SOME NEW RESULTS RELATED TO BESSEL
AND GRÜSS INEQUALITIES IN 2-INNER
PRODUCT SPACES AND APPLICATIONS

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ABSTRACT. Some new reverses of Bessel's inequality for orthonormal families in real or complex 2-inner product spaces are pointed out. Applications for some Grüss type inequalities and for determinantal integral inequalities are given as well.

1. Introduction

The concepts of 2-inner products and 2-inner product spaces have been intensively studied by many authors in the last three decades. A systematic presentation of the recent results related to the theory of 2-inner product spaces as well as an extensive list of the related references can be found in [1]. Here we give the basic definitions and the elementary properties of 2-inner product spaces.

Let $X$ be a linear space of dimension greater than 1 over the field $\mathbb{K} = \mathbb{R}$ of real numbers or the field $\mathbb{K} = \mathbb{C}$ of complex numbers. Suppose that $(\cdot, \cdot|\cdot)$ is a $\mathbb{K}$-valued function defined on $X \times X \times X$ satisfying the following conditions:

$(2I_1)$ $(x, x|z) \geq 0$ and $(x, x|z) = 0$ if and only if $x$ and $z$ are linearly dependent,
$(2I_2)$ $(x, x|z) = (z, z|x)$,
$(2I_3)$ $(y, x|z) = (x, y|z)$,
$(2I_4)$ $(\alpha x, y|z) = \alpha(x, y|z)$ for any scalar $\alpha \in \mathbb{K}$,
$(2I_5)$ $(x + x', y|z) = (x, y|z) + (x', y|z)$.

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(·, ·|·) is called a 2-inner product on X and \((X, (·, ·|·))\) is called a 2-inner product space (or 2-pre-Hilbert space). Some basic properties of 2-inner product spaces can be immediately obtained as follows [2]:

1. If \(K = \mathbb{R}\), then \((2I_3)\) reduces to
\[
(y, x|z) = (x, y|z).
\]

2. From \((2I_3)\) and \((2I_4)\), we have
\[
(0, y|z) = 0, \quad (x, 0|z) = 0
\]
and also
\[
(1.1) \quad (x, \alpha y|z) = \bar{\alpha}(x, y|z).
\]

3. Using \((2I_2)\)–\((2I_5)\), we have
\[
(z, z|x \pm y) = (x \pm y, x \pm y|z) = (x, x|z) + (y, y|z) \pm 2 \text{Re}(x, y|z)
\]
and
\[
(1.2) \quad \text{Re}(x, y|z) = \frac{1}{4} \left[ (z, z|x + y) - (z, z|x - y) \right].
\]

In the real case \(K = \mathbb{R}\), \((1.2)\) reduces to
\[
(1.3) \quad (x, y|z) = \frac{1}{4} \left[ (z, z|x + y) - (z, z|x - y) \right]
\]
and, using this formula, it is easy to see that, for any \(\alpha \in \mathbb{R}\),
\[
(1.4) \quad (x, y|\alpha z) = \alpha^2 (x, y|z).
\]

In the complex case, using \((1.1)\) and \((1.2)\), we have
\[
\text{Im}(x, y|z) = \text{Re}[-i(x, y|z)] = \frac{1}{4} \left[ (z, z|x + iy) - (z, z|x - iy) \right],
\]
which, in combination with \((1.2)\), yields
\[
(1.5) \quad (x, y|z) = \frac{1}{4} \left[ (z, z|x + y) - (z, z|x - y) \right] + \frac{i}{4} \left[ (z, z|x + iy) - (z, z|x - iy) \right].
\]

Using the above formula and \((1.1)\), we have, for any \(\alpha \in \mathbb{C}\),
\[
(1.6) \quad (x, y|\alpha z) = |\alpha|^2 (x, y|z).
\]

However, for \(\alpha \in \mathbb{R}\), \((1.6)\) reduces to \((1.4)\). Also, from \((1.6)\) it follows that
\[
(x, y|0) = 0.
\]

4. For any three given vectors \(x, y, z \in X\), consider the vector \(u = (y, y|z)x - (x, y|z)y\). By \((2I_1)\), we know that \((u, u|z) \geq 0\) with the equality
if and only if $u$ and $z$ are linearly dependent. The inequality $(u, u|z) \geq 0$ can be rewritten as

$$(y, y|z) \left[ (x, x|z)(y, y|z) - |(x, y|z)|^2 \right] \geq 0.$$  

For $x = z$, (1.7) becomes

$$-(y, y|z)(z, y|z)^2 \geq 0,$$

which implies that

$$(z, y|z) = (y, z|z) = 0$$  

provided $y$ and $z$ are linearly independent. Obviously, when $y$ and $z$ are linearly dependent, (1.8) holds too. Thus (1.8) is true for any two vectors $y, z \in X$. Now, if $y$ and $z$ are linearly independent, then $(y, y|z) > 0$ and, from (1.7), it follows that

$$(x, y|z)^2 \leq (x, x|z)(y, y|z).$$

Using (1.8), it is easy to check that (1.9) is trivially fulfilled when $y$ and $z$ are linearly dependent. Therefore, the inequality (1.9) holds for any three vectors $x, y, z \in X$ and is strict unless the vectors $u = (y, y|z)x - (x, y|z)y$ and $z$ are linearly dependent. In fact, we have the equality in (1.9) if and only if the three vectors $x, y$ and $z$ are linearly dependent.

In any given 2-inner product space $(X, \langle \cdot, \cdot \rangle)$, we can define a function $\| \cdot | \cdot \|$ on $X \times X$ by

$$(x, z) = \sqrt{(x, x|z)}$$

for all $x, z \in X$.

It is easy to see that this function satisfies the following conditions:

$(2N_1) \|x|z\| \geq 0$ and $\|x|z\| = 0$ if and only if $x$ and $z$ are linearly dependent,

$(2N_2) \|zx\| = |z||x|z|$

$(2N_3) \|\alpha x\| = |\alpha||x|z|$

for any scalar $\alpha \in \mathbb{K}$,

$(2N_4) \|x + x'\| \leq \|x|z\| + \|x'|z\|.$

Any function $\| \cdot | \cdot \|$ defined on $X \times X$ and satisfying the conditions $(2N_1)-(2N_4)$ is called a 2-norm on $X$ and $(X, \| \cdot | \cdot \|)$ is called a linear 2-normed space [5]. Whenever a 2-inner product space $(X, \langle \cdot, \cdot \rangle)$ is given, we consider it as a linear 2-normed space $(X, \| \cdot | \cdot \|)$ with the 2-norm defined by (1.10).

Let $(X; \langle \cdot, \cdot \rangle)$ be a 2-inner product space over the real or complex number field $\mathbb{K}$. If $(f_i)_{1 \leq i \leq n}$ are linearly independent vectors in the 2-inner product space $X$, and, for a given $z \in X$, $(f_i, f_j|z) = \delta_{ij}$ for all $i, j \in \{1, \ldots, n\}$, where $\delta_{ij}$ is the Kronecker delta (we say that the family
(\(f_i\))_{1 \leq i \leq n} is \textit{z-orthonormal}), then the following inequality is the corresponding \textit{Bessel’s inequality} (see for example [2]) for the \textit{z-orthonormal family} \((f_i)_{1 \leq i \leq n}\) in the 2-inner product space \((X; \langle \cdot, \cdot \rangle)\):

\[
(1.11) \quad \sum_{i=1}^{n} |\langle x, f_i | z \rangle|^2 \leq \|x|z\|^2
\]

for any \(x \in X\). For more details on this inequality, see the recent paper [2] and the references therein.

The following reverse of Bessel’s inequality in 2-inner product spaces has been obtained in [4]:

\textbf{THEOREM 1.} Let \(\{e_i\}_{i \in I}\) be a family of \textit{z-orthonormal vectors} in \(X\) and \(F\) a finite part of \(I\), \(\phi_i, \Phi_i\) \((i \in F)\) real or complex numbers and \(x, z \in X\) be so that either

\(\text{i.} \quad \text{Re} \left( \sum_{i \in F} \Phi_i e_i - x, x - \sum_{i \in F} \phi_i e_i |z\right) \geq 0\)


or, equivalently,

\(\text{ii.} \quad \left\| x - \sum_{i \in F} \frac{\Phi_i + \phi_i}{2} e_i |z\right\| \leq \frac{1}{2} \left( \sum_{i \in F} \left| \Phi_i - \phi_i \right|^2 \right)^{\frac{1}{2}}\)

holds. Then we have the inequality:

\[
0 \leq \|x|z\|^2 - \sum_{i \in F} |\langle x, e_i | z \rangle|^2
\]

\[
\leq \frac{1}{4} \sum_{i \in F} \left| \Phi_i - \phi_i \right|^2 - \text{Re} \left( \sum_{i \in F} \Phi_i e_i - x, x - \sum_{i \in F} \phi_i e_i |z\right)
\]

\[
\left( \leq \frac{1}{4} \sum_{i \in F} \left| \Phi_i - \phi_i \right|^2 \right).
\]

The constant \(\frac{1}{4}\) is best possible.

The following different reverse of Bessel’s inequality has been obtained in [3].

\textbf{THEOREM 2.} Let \(\{e_i\}_{i \in I}\) be a family of \textit{z-orthonormal vectors} in \(X\), \(F\) a finite part of \(I\), \(\phi_i, \Phi_i\) \((i \in I)\) real or complex numbers. For \(x \in X\), if either (i) or (ii) from Theorem 1 holds, then the following reverse of
Bessel’s inequality

\[ 0 \leq \|x\|_z^2 - \sum_{i \in F} |(x, e_i)_z|^2 \]
\[ \leq \frac{1}{4} \sum_{i \in F} |\Phi_i - \phi_i|^2 - \sum_{i \in F} \left| \frac{\phi_i + \Phi_i}{2} - (x, e_i)_z \right|^2 \]
\[ \leq \frac{1}{4} \sum_{i \in F} |\Phi_i - \phi_i|^2 \]

is valid. The constant \( \frac{1}{4} \) is best possible.

The main aim of the present paper is to establish a different reverse inequality for (1.11) than those incorporated in the above two theorems. Some companion results and applications for determinantal integral inequalities are also given.

2. A new reverse of Bessel’s inequality

The following reverse of Bessel’s inequality holds.

**Theorem 3.** Let \( \{e_i\}_{i \in I} \) be a family of \( z \)-orthonormal vectors in \( X \), \( F \) a finite part of \( I \) and \( \phi_i, \Phi_i \ (i \in F) \) real or complex numbers such that 
\[ \sum_{i \in F} \text{Re} (\Phi_i \phi_i) > 0. \]
If \( x \in X \) is such that either

(i) \( \text{Re} \left( \sum_{i \in F} \Phi_i e_i - x, x - \sum_{i \in F} \phi_i e_i \right)_z \geq 0 \)

or, equivalently,

(ii) \[ \left\| x - \sum_{i \in F} \frac{\phi_i + \Phi_i}{2} e_i \right\| \leq \frac{1}{2} \left( \sum_{i \in F} |\Phi_i - \phi_i|^2 \right)^{\frac{1}{2}} \]

holds, then one has the inequality

\[ \|x\|_z^2 \leq \frac{1}{4} \cdot \sum_{i \in F} \frac{|\phi_i + \Phi_i|^2}{\text{Re} (\Phi_i \phi_i)} \sum_{i \in F} |(x, e_i)_z|^2. \]

The constant \( \frac{1}{4} \) is best possible in the sense that it cannot be replaced by a smaller constant.

**Proof.** Firstly, we observe that, for \( y, a, A \in X \), the following are equivalent

(2.2) \[ \text{Re} \ (A - y, y - a|z) \geq 0 \]

and

(2.3) \[ \left\| y - \frac{a + A}{2} z \right\| \leq \frac{1}{2} \|A - a|z\|. \]
Now, for \( a = \sum_{i \in F} \phi_i e_i \) and \( A = \sum_{i \in F} \Phi_i e_i \), we have

\[ \| A - a \| z = \left\| \sum_{i \in F} (\Phi_i - \phi_i) e_i \| z \right\| = \left[ \left\{ \sum_{i \in F} (\Phi_i - \phi_i) e_i \| z \right\}^2 \right]^{\frac{1}{2}} \]

\[ = \left( \sum_{i \in F} |\Phi_i - \phi_i|^2 \| e_i \| z \right)^{\frac{1}{2}} \]

\[ = \left( \sum_{i \in F} |\Phi_i - \phi_i|^2 \right)^{\frac{1}{2}}, \]

which gives, for \( y = x \), the desired equivalence. On the other hand, we have the identity

\[ \text{Re} \left( \sum_{i \in F} \Phi_i e_i - x, x - \sum_{i \in F} \phi_i e_i \| z \right) \]

\[ = \sum_{i \in F} \text{Re} \left[ \Phi_i (x, e_i \| z) + \phi_i (x, e_i \| z \right] - \| x \| z \|^2 - \sum_{i \in F} \text{Re} (\Phi_i \phi_i), \]

which gives, from (i), that

\[ \| x \| z \|^2 + \sum_{i \in F} \text{Re} (\Phi_i \phi_i) \leq \sum_{i \in F} \text{Re} \left[ \Phi_i (x, e_i \| z) + \phi_i (x, e_i \| z) \right]. \]

(2.4)

Utilizing the elementary inequality

\[ \alpha p^2 + \frac{1}{\alpha} q^2 \geq 2pq, \quad \alpha > 0, \ p, q \geq 0, \]

we deduce

\[ 2 \| x \| z \| \leq \frac{\| x \| z \|^2}{[\sum_{i \in F} \text{Re} (\Phi_i \phi_i)]^{\frac{1}{2}}} + \left[ \sum_{i \in F} \text{Re} (\Phi_i \phi_i) \right]^{\frac{1}{2}}. \]

(2.5)

Dividing (2.4) by \( [\sum_{i \in F} \text{Re} (\Phi_i \phi_i)]^{\frac{1}{2}} > 0 \) and using (2.5), we obtain

\[ \| x \| z \| \leq \frac{1}{2} \sum_{i \in F} \text{Re} \left[ \Phi_i (x, e_i \| z) + \phi_i (x, e_i \| z) \right] \]

\[ = \frac{1}{2} \sum_{i \in F} \text{Re} \left[ (\Phi_i + \phi_i) (x, e_i \| z) \right] \]

(2.6)

\[ \left[ \sum_{i \in F} \text{Re} (\Phi_i \phi_i) \right]^{\frac{1}{2}} \]
since it is obvious that

\[
\text{Re} \left[ \Phi_i(x, e_i|z) \right] = \text{Re} \left[ \overline{\Phi}_i(x, e_i|z) \right].
\]

Note that (2.6) is also an interesting inequality in itself.

Utilizing the Cauchy-Bunyakovsky-Schwarz inequality for real numbers, we get

\[
\sum_{i \in F} \text{Re} \left[ (\overline{\Phi}_i + \phi_i)(x, e_i|z) \right] \\
\leq \sum_{i \in F} \left| (\overline{\Phi}_i + \phi_i)(x, e_i|z) \right| \\
\leq \sum_{i \in F} (|\Phi_i + \phi_i|) |(x, e_i|z)| \\
\leq \left[ \sum_{i \in F} |\Phi_i + \phi_i|^2 \right]^{\frac{1}{2}} \left[ \sum_{i \in F} |(x, e_i|z)|^2 \right]^{\frac{1}{2}}. 
\]

(2.7)

Making use of (2.6) and (2.7), we deduce the desired result (2.1).

To prove the sharpness of the constant \(\frac{1}{4}\), let us assume that (2.1) holds with a constant \(c > 0\), i.e.,

\[
\|x|z\|^2 \leq c \cdot \frac{\sum_{i \in F} |\Phi_i + \phi_i|^2}{\sum_{i \in F} \text{Re} (\Phi_i \phi_i) \sum_{i \in F} |(x, e_i|z)|^2}, 
\]

(2.8)

provided \(x, \phi_i, \Phi_i (i \in F)\) satisfy (i).

Suppose that \(F = \{1\}, e_1 = e, \|e|z\| = 1, \Phi_1 = \Phi > 0, \phi_1 = \phi > 0\). If we choose \(x = \Phi e\), then the condition (i) holds true and, by (2.8), for \(F = \{1\}\), we get

\[
\Phi^2 \leq c \cdot \frac{(\Phi + \phi)^2}{\Phi \phi} \Phi^2, 
\]

i.e., \(\Phi \phi \leq c (\Phi + \phi)^2\) for any \(\Phi, \phi > 0\). Now, if we choose \(\Phi = 1 + \varepsilon, \phi = 1 - \varepsilon\) with \(\varepsilon \in (0, 1)\) in the last inequality and make \(\varepsilon \to 0^+\), then we get \(c \geq \frac{1}{4}\) and so the proof is completed.

\[\square\]

**Remark 1.** By the use of (2.6), the second inequality in (2.7) and the Hölder inequality, we may state the following reverses of Bessel’s
inequality as well:
\[
\|x\| \leq \frac{1}{2} \cdot \frac{1}{\left[ \sum_{i \in F} \text{Re} \left( \Phi_i \bar{\phi}_i \right) \right]^{\frac{1}{2}}} \left\{ \max_{i \in F} \left\{ |\Phi_i + \phi_i| \right\} \sum_{i \in F} |(x, e_i|z)| \right. \\
\left. \times \left[ \sum_{i \in F} |\Phi_i + \phi_i|^p \right]^{\frac{1}{p}} \left( \sum_{i \in F} |(x, e_i|z)|^q \right)^{\frac{1}{q}}, \right. \\
\text{for } p > 1, \quad \frac{1}{p} + \frac{1}{q} = 1; \\
\right. \\
\left. \max_{i \in F} |(x, e_i|z)| \sum_{i \in F} |\Phi_i + \phi_i|. \right.
\]

(2.9)

The following corollary holds.

**Corollary 1.** With the assumption of Theorem 3 and, if either (i) or (ii) holds, then

\[
(2.10) \quad 0 \leq \|x\|^2 - \sum_{i \in F} |(x, e_i|z)|^2 \leq \frac{1}{4} \cdot \frac{\sum_{i \in F} |\Phi_i - \phi_i|^2}{\sum_{i \in F} \text{Re} \left( \Phi_i \bar{\phi}_i \right)} \sum_{i \in F} |(x, e_i|z)|^2.
\]

The constant $\frac{1}{4}$ is best possible.

**Proof.** The inequality (2.10) follows by (2.1) on subtracting the same quantity $\sum_{i \in F} |(x, e_i|z)|^2$ from both sides.

The best constant may be shown in a similar way to the one in the above Theorem 3 and we omit the details. \square

**Remark 2.** If $\{e_i\}_{i \in I}$ is an $z$-orthonormal family in the real 2-inner product space $(X; (\cdot, \cdot))$ and $M_i, m_i \in \mathbb{R}$, $i \in F$ ($F$ is a finite part of $I$) and $x \in X$ are such that $M_i, m_i \geq 0$ for $i \in F$ with $\sum_{i \in F} M_i m_i > 0$ and

\[
\left( \sum_{i \in F} M_i e_i - x, x - \sum_{i \in F} m_i e_i \right) \geq 0,
\]

then we have the inequality

\[
(2.11) \quad 0 \leq \|x\|^2 - \sum_{i \in F} |(x, e_i|z)|^2 \\
\leq \frac{1}{4} \cdot \frac{\sum_{i \in F} (M_i - m_i)^2}{\sum_{i \in F} M_i m_i} \cdot \sum_{i \in F} |(x, e_i|z)|^2.
\]

The constant $\frac{1}{4}$ is best possible.
The following reverse of the Schwarz's inequality in 2-inner product spaces holds.

**Corollary 2.** Let $x, y \in X$ and $\delta, \Delta \in \mathbb{K}$ ($\mathbb{K} = \mathbb{C}, \mathbb{R}$) with the property that $\text{Re}(\Delta \overline{\delta}) > 0$. If either

$$
(2.12) \quad \text{Re}(\Delta y - x, x - \delta y|z) \geq 0
$$

or, equivalently,

$$
(2.13) \quad \left\| x - \frac{\delta + \Delta}{2} \cdot y|z \right\| \leq \frac{1}{2} |\Delta - \delta| \| y|z \|
$$

holds, then we have the inequalities

$$
(2.14) \quad \| x|z \| \| y|z \| \leq \frac{1}{2} \cdot \frac{\text{Re} \left( (\Delta + \overline{\delta}) (x, y|z) \right)}{\sqrt{\text{Re}(\Delta \overline{\delta})}}
$$

$$
\quad \quad \leq \frac{1}{2} \cdot \frac{|\Delta + \delta|}{\sqrt{\text{Re}(\Delta \overline{\delta})}} |(x, y|z)|,
$$

$$
(2.15) \quad 0 \leq \| x|z \| \| y|z \| - |(x, y|z)|
$$

$$
\quad \quad \leq \frac{1}{2} \cdot \frac{|\Delta + \delta| - 2\sqrt{\text{Re}(\Delta \overline{\delta})}}{\sqrt{\text{Re}(\Delta \overline{\delta})}} |(x, y|z)|,
$$

$$
(2.16) \quad \| x|z \|^2 \| y|z \|^2 \leq \frac{1}{4} \cdot \frac{|\Delta + \delta|^2}{\text{Re}(\Delta \overline{\delta})} |(x, y|z)|^2
$$

and

$$
(2.17) \quad 0 \leq \| x|z \|^2 \| y|z \|^2 - |(x, y|z)|^2 \leq \frac{1}{4} \cdot \frac{|\Delta - \delta|^2}{\text{Re}(\Delta \overline{\delta})} |(x, y|z)|^2.
$$

The constants $\frac{1}{2}$ and $\frac{1}{4}$ are best possible.

**Proof.** The inequality (2.14) follows from (2.6) on choosing $F = \{1\}$, $e_1 = e = \frac{y}{\| y|z \|}$, $\Phi_1 = \Phi = \Delta \| y|z \|$, $\phi_1 = \phi = \delta \| y|z \|$ ($y, z$ are linearly independent). The inequality (2.15) is equivalent with (2.14). The inequality (2.16) follows from (2.1) for $F = \{1\}$ and the same choices as above. Finally, (2.17) is obviously equivalent with (2.16).
3. Some Grüss type inequalities

The following result holds.

**Theorem 4.** Let \( \{e_i\}_{i \in I} \) be a family of \( z \)-orthonormal vectors in \( X \), \( F \) a finite part of \( I \), \( \phi_{i,j}, \phi_{i,j} \in \mathbb{K} \) (\( i \in F, j = 1, 2 \)) and \( x, y \in X \). If either

\[
(3.1) \quad \text{Re} \left( \sum_{i \in F} \Phi_{i,j} e_i - x, x - \sum_{i \in F} \phi_{i,j} e_i |z \right) \geq 0
\]

or, equivalently,

\[
(3.2) \quad \left\| x - \sum_{i \in F} \frac{\Phi_{i,j} + \phi_{i,j}}{2} e_i |z \right\| \leq \frac{1}{2} \left( \sum_{i \in F} |\Phi_{i,j} - \phi_{i,j}|^2 \right)^{1/2}
\]

for \( j = 1, 2 \) hold, then we have the inequality

\[
0 \leq \left| (x, y | z) - \sum_{i \in F} (x, e_i | z) (e_i, y | z) \right|
\]

\[
(3.3) \quad \leq \frac{1}{4} \left( \frac{\sum_{i \in F} |\Phi_{i,1} - \phi_{i,1}|^2 \sum_{i \in F} |\Phi_{i,2} - \phi_{i,2}|^2}{\sum_{i \in F} \text{Re} (\Phi_{i,1} \phi_{i,1}) \sum_{i \in F} \text{Re} (\Phi_{i,2} \phi_{i,2})} \right)^{1/2}
\]

\[
\times \left( \sum_{i \in F} |(x, e_i | z)|^2 \right)^{1/2} \left( \sum_{i \in F} |(y, e_i | z)|^2 \right)^{1/2}
\]

The constant \( \frac{1}{4} \) is best possible.

**Proof.** If we use Schwarz's inequality in 2-inner product space \( (X, \langle \cdot, \cdot \rangle) \), one has

\[
(3.4) \quad \left\| \left( x - \sum_{i \in F} (x, e_i | z) e_i, y - \sum_{i \in F} (y, e_i | z) e_i | z \right) \right\|^2
\]

\[
\leq \left\| x - \sum_{i \in F} (x, e_i | z) e_i | z \right\|^2 \left\| y - \sum_{i \in F} (y, e_i | z) e_i | z \right\|^2
\]

and, since a simple calculation shows that

\[
(x - \sum_{i \in F} (x, e_i | z) e_i, y - \sum_{i \in F} (y, e_i | z) e_i | z) = (x, y | z) - \sum_{i \in F} (x, e_i | z) (e_i, y | z)
\]
and
\[ \left\| x - \sum_{i \in F} (x, e_i | z) e_i | z \right\|^2 = \| x | z \|^2 - \sum_{i \in F} |(x, e_i | z)|^2 \]
for any \( x, y \in X \), by (3.4) and by the reverse of Bessel's inequality in Corollary 1, we have
\[ \left| (x, y | z) - \sum_{i \in F} (x, e_i | z) (e_i, y | z) \right|^2 \]
\[ \leq \left( \| x | z \|^2 - \sum_{i \in F} |(x, e_i | z)|^2 \right) \left( \| y | z \|^2 - \sum_{i \in F} |(y, e_i | z)|^2 \right) \]
(3.5)
\[ \leq \frac{1}{4} \sum_{i,j \in F} |\Phi_{i,j} - \Phi_{i,j}|^2 \sum_{i \in F} |(x, e_i | z)|^2 \]
\[ \times \sum_{i \in F} |\Phi_{i,j} - \Phi_{i,j}|^2 \sum_{i \in F} |(y, e_i | z)|^2 . \]

Taking the square root in (3.5), we deduce (3.3).

The fact that \( \frac{1}{4} \) is the best possible constant follows by Corollary 1 and we omit the details.

The following corollary for real 2-inner product spaces holds.

**Corollary 3.** Let \( \{e_i\}_{i \in I} \) be a family of \( z \)-orthonormal vectors in \( X, F \) a finite part of \( I \), \( M_{i,j}, m_{i,j} \geq 0 \) \((i \in F, j = 1, 2)\) and \( x, y \in X \) such that \( \sum_{i \in F} M_{i,j} m_{i,j} > 0 \) \((j = 1, 2)\) and
\[ \left( \sum_{i \in F} M_{i,j} e_i - x, x - \sum_{i \in F} m_{i,j} e_i | z \right) \geq 0 . \]
(3.6)

Then we have the inequality
\[ 0 \leq \left| (x, y | z) - \sum_{i \in F} (x, e_i | z) (y, e_i | z) \right|^2 \]
\[ \leq \frac{1}{16} \cdot \sum_{i \in F} (M_{i,1} - m_{i,1})^2 \sum_{i \in F} (M_{i,2} - m_{i,2})^2 \]
\[ \times \sum_{i \in F} |(x, e_i | z)|^2 \sum_{i \in F} |(y, e_i | z)|^2 . \]
(3.7)
The constant $\frac{1}{16}$ is best possible.

In the case where the family $\{e_i\}_{i \in I}$ reduces to a single vector, we may deduce from Theorem 4 the following particular case:

**Corollary 4.** Let $e \in X$, $\|e\| = 1$, $\phi_j, \Phi_j \in K$ with $\text{Re}(\Phi_j \overline{\phi_j}) > 0$ $(j = 1, 2)$ and $x, y \in X$ such that either

$$\text{Re}(\Phi_j e - x, x - \phi_j e|z|) \geq 0$$

or, equivalently,

$$\left\|x - \frac{\phi_j + \Phi_j}{2} e|z|\right\| \leq \frac{1}{2} |\Phi_j - \phi_j|$$

holds, then

$$0 \leq |(x, y|z) - (x, e|z) (e, y|z)|$$

$$\leq \frac{1}{4} \frac{|\Phi_1 - \phi_1|}{\sqrt{\text{Re}(\Phi_1 \overline{\phi_1})}} \frac{|\Phi_2 - \phi_2|}{\sqrt{\text{Re}(\Phi_2 \overline{\phi_2})}} |(x, e|z) (e, y|z)|.$$  

The constant $\frac{1}{4}$ is best possible.

**Remark 3.** If $X$ is real, $e \in X$, $\|e|z\| = 1$ and $a, b, A, B \in \mathbb{R}$ are such that $A > a > 0$, $B > b > 0$ and

$$\left\|x - \frac{a + A}{2} e|z|\right\| \leq \frac{1}{2} (A - a), \quad \left\|y - \frac{b + B}{2} e|z|\right\| \leq \frac{1}{2} (B - b),$$

then

$$| (x, y|z) - (x, e|z) (e, y|z) |$$

$$\leq \frac{1}{4} \frac{(A - a)(B - b)}{\sqrt{abAB}} |(x, e|z) (e, y|z)|.$$  

The constant $\frac{1}{4}$ is best possible.

If $(x, e|z), (y, e|z) \neq 0$, then the following equivalent form of (3.12) also holds

$$\left| \frac{(x, y|z)}{(x, e|z) (e, y|z)} - 1 \right| \leq \frac{1}{4} \frac{(A - a)(B - b)}{\sqrt{abAB}}.$$  

4. Some companion inequalities

The following companion of the Grüss inequality also holds.
Theorem 5. Let \( \{e_i\}_{i \in I} \) be a family of \( z \)-orthonormal vectors in \( X \), \( F \) a finite part of \( I \), \( \phi_i, \Phi_i \in \mathbb{K} \) \((i \in F)\), \( x, y \in X \) and \( \lambda \in (0, 1) \) such that either

\[
\text{Re} \left( \sum_{i \in F} \Phi_i e_i - (\lambda x + (1 - \lambda) y), \lambda x + (1 - \lambda) y - \sum_{i \in F} \phi_i e_i |z\right) \\
\geq 0
\]

or, equivalently,

\[
\left\| \lambda x + (1 - \lambda) y - \sum_{i \in F} \Phi_i + 2 \phi_i \cdot e_i |z\right\| \leq \frac{1}{2} \left( \sum_{i \in F} |\Phi_i - \phi_i|^2 \right)^{\frac{1}{2}}
\]

holds. Then we have the inequality

\[
\text{Re} \left[ (x, y|z) - \sum_{i \in F} (x, e_i|z)(e_i, y|z) \right] \\
\leq \frac{1}{16} \cdot \frac{1}{\lambda (1 - \lambda) \sum_{i \in F} \text{Re} (\Phi_i \Phi_i)} \sum_{i \in F} (|\lambda x + (1 - \lambda) y, e_i|z)|^2.
\]

The constant \( \frac{1}{16} \) is the best possible constant in (4.3) in the sense that it cannot be replaced by a smaller constant.

Proof. Using the known inequality

\[
\text{Re} (z, u|v) \leq \frac{1}{4} \|z + u|v\|^2,
\]

we may state, for any \( a, b \in X \) and \( \lambda \in (0, 1) \), that

\[
\text{Re} (a, b|z) \leq \frac{1}{4\lambda (1 - \lambda)} \|\lambda a + (1 - \lambda) b|z\|^2.
\]

Since

\[
(x, y|z) - \sum_{i \in F} (x, e_i|z)(e_i, y|z) = \left( x - \sum_{i \in F} (x, e_i|z), y - \sum_{i \in F} (y, e_i|z) e_i |z\right)
\]

for any \( x, y \in X \), by (4.4), we get

\[
\text{Re} \left[ (x, y|z) - \sum_{i \in F} (x, e_i|z)(e_i, y|z) \right] \\
= \text{Re} \left[ \left( x - \sum_{i \in F} (x, e_i|z) e_i, y - \sum_{i \in F} (y, e_i|z) e_i |z\right) \right]
\]
\[
\begin{align*}
\leq & \frac{1}{4\lambda (1 - \lambda)} \left\| \lambda \left( x - \sum_{i \in F} (x, e_i | z) e_i \right) \right. \\
& + (1 - \lambda) \left( y - \sum_{i \in F} (y, e_i | z) e_i \right) \left\| z \right. \\
= & \frac{1}{4\lambda (1 - \lambda)} \left\| \lambda x + (1 - \lambda) y - \sum_{i \in F} (\lambda x + (1 - \lambda) y, e_i | z) e_i | z \right. \\
= & \frac{1}{4\lambda (1 - \lambda)} \left[ \left\| \lambda x + (1 - \lambda) y | z \right. \right. \\
\left. \left. \right\|^2 - \sum_{i \in F} |(\lambda x + (1 - \lambda) y, e_i | z)|^2 \right].
\end{align*}
\]

If we apply the reverse of Bessel's inequality from Corollary 1 for \(\lambda x + (1 - \lambda) y\), we may state that

\[
\left\| \lambda x + (1 - \lambda) y | z \right. \right. \\
\left. \left. \right\|^2 - \sum_{i \in F} |(\lambda x + (1 - \lambda) y, e_i | z)|^2 \right.
\]

(4.6)
\[
\leq \frac{1}{4} \sum_{i \in F} |\Phi_i - \phi_i|^2 \sum_{i \in F} |(\lambda x + (1 - \lambda) y, e_i | z)|^2.
\]

Now, by making use of (4.5) and (4.6), we deduce (4.3).

The fact that \(\frac{1}{16}\) is the best possible constant in (4.3) follows by the fact that, if in (4.1) we choose \(x = y\), then it becomes (i) of Theorem 3, implying for \(\lambda = \frac{1}{2}\) the inequality (2.10), for which we have shown that \(\frac{1}{4}\) is the best constant.

\[
\begin{proof}
\end{proof}
\]

Remark 4. If, in Theorem 5, we choose \(\lambda = \frac{1}{2}\), then we get

\[
\begin{align*}
\text{Re} \left[ (x, y | z) - \sum_{i \in F} (x, e_i | z) (e_i, y | z) \right] \\
\leq & \frac{1}{4} \sum_{i \in F} |\Phi_i - \phi_i|^2 \sum_{i \in F} \left| \left( \frac{x + y}{2}, e_i | z \right) \right|^2
\end{align*}
\]

(4.7)

provided

\[
\text{Re} \left( \sum_{i \in F} \Phi_i e_i - \frac{x + y}{2}, \frac{x + y}{2} - \sum_{i \in F} \phi_i e_i | z \right) \geq 0
\]
or, equivalently,

\[
\left| \frac{x + y}{2} - \sum_{i \in F} \frac{\Phi_i + \phi_i}{2} e_i | z \right| \leq \frac{1}{2} \left( \sum_{i \in F} |\Phi_i - \phi_i|^2 \right)^{\frac{1}{2}}.
\]

(4.8)
5. Applications for determinantal integral inequalities

Let \((\Omega, \Sigma, \mu)\) be a measure space consisting of a set \(\Omega\), \(\Sigma\) a \(\sigma\)-algebra of subsets of \(\Omega\) and \(\mu\) a countably additive and positive measure on \(\Sigma\) with values in \(\mathbb{R} \cup \{ \infty \}\).

Denote by \(L^2_\rho(\Omega)\) the Hilbert space of all real-valued functions \(f\) defined on \(\Omega\) that are \(2-\rho\)-integrable on \(\Omega\), i.e., \(\int_\Omega \rho(s) |f(s)|^2 \, d\mu(s) < \infty\), where \(\rho : \Omega \to [0, \infty)\) is a measurable function on \(\Omega\).

We can introduce the following 2-inner product on \(L^2_\rho(\Omega)\) by formula

\[
(5.1) \quad (f, g|h)_\rho := \frac{1}{2} \int_\Omega \int_\Omega \rho(s)\rho(t) \begin{vmatrix} f(s) & f(t) \\ h(s) & h(t) \end{vmatrix} \begin{vmatrix} g(s) & g(t) \\ h(s) & h(t) \end{vmatrix} d\mu(s)d\mu(t),
\]

where

\[
\begin{vmatrix} f(s) & f(t) \\ h(s) & h(t) \end{vmatrix}
\]

denotes the determinant of the matrix

\[
\begin{bmatrix} f(s) & f(t) \\ h(s) & h(t) \end{bmatrix},
\]

which generates the 2-norm on \(L^2_\rho(\Omega)\) expressed by

\[
(5.2) \quad \| f|h \|_\rho := \left( \frac{1}{2} \int_\Omega \int_\Omega \rho(s)\rho(t) \begin{vmatrix} f(s) & f(t) \\ h(s) & h(t) \end{vmatrix}^2 d\mu(s)d\mu(t) \right)^{1/2}.
\]

A simple calculation with integrals reveals that

\[
(5.3) \quad (f, g|h)_\rho = \begin{vmatrix} \int_\Omega \rho fg \, d\mu & \int_\Omega \rho fh \, d\mu \\ \int_\Omega \rho gh \, d\mu & \int_\Omega \rho h^2 \, d\mu \end{vmatrix}
\]

and

\[
(5.4) \quad \| f|h \|_\rho = \left( \int_\Omega \rho f^2 \, d\mu \int_\Omega \rho fh \, d\mu \int_\Omega \rho h^2 \, d\mu \right)^{1/2},
\]

where, for simplicity, instead of \(\int_\Omega \rho(s) f(s)g(s)d\mu(s)\), we have written \(\int_\Omega \rho fgd\mu\).

We recall that the pair of functions \((q, p) \in L^2_\rho(\Omega) \times L^2_\rho(\Omega)\) is called synchronous if

\[
(q(x) - q(y))(p(x) - p(y)) \geq 0
\]

for a.e. \(x, y \in \Omega\).
We note that, if $\Omega = [a, b]$, then a sufficient condition for synchronicity is that the functions are both monotonic increasing or decreasing. This condition is not necessary.

Now, suppose that $h \in L^2_\rho(\Omega)$ is such that $h(x) \neq 0$ for $\mu$ - a.e. $x \in \Omega$. Then, by the definition of 2-inner product $(f, g|h)_\rho$, we have

$$
(f, g|h)_\rho = \frac{1}{2} \int_\Omega \int_\Omega \rho(s)\rho(t)h^2(s)h^2(t)\left(\frac{f(s)}{h(s)} - \frac{f(t)}{h(t)}\right)\left(\frac{g(s)}{h(s)} - \frac{g(t)}{h(t)}\right) d\mu(s)d\mu(t)
$$

and thus a sufficient condition for the inequality

$$
(f, g|h)_\rho \geq 0
$$

to hold, that is, the functions $(f/h, g/h)$ are synchronous. It is obvious that this condition is not necessary.

Using the representations (5.3), (5.4) and the inequalities for 2-inner products and 2-norms established in the previous sections, we have some interesting determinantal integral inequalities.

**Proposition 1.** Let $h \in L^2_\rho(\Omega)$ be such that $h(x) \neq 0$ for $\mu$ - a.e. $x \in \Omega$ and $(f_i)_{i \in I}$ a family of functions in $L^2_\rho(\Omega)$ with the property that

$$
\begin{vmatrix}
\int_\Omega \rho f_i f_j d\mu & \int_\Omega \rho f_i h d\mu \\
\int_\Omega \rho f_j h d\mu & \int_\Omega \rho h^2 d\mu
\end{vmatrix} = \delta_{i,j}
$$

for any $i, j \in I$, where $\delta_{i,j}$ is the Kronecker delta.

If we assume that there exists the real numbers $M_i, m_i$ $(i \in F)$ with $\sum_{i \in F} M_i m_i > 0$, where $F$ is a given finite part of $I$, such that the functions

$$
\sum_{i \in F} M_i \cdot \frac{f_i}{h} - \frac{f}{h}, \quad \sum_{i \in F} m_i \cdot \frac{f_i}{h}
$$

are synchronous on $\Omega$, then we have the inequalities

$$
\begin{align*}
&\left|\int_\Omega \rho f^2 d\mu - \int_\Omega \rho f h d\mu\right| \leq \frac{1}{4} \sum_{i \in F} (M_i + m_i)^2 \sum_{i \in F} \left|\int_\Omega \rho f_i f d\mu - \int_\Omega \rho f_i h d\mu\right|^2 \\
&\left|\int_\Omega \rho f h d\mu - \int_\Omega \rho h^2 d\mu\right| \leq \frac{1}{4} \sum_{i \in F} M_i m_i \sum_{i \in F} \left|\int_\Omega \rho f h d\mu - \int_\Omega \rho h^2 d\mu\right|
\end{align*}
$$
and

\[
0 \leq \left| \int_{\Omega} \rho f^2 d\mu - \int_{\Omega} \rho h d\mu \right| \leq \sum_{i \in F} \left( \int_{\Omega} \rho f_i f d\mu - \int_{\Omega} \rho h d\mu \right)^2 \sum_{i \in F} \left( \int_{\Omega} \rho f_i h d\mu - \int_{\Omega} \rho h d\mu \right)^2 \sum_{i \in F} \left( \int_{\Omega} \rho f_i h d\mu - \int_{\Omega} \rho h d\mu \right)^2
\]

The constant \(\frac{1}{4}\) is best possible in both inequalities.

The proof follows by Theorem 3 and Corollary 1 applied for the 2-inner product \((\cdot, \cdot)_\rho\) and we omit the details.

Similar determinantal integral inequalities may be stated if one uses the other results for 2-inner products obtained above, but we do not present them here.

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**References**


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