_\alpha$ in $K$, there is a projective Schur $k$-algebra $A'$ such that $A \cong K \otimes_k A'$.

Proof. Let $\chi_\alpha$ be the $\alpha$-character of $G$ over an algebraic closure $E$ that corresponds to the simple $KG^\alpha$-algebra $A$. Let $\Omega_\alpha = \langle \mu, \{\delta_g | g \in G \rangle \rangle$ be the subset of $E^*$, where $\mu$ is the set of $|G|$-th root of unity in $E$ and $\delta_g^{o(g)} = \prod_{i=1}^{o(g)} \alpha(g^i, g)$ (see Theorem 3). Then there is a tower of fields $K \subseteq K(\chi_\alpha) \subseteq K(\Omega_\alpha) \subseteq E$, and by Theorem 7 we are able to write

$$A \cong KG^\alpha e(\chi_\alpha) \text{ and } K(\chi_\alpha) = K,$$

where $e(\chi_\alpha) = (\chi_\alpha(1)/|G|) \sum_{g \in G} \alpha^{-1}(g, g^{-1}) \chi_\alpha(g^{-1}) a_g$ is the block idempotent of $EG^\alpha$ (see (4)).

Consider two extension fields $k(\chi_\alpha)$ and $k(\Omega_\alpha)$ of $k$ adjoined by the values of $\chi_\alpha$ and the set $\Omega_\alpha$ to $k$ respectively. Then $k \subseteq k(\chi_\alpha) \subseteq k(\Omega_\alpha) \subseteq E$, and $k(\Omega_\alpha)$ is a finite radical Galois extension of $k$ because $\delta_g^{o(g)} = \prod_{i=1}^{o(g)} \alpha(g^i, g) \in M_\alpha \subseteq k$. We denote $\text{Gal}(k(\Omega_\alpha)/k)$ by $G'$.

Obviously $KG^\alpha$ is definable over $k$ since all values of $\alpha$ are in $M_\alpha \subseteq k$. Thus $KG^\alpha = K \otimes_k kG^\alpha$, where the $K$-basis $a_g$ of $KG^\alpha$ is also considered as a $k$-basis of $kG^\alpha$, and the block idempotent $e(\chi_\alpha)$ of $EG^\alpha$ belongs to $k(\chi_\alpha)G^\alpha \subseteq k(\Omega_\alpha)G^\alpha$.

Let $A'$ be the simple component of $kG^\alpha$ corresponding to $\chi_\alpha$. Due to Theorem 5, $\nu'(\chi_\alpha) = \sum_{\tau} e(\chi_\alpha^\tau)$ where $\tau$ runs over $G'$ such that all $e(\chi_\alpha^\tau)$ are distinct is a block idempotent of $kG^\alpha$ associated with $\chi_\alpha$. Here we may assume $e(\chi_\alpha) \subseteq \nu'(\chi_\alpha)$. And the simple component $A'$ is isomorphic to $kG^\alpha \nu'(\chi_\alpha)$. We are now enough to show that $A'$ is a projective Schur $k$-algebra satisfying $A \cong K \otimes_k A'$.

Clearly $K \cap k(\Omega_\alpha)$ is a radical extension of $k$ contained in $K$, thus $K \cap k(\Omega_\alpha)$ is also radical over $M_\alpha$ because $k$ is radical over $M_\alpha$ (see [2, (3.10.1)]). But since $k$ is a maximal radical extension of $M_\alpha$ in $K$, it follows that $k \subseteq K \cap k(\chi_\alpha) \subseteq K \cap k(\Omega_\alpha) \subseteq k$, and they are all same.

Every value of $\chi_\alpha$ is contained in $K$, for $K = K(\chi_\alpha)$. And $\chi_\alpha(g) \in k(\chi_\alpha)$ for all $g \in G$. Thus $\chi_\alpha(g) \in K \cap k(\chi_\alpha) = k$, so $k = k(\chi_\alpha)$. Hence by making use of Corollary 6, $A'$ is a central simple $k$-algebra.

Since $e(\chi_\alpha)$ belongs to $k(\chi_\alpha)G^\alpha = kG^\alpha$, every $\tau \in G' = \text{Gal}(k(\Omega_\alpha)/k)$ leaves $e(\chi_\alpha)$ fixed, so $\nu'(\chi_\alpha) = e(\chi_\alpha)$. Thus the simple algebra $A' \cong kG^\alpha \nu'(\chi_\alpha)$ is isomorphic to $kG^\alpha e(\chi_\alpha)$, hence $A'$ is a projective Schur $k$-algebra due to Theorem 7. Therefore our required situation follows
immediately that

\[ A \cong KG^\alpha e(\chi_\alpha) \cong K \otimes_k kG^\alpha e(\chi_\alpha) \cong K \otimes_k A'. \]

Without loss of generality we may assume that \( M_\alpha \) is the smallest subfield of \( K \) containing all values of \( \alpha \). We showed that a projective Schur \( K \)-algebra can be obtained by \( K \)-scalar extension of a projective Schur \( k \)-algebra where \( k \) is a certain subfield of \( K \). This observation will be clear if we assume the following case.

**Corollary 9.** Let \( A \) be a projective Schur \( K \)-algebra which is a homomorphic image of \( KG^\alpha \). With the same context in Theorem 8, let \( M_\alpha(\Omega_\alpha) \) be the extension field of \( M_\alpha \) adjoining the set \( \Omega_\alpha \). If \( k = K \cap M_\alpha(\Omega_\alpha) \), then \( A \) is a \( K \)-scalar extension of a projective Schur \( k \)-algebra.

**Proof.** Due to Theorem 7, we may write \( A \cong KG^\alpha e(\chi_\alpha) \) and \( K = K(\chi_\alpha) \). Since values of \( \alpha \) are contained in both \( K \) and \( M_\alpha \), \( KG^\alpha \) is definable over \( k \) so that \( KG^\alpha \cong K \otimes_k kG^\alpha \).

From \( k(\chi_\alpha) = K(\chi_\alpha) \cap M_\alpha(\Omega_\alpha) = K \cap M_\alpha(\Omega_\alpha) = k \), \( e(\chi_\alpha) \in EG^\alpha \) is contained in \( kG^\alpha \) and is left fixed by all \( \tau \in \text{Gal}(k(\Omega_\alpha)/k) \). Hence the block idempotent \( v'(\chi_\alpha) \) in \( kG^\alpha \) which is a sum of distinct \( e(\chi_\alpha^\tau) \)'s for \( \tau \in \text{Gal}(k(\Omega_\alpha)/k) \) is equal to \( e(\chi_\alpha) \). Thus the central simple \( k \)-algebra \( A' \cong kG^\alpha v'(\chi_\alpha) \) associated with \( \chi_\alpha \) is isomorphic to \( kG^\alpha e(\chi_\alpha) \), and it follows that \( A \cong KG^\alpha e(\chi_\alpha) = K \otimes_k kG^\alpha e(\chi_\alpha) \cong K \otimes_k A' \).

In Corollary 9, if \( \alpha = 1 \) then \( k = K \cap M_\alpha(\Omega_\alpha) \) equals \( K \cap \mathbb{Q}(\mu) \), which is the same field chosen in Theorem 2 for Schur algebra. Theorem 8 provides a partial analog of Theorem 2 that \( A \cong K \otimes_k A' \) for \( [A] \in PS(K) \) and \( [A'] \in PS(k) \). However it does not imply the equality \( PS(K) = K \otimes_k PS(k) \), even it is not true. For instance, if \( K \) is an algebraic number field then \( PS(K) \) is the whole Brauer group \( B(K) \) due to [4], hence the equality would mean that every element in \( B(K) \) comes from \( B(\mathbb{Q}) \), which is not correct.

**Theorem 10.** Let \( K, E, \alpha \in Z^2(G, K^*) \), \( \chi_\alpha \) and \( v(\chi_\alpha) \) be the same as in Theorem 8. Let \( A \cong KG^\alpha v(\chi_\alpha) \) be a simple component of \( KG^\alpha \) corresponding to \( \chi_\alpha \). If \( \beta \in Z^2(G, K^*) \) is cohomologous to \( \alpha \) (denote it by \( \alpha \sim \beta \)) then there is an irreducible \( \beta \)-character \( \chi_\beta \) of \( G \) over \( E \) such that \( K(\chi_\alpha) = K(\chi_\beta) \) and \( v(\chi_\alpha) = v(\chi_\beta) \), so the simple component \( B \) of \( KG^\beta \) corresponding to \( \chi_\beta \) is isomorphic to \( A \), as \( K \)-algebras.

**Proof.** Let \( \rho \) be an irreducible \( \alpha \)-representation of \( G \) over \( E \) which affords \( \chi_\alpha \). Let \( \beta(g, x) = \alpha(g, x)t(g)t(x)t^{-1}(gx) \) with a map \( t : G \to K^* \) \((t(1) = 1)\) for \( g, x \in G \). Then it is easy to see that \( \rho' \) and \( \chi_\beta \) defined by
\( \rho'(g) = t(g) \rho(g) \) and \( \chi_\beta(g) = t(g) \chi_\alpha(g) \) are irreducible \( \beta \)-representation and \( \beta \)-character of \( G \) respectively, and \( \rho' \) affords \( \chi_\beta \). Since \( \chi_\beta(g) = t(g) \chi_\alpha(g) \in K(\chi_\alpha), K(\chi_\beta) \subseteq K(\chi_\alpha) \) and they are equal. Moreover since

\[
\prod_{i=1}^{o(g)} \beta(g^i, g) = \prod_{i=1}^{o(g)} \alpha(g^i, g) t(g^i) t(g) t^{-1}(g^{i+1}) = \lambda_g t(g)^{o(g)} \quad \text{for } g \in G,
\]

where \( \lambda_g \) is in (3), we may take \( o(g) \)-th root \( \delta'_g \) of \( \prod_{i=1}^{o(g)} \beta(g^i, g) \) as \( \delta'_g t(g) \), where \( \delta^{o(g)}_g = \lambda_g \). Hence

\[
K(\Omega_\beta) = K(\{ \mu, \{ \delta'_g g \in G \} \}) = K(\{ \mu, \{ \delta_g g \in G \} \}) = K(\Omega_\alpha),
\]

so we shall denote it by \( K(\Omega) = K(\Omega_\alpha) = K(\Omega_\beta) \).

Let \( \{ a_g | g \in G \} \) be a \( K \)-basis of \( KG^\alpha \). Then \( b_g = t(g) a_g \) forms a basis of \( KG^\beta \), and \( KC^\alpha \cong KG^\beta \) as \( K \)-algebras under \( a_g \mapsto t(g^{-1}) b_g \) \( g \in G \). Moreover the block idempotent \( e(\chi_\alpha) \) of \( EG^\beta \) is equal to \( e(\chi_\alpha) \) of \( EG^\alpha \), because

\[
e(\chi_\beta) = \frac{\chi_\beta(1)}{|G|} \sum_{g \in G} \beta^{-1}(g, g^{-1}) \chi_\beta(g^{-1}) b_g
\]

\[
= \frac{t(1) \chi_\alpha(1)}{|G|} \sum_{g \in G} \alpha^{-1}(g, g^{-1}) t^{-1}(g) t^{-1}(g^{-1}) t(g g^{-1}) t(g^{-1}) \chi_\alpha(g^{-1}) t(g) a_g
\]

\[
= \frac{\chi_\alpha(1)}{|G|} \sum_{g \in G} \alpha^{-1}(g, g^{-1}) \chi_\alpha(g^{-1}) a_g = e(\chi_\alpha),
\]

thus \( \nu(\chi_\beta) \) the sum of distinct \( e(\chi_\beta) \) for \( \tau \in \text{Gal}(K(\Omega)/K) \) is equal to \( \nu(\chi_\alpha) \). Hence it follows immediately that the simple component \( B \) of \( KG^\beta \) corresponding to \( \chi_\beta \) is isomorphic to \( KG^\beta \nu(\chi_\beta) \cong KG^\alpha \nu(\chi_\alpha) \cong A \).

Let \( A \) be a projective Schur \( K \)-algebra. Then due to Theorem 7, \( A \cong KG^\alpha e(\chi_\alpha) \) and \( K(\chi_\alpha) = K \) with \( \alpha \in Z^2(G, K^*) \) and an irreducible \( \alpha \)-character \( \chi_\alpha \) of a finite group \( G \). If we consider \( \beta \in Z^2(G, K^*) \) such that \( \alpha \sim \beta \) then \( A \) is isomorphic to a simple algebra \( B \cong KG^\beta e(\chi_\beta) \) for some irreducible \( \beta \)-character \( \chi_\beta \) due to Theorem 10. Furthermore since \( K(\chi_\beta) = K(\chi_\alpha) = K \), \( B \) is a projective Schur \( K \)-algebra. Now applying Theorem 8 to both \( A \) and \( B \), there exists a projective Schur \( k_\alpha \)-algebra \( A' \) and a projective Schur \( k_\beta \)-algebra \( B' \) such that

\[
K \otimes k_\alpha A' \cong A \cong B \cong K \otimes k_\beta B',
\]
where $M_{\alpha}$ [resp. $M_{\beta}$] is the (smallest) subfield of $K$ containing all values of $\alpha$ [resp. $\beta$], and $k_{\alpha}$ [resp. $k_{\beta}$] is the maximal radical extension of $M_{\alpha}$ [resp. $M_{\beta}$] contained in $K$. We observe that though $K(\chi_\alpha) = K(\chi_\beta)$ and $K(\Omega_\alpha) = K(\Omega_\beta)$, it is not necessarily $M_{\alpha}$ and $M_{\beta}$, and $k_{\alpha}$ and $k_{\beta}$ are same respectively.

**Theorem 11.** Let $K$ be an algebraic number field, and $A$ and $B$ be any projective Schur $K$-algebras. Then there exist a subfield $k$ of $K$ and projective Schur $k$-algebras $A_0$ and $B_0$ such that $A \cong K \otimes_k A_0$ and $B \cong K \otimes_k B_0$.

**Proof.** Let $A$ and $B$ be homomorphic images of $KG^{\alpha}$ and $KH^{\beta}$ respectively where $G$ and $H$ are finite groups, $\alpha \in Z^2(G, K^*)$ and $\beta \in Z^2(H, K^*)$. Due to Theorem 8 there are $M_{\alpha}$ [resp. $M_{\beta}$] which is the smallest subfield of $K$ containing all values of $\alpha$ [resp. $\beta$], and $k_{\alpha}$ [resp. $k_{\beta}$] which is the maximal radical extension of $M_{\alpha}$ [resp. $M_{\beta}$] in $K$, satisfying

$$A \cong K \otimes_{k_{\alpha}} A' \quad \text{and} \quad B \cong K \otimes_{k_{\beta}} B',$$

where $A'$ and $B'$ are projective Schur $k_{\alpha}$ and $k_{\beta}$-algebras respectively.

Let $F$ be a subfield of $K$ containing the values of both $\alpha$ and $\beta$. And let $k$ be the maximal radical extension of $F$ in $K$. Obviously $M_{\alpha} \subseteq F$ and $M_{\beta} \subseteq F$.

It is easy to see that $k_{\alpha}$ and $k_{\beta}$ are contained in $k$. In fact, since $k_{\alpha}$ is a radical extension of $M_{\alpha}$, we may write $k_{\alpha} = M_{\alpha}(\Delta_{\alpha})$ with a subset $\Delta_{\alpha}$ of $k_{\alpha}^*$ such that $\Delta_{\alpha}M_{\alpha}^*/M_{\alpha}^*$ is torsion. Clearly $M_{\alpha} \subseteq F \subseteq k$. Moreover if $x \in \Delta_{\alpha}$ then $x^m \in M_{\alpha} \subseteq F$ for some $m > 0$, thus $xF^*$ is of finite order in $K^*/F^*$. Due to the maximality of $k$ in $K$, we have $x \in k$ and $\Delta_{\alpha} \subseteq k$, thus $k_{\alpha} \subseteq k$. Similarly we have $k_{\beta} \subseteq k$.

Now since both $KG^{\alpha}$ and $KH^{\beta}$ are definable over $k$, we have $KG^{\alpha} = K \otimes_k kG^{\alpha}$ and $KH^{\beta} = K \otimes_k kH^{\beta}$. And by applying Theorem 8 to $F$ and its maximal radical extension $k$ in $K$, we can conclude that there exist projective Schur $k$-algebras $A_0$ and $B_0$ such that $A \cong K \otimes_k A_0$ and $B \cong K \otimes_k B_0$. \qed

Theorem 11 motivates to construct a subset $PS_F(K)$ of $PS(K)$ for a subfield $F$ of $K$: let $F \subseteq K$ and let $PS_F(K)$ be the set of similar classes $[S]$ of $K$-algebras where $[S]$ contains a projective Schur $K$-algebra that is an image of $KG^{\alpha}$ definable over $F$ for some finite group $G$ and $\alpha \in Z^2(G, K^*)$.

Obviously, $PS_F(K)$ is a subgroup of $PS(K)$ for, let $[S_i] \in PS_F(K)$ be with $A_i \in [S_i]$ ($i = 1, 2$) where $A_i$ is an image of $KG_{G_i}^{\alpha_i}$ and $KG_{G_i}^{\alpha_i}$ is definable over $F$. Then $A_1 \otimes_K A_2$ is represented by $K(G_1 \times G_2)^{\alpha_1 \times \alpha_2}$, where
\( \alpha_1 \times \alpha_2 \) is defined by \( \alpha_1 \times \alpha_2((g_1, g_2), (x_1, x_2)) = \alpha_1(g_1, x_1)\alpha_2(g_2, x_2) \) for \( g_i, x_i \in G_i \) \((i = 1, 2)\). Moreover \( K(G_1 \times G_2)_{\alpha_1 \times \alpha_2} \) is definable over \( F \) because \( K(G_1 \times G_2)_{\alpha_1 \times \alpha_2} = KG_{\alpha_1} \otimes_K KG_{\alpha_2} = (K \otimes_F FG_{\alpha_1}) \otimes_K (K \otimes_F FG_{\alpha_2}) = K \otimes_F (FG_{\alpha_1} \otimes_F FG_{\alpha_2}) = K \otimes_F F(G_1 \times G_2)_{\alpha_1 \times \alpha_2} \). Thus \( A_1 \otimes_K A_2 \in [S_1][S_2] \) and \([S_1][S_2] \in PS_F(K)\). In particular if \( \alpha_i \) has values in \( F \) then so does \( \alpha_1 \times \alpha_2 \).

Let \([S]\) be any element in \( PS_F(K) \) and \( A \in [S]\) be an image of \( KG_{\alpha} \) for some \( \alpha \in Z^2(G, K^*)\). Then there is an irreducible \( \alpha \)-character \( \chi_\alpha \) such that \( K = K(\chi_\alpha) \) and \( A \cong KG_{\alpha}e(\chi_\alpha) \) by Theorem 7, where \( e(\chi_\alpha) \) is as in (4). We note that since \( KG_{\alpha} \) is definable over \( F \), it is also definable over the maximal radical extension field \( k \) of \( F \) in \( K \). Moreover due to Theorem 8, \( A \cong K \otimes_k kG_{\alpha}e(\chi_\alpha) \). But since \([kG_{\alpha}e(\chi_\alpha)] \in PS_F(k)\), \([A] = K \otimes_k [kG_{\alpha}e(\chi_\alpha)] \) belongs to \( K \otimes_k PS_F(k) \).

Hence the following theorem is straightforward.

**Theorem 12.** Let \( K \) be an algebraic number field and \( F \subset K \). Then \( PS_F(K) \) is a subgroup of \( PS(K) \) and \( PS_F(K) = K \otimes_k PS_F(k) \) for the maximal radical extension field \( k \) of \( F \) in \( K \).

As an application to a special class of projective Schur algebras, we consider the radical (abelian) algebra \([1]\) which is a crossed product algebra \((L/K, \alpha')\) where \( L = K(\Omega) \) is a radical (abelian) \( G \)-Galois extension of \( K \) (that is, \( \Omega \) is a \( G = \text{Gal}(L/K) \)-invariant subgroup of \( L^* \) (i.e., \( \sigma(\Omega) \subseteq \Omega \) for any \( \sigma \in G \)) such that \( \Omega K^*/K^* \) is a torsion group), and \( \alpha' \in Z^2(G, L^*) \) is the image of some \( \alpha \in Z^2(G, \Omega) \) under the inclusion \( \Omega \hookrightarrow L^* \). The radical algebra is an analogue of the cyclotomic algebra in the context of projective Schur algebra, and every projective Schur division algebra is itself a radical abelian algebra. The set of similarity classes of radical \( K \)-algebra forms a radical group \( \text{Rad}(K) \) which is a subgroup of \( PS(K) \).

**Theorem 13.** Let \( k \) be a maximal radical extension of \( \mathbb{Q} \) contained in a field \( K \). Then for any \([S] \in \text{Rad}(K)\), \([S]^h \) is a \( K \)-scalar extension of an element in \( \text{Rad}(k) \) for some \( h > 0 \).

**Proof.** Let \( A = (K(\Omega)/K, \alpha') \) be a radical \( K \)-algebra contained in \([S]\). Then \([A] = [S] \), \( K(\Omega) \) is a radical \( G \)-Galois extension of \( K \) with \( G = \text{Gal}(K(\Omega)/K) \), and \( \alpha' \in Z^2(G, K(\Omega)^*) \) is the image of \( \alpha \in Z^2(G, \Omega) \). And for any \( \sigma \in G \) and \( \omega \in \Omega \), \( \omega^n \in K^* \) for some integer \( n > 0 \) and \( \sigma(\omega) \) belongs to \( \Omega \). Now let

\[
\Omega_0 = \{ \omega \in \Omega | \omega^n \in k^* \text{ for some } n > 0 \}.
\]
Then $\Omega_0 < \Omega$, $\Omega_0 k^*/k^*$ is a torsion group and $k(\Omega_0)$ is a radical extension of $k$. Since $k$ is the maximal radical extension contained in $K$, we have $K \cap k(\Omega_0) = k$.

Consider the field extensions $K \subseteq K(\Omega_0) \subseteq K(\Omega)$ and $k \subseteq k(\Omega_0) \subseteq k(\Omega)$. Let $\omega$ be any element in $\Omega_0$. Then $\omega^n \in k^* \subset K$, and $\sigma(\omega^n) = \sigma(\omega^n) = (i.e., \sigma(\Omega_0) \subset \Omega_0)$ for any $\sigma \in \text{Aut}_K K(\Omega_0)$. Let $x$ be any element in $K(\Omega_0) - K$. Then $x \in K(\Omega) - K$ and there is $\tau \in \text{Aut}_K K(\Omega)$ such that $\tau(x) \neq x$, for $K(\Omega)/K$ is Galois. Denote $\tau|_{K(\Omega_0)}$ by $\tau_0$. If we write any element $y \in K(\Omega_0)$ by $y = \sum a_i \omega_i$ with $a_i \in K$, $\omega_i \in \Omega_0$ then $\tau_0(y) = \tau(y) = \sum a_i \tau(\omega_i) \in K(\Omega_0)$. This shows that $\tau_0$ can be regarded as an element in $\text{Aut}_K K(\Omega_0)$ satisfying $\tau_0(x) = \tau(x) \neq x$. Therefore $K(\Omega_0)$ is a radical $G_0$-Galois extension of $K$ where $G_0 = \text{Gal}(K(\Omega_0)/K)$.

Since $\text{Gal}(K(\Omega_0)/K) \cong \text{Gal}(k(\Omega_0)/(K \cap k(\Omega_0))) = \text{Gal}(k(\Omega_0)/k)$, $k(\Omega_0)$ is also $G_0$-Galois radical over $k$; we shall denote $\text{Gal}(k(\Omega_0)/k)$ by the same notation $G_0$. If we write $H = \text{Gal}(K(\Omega)/K(\Omega_0))$ then $G/H$ is isomorphic to $G_0$.

From $A = (K(\Omega)/K, \alpha')$, let $\Gamma_\alpha$ be the group extension of $\Omega$ by $G$

$$\alpha : 1 \rightarrow \Omega \rightarrow \Gamma_\alpha \xrightarrow{j} G \rightarrow 1,$$

which corresponds to $\alpha \in Z^2(G, \Omega)$. Then $A = K(\Gamma_\alpha)$ as a $K$-vector space.

Consider the homomorphism [8, (5.3.2)]

$$\nu_{G \rightarrow G/H} : H^2(G, \Omega) \rightarrow H^2(G/H, \Omega^H),$$

defined in the following manner. Let $j^{-1}(H) = W$ and let $W_c$ be the commutator subgroup of $W$. Then there is a group extension

$$\alpha_c : 1 \rightarrow W/W_c \rightarrow \Gamma_\alpha/W_c \rightarrow G/H \rightarrow 1$$

having a factor set $\alpha_c$ in $Z^2(G/H, W/W_c)$. Denote by $\Lambda$ the reduced group theoretical transfer map $W/W_c \rightarrow \Omega^H$.

The $\Lambda$ is a $G/H$-homomorphism and induces a homomorphism of cohomology groups $\Lambda : H^2(G/H, W/W_c) \rightarrow H^2(G/H, \Omega^H)$. Then $\nu_{G \rightarrow G/H}$ is defined by $\nu_{G \rightarrow G/H}(\Lambda) = \Lambda(\alpha_c)$, where $\alpha \in H^2(G, \Omega)$ is the cohomology class of $\alpha$. It can be seen that $\nu_{G \rightarrow G/H}$ is a homomorphism. And we denote $\nu_{G \rightarrow G/H}(\alpha)$ by $\beta \in H^2(G/H, \Omega^H)$.

We observe $\Omega^H = \Omega_0$. In fact if $\omega \in \Omega^H$ then $\omega \in \Omega$ is fixed by all elements in $H = \text{Gal}(K(\Omega)/K(\Omega_0))$, so $\omega \in \Omega_0$. Conversely if $\omega \in \Omega_0$ then $\omega \in \Omega \cap K(\Omega_0)$ is fixed by $H$. Hence we may regard $\beta$ as an element
in $Z^2(G_0, \Omega_0)$, and we have a group extension $\Gamma_{\beta}$ of $\Omega_0$ by $G_0$:
$$\beta : 1 \rightarrow \Omega_0 \rightarrow \Gamma_{\beta} \rightarrow G_0 \rightarrow 1.$$ 

If let $\beta' \in Z^2(G_0, k(\Omega_0)^*)$ be an image of $\beta$ under the inclusion $\Omega_0 \hookrightarrow k(\Omega_0)^*$ and let $B = (k(\Omega_0)/k, \beta')$ be the crossed product algebra then $B = k(\Gamma_{\beta})$ is a radical $k$-algebra, so $[B] \in \text{Rad}(k)$.

In connection with the inflation map $H^2(G_0, \Omega_0) \cong H^2(G/H, \Omega^H) \xrightarrow{\inf} H^2(G, \Omega)$, the composition map $(\inf \cdot v_{G \rightarrow G/H})$ on $H^2(G, \Omega)$ defines
$$\bar{\alpha}^{[H]} = (\inf \cdot v_{G \rightarrow G/H})(\bar{\alpha}) = \inf(\bar{\beta})$$
[8, (5.3.3)]. Hence $\inf\beta$ is cohomologous to $\alpha^{[H]}$. Thus due to [6, (29,13), (29,16)], we have the following isomorphisms of crossed product algebras:

$$K \otimes [B] = K \otimes [(k(\Omega_0)/k, \beta')] = [(K(\Omega_0)/k, \beta')] = [(K(\Omega)/k, \inf(\beta'))] = [(K(\Omega)/K, \alpha^{[H]})] = [(K(\Omega)/K, \alpha')]^{[H]} = [A]^{[H]} = [S]^{[H]}.$$ 

In particular when $|H| = 1$ (i.e., $K(\Omega) = K(\Omega_0)$), a radical $K$-algebra can be extended from a radical $k$-algebra where $k$ is the maximal radical extension in $K$.

References


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