INCLUSION RELATIONS FOR $k$-UNIFORMLY STARLIKE AND RELATED FUNCTIONS UNDER CERTAIN INTEGRAL OPERATORS

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ABSTRACT. Inclusion relations under certain integral operators are proved for $k$-uniformly starlike functions. These results are also extended to $k$-uniformly convex, close-to-convex, and quasi-convex functions.

1. Introduction

Let $A$ denote the class of functions of the form $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ which are analytic in the open unit disc $U = \{z : |z| < 1\}$. A function $f \in A$ is said to be in $UST(k, \gamma)$, the class of $k$-uniformly starlike functions of order $\gamma$, $0 \leq \gamma < 1$, if $f$ satisfies the condition

\[
\Re\left(\frac{zf'(z)}{f(z)}\right) > k \left| \frac{zf''(z)}{f'(z)} - 1 \right| + \gamma, \quad k \geq 0.
\]

Replacing $f$ in (1.1) by $zf'$ we obtain the condition

\[
\Re\left\{1 + \frac{zf''(z)}{f'(z)}\right\} > k \left| \frac{zf''(z)}{f'(z)} \right| + \gamma, \quad k \geq 0
\]

required for function $f$ to be in the subclass $UCV(k, \gamma)$ of $k$-uniformly convex functions of order $\gamma$. Uniformly starlike and convex functions were first introduced by Goodman[2] and then studied by various authors. For a wealth of reference, see Ronning[5]. Setting $\Omega_{k, \gamma} = \{u + iv; u > k\sqrt{(u - 1)^2 + v^2} + \gamma\}$, with $p(z) = \frac{zf'(z)}{f(z)}$ or $p(z) = 1 + \frac{zf''(z)}{f'(z)}$ and considering the functions which maps $U$ on to the conic domain.
\( \Omega_{k, \gamma} \), such that \( 1 \in \Omega_{k, \gamma} \), we may rewrite the conditions (1.1) or (1.2) in the form

(1.3) \quad p(z) < q_{k, \gamma}(z).

Note that the explicit forms of function \( q_{k, \gamma} \) for \( k = 0 \) and \( k = 1 \) are

\[
q_{0, \gamma}(z) = \frac{1 + (1 - 2\gamma)z}{1 - z}, \quad \text{and} \quad q_{1, \gamma}(z) = 1 + \frac{2(1 - \gamma)}{\pi^2} \left( \frac{\log \frac{1 + \sqrt{z}}{1 - \sqrt{z}}}{1 - \sqrt{z}} \right)^2.
\]

For \( 0 < k < 1 \) we obtain

\[
q_{k, \gamma}(z) = \frac{1 - \gamma}{1 - k^2} \cos \left( \frac{2}{\pi} (\cos^{-1} k) i \log \frac{1 + \sqrt{z}}{1 - \sqrt{z}} \right) - \frac{k^2 - \gamma}{1 - k^2},
\]

and if \( k > 1 \), then \( q_{k, \gamma} \) has the form

\[
q_{k, \gamma}(z) = \frac{1 - \gamma}{k^2 - 1} \sin \left( \frac{\pi}{2K(k)} \int_0^{\tfrac{u(z)}{\sqrt{k}}} \frac{dt}{\sqrt{1 - t^2/1 - k^2 t^2}} \right) + \frac{k^2 - \gamma}{k^2 - 1},
\]

where \( u(z) = \frac{z - \sqrt{k}}{1 - \sqrt{k} z} \) and \( K \) is so that \( k = \cosh \frac{\pi K'(z)}{4K(z)} \).

By virtue of (1.3) and the properties of the domains \( \Omega_{k, \gamma} \) we have

(1.4) \quad \Re(p(z)) > \Re(q_{k, \gamma}(z)) > \frac{k + \gamma}{k + 1}.

Define \( UCC(k, \gamma, \beta) \) to be the family of functions \( f \in A \) so that

\[
\Re \left( \frac{zf'(z)}{g(z)} \right) \geq k \left| \frac{zf'(z)}{g(z)} - 1 \right| + \gamma, \quad k \geq 0, \ 0 \leq \gamma < 1
\]

for some \( g \in UST(k, \beta) \).

Similarly, we define \( UQC(k, \gamma, \beta) \) to be the family of function \( f \in A \) so that

\[
\Re \left( \frac{(zf'(z))'}{g'(z)} \right) \geq k \left| \frac{(zf'(z))'}{g'(z)} - 1 \right| + \gamma, \quad k \geq 0, \ 0 \leq \gamma < 1
\]

for some \( g \in UCV(k, \beta) \).
If \( k = 0 \) then \( UCC(0, \gamma, \beta) \) is the class of close-to-convex functions of order \( \gamma \) and type \( \beta \) and \( UQC(0, \gamma, \beta) \) is the class of quasi-convex functions of order \( \gamma \) and type \( \beta \).

The aim of this note is to study the inclusion properties of the above mentioned classes of functions under the following one-parameter family of integral operator (see Jung, Kim, and Srivastava[3])

\[
I^\alpha = I^\alpha f(z) = \frac{2^\alpha}{z \Gamma(\alpha)} \int_0^z (\log \frac{z}{t})^{\alpha-1} f(t) dt, \quad \alpha > 0,
\]

and the generalized Bernardi-Libera-Livingston integral operator

\[
L_c(f) = L_c(f(z)) = \frac{c+1}{z^c} \int_0^z t^{c-1} f(t) dt, \quad c > -1.
\]

2. Main results

First we state and prove an inclusion theorem for \( UST(k, \gamma) \) under \( I^\alpha \).

**Theorem 1.** If \( I^\alpha \in UST(k, \gamma) \) then \( I^{\alpha+1} \in UST(k, \gamma) \).

In order to prove the above theorem we shall need the following lemma which is due to Eenigenburg, Miller, Mocanu, and Read[1].

**Lemma A.** Let \( \beta, \gamma \) be complex constants and \( h \) be univalently convex in the unit disk \( U \) with \( h(0) = c \) and \( \Re(\beta h(z) + \gamma) > 0 \). Let \( g(z) = c + \sum_{n=1}^{\infty} p_n z^n \) be analytic in \( U \). Then

\[
g(z) + \frac{zg'(z)}{\beta g(z) + \gamma} < h(z) \Rightarrow g(z) < h(z).
\]

**Proof of Theorem 1.** Since \( I^\alpha \in UST(k, \gamma) \), by definition, we have

\[
z(I^{\alpha+1} f(z))' = 2I^\alpha f(z) - I^{\alpha+1} f(z).
\]

Setting \( p(z) = z(I^{\alpha+1} f(z))/(I^{\alpha+1} f(z)) \) in (2.1) we can write

\[
\frac{I^\alpha f(z)}{I^{\alpha+1} f(z)} = \frac{1}{2} \left( \frac{z(I^{\alpha+1} f(z))'}{I^{\alpha+1} f(z)} + 1 \right) = \frac{1}{2}(p(z) + 1).
\]

Differentiating (2.2) yields

\[
\frac{z(I^\alpha f(z))'}{I^\alpha f(z)} = \frac{z(I^{\alpha+1} f(z))'}{I^{\alpha+1} f(z)} + \frac{zp'(z)}{p(z) + 1} = p(z) + \frac{zp'(z)}{p(z) + 1}.
\]
From this and the argument given in Section 1 we may write

\[ p(z) + \frac{zp'(z)}{p(z) + 1} < q_{k, \gamma}(z). \]

Therefore the theorem follows by Lemma A and the condition (1.4) since \( q_{k, \gamma} \) is univalent and convex in \( U \) and \( \Re(q_{k, \gamma}) > \frac{k + \gamma}{k + 1} \).

Using a similar argument we can prove

**Theorem 2.** If \( I^{\alpha} \in UCV(k, \gamma) \) then \( I^{\alpha+1} \in UCV(k, \gamma) \).

We next prove

**Theorem 3.** If \( I^{\alpha} \in UCC(k, \gamma, \beta) \) then \( I^{\alpha+1} \in UCC(k, \gamma, \beta) \).

We shall need the following lemma which is due to Miller and Mocanu[4].

**Lemma B.** Let \( h \) be convex in the unit disk \( U \) and let \( A \geq 0 \). Suppose \( B(z) \) is analytic in \( U \) with \( \Re(B(z)) \geq A \). If \( g \) is analytic in \( U \) and \( g(0) = h(0) \). Then

\[ \begin{align*}
Ax^2g''(z) + B(z)zg'(z) + g(z) &< h(z) \quad \Rightarrow \quad g(z) < h(z).
\end{align*} \]

**Proof of Theorem 3.** Since \( I^{\alpha} \in UCC(k, \gamma, \beta) \), by definition, we can write

\[ \frac{z(I^{\alpha}f(z))'}{k(z)} < q_{k, \gamma}(z) \]

for some \( k(z) \in UST(k, \beta) \). For \( g \) so that \( I^{\alpha}g(z) = k(z) \), we have

\[ \frac{z(I^{\alpha}f(z))'}{I^{\alpha}g(z)} < q_{k, \gamma}(z). \]

Letting \( h(z) = \frac{z(I^{\alpha+1}f(z))'}{I^{\alpha+1}g(z)} \) and \( H(z) = \frac{z(I^{\alpha+1}g(z))'}{I^{\alpha+1}g(z)} \) we observe that \( h \) and \( H \) are analytic in \( U \) and \( h(0) = H(0) = 1 \). Now, by Theorem 1, \( I^{\alpha+1}g(z) \in UST(k, \beta) \) and so \( \Re(H(z)) > \frac{k + \beta}{k + 1} \). Also, note that

\[ z(I^{\alpha+1}f(z))' = (I^{\alpha+1}g(z))h(z). \]

Differentiating both sides of (2.5) yields

\[ \frac{z(I^{\alpha+1}(zf'(z)))'}{I^{\alpha+1}g(z)} = \frac{z(I^{\alpha+1}g(z))'}{I^{\alpha+1}g(z)}h(z) + zh'(z) = H(z)h(z) + zh'(z). \]
Now using the identity (2.1) we obtain
\[
\frac{z(I^\alpha f(z))'}{I^\alpha g(z)} = \frac{I^\alpha(z f'(z))}{I^\alpha g(z)}
\]
\[
= \frac{z(I^{\alpha+1}(z f'(z)))' + I^{\alpha+1}(z f'(z))}{z(I^{\alpha+1}g(z))' + I^{\alpha+1}g(z)}
\]
\[
= \frac{z(I^{\alpha+1}(z f'(z)))' + I^{\alpha+1}(z f'(z))}{I^{\alpha+1}g(z)} + 1
\]
\[
= \frac{H(z)h(z) + zh'(z) + h(z)}{H(z) + 1}
\]
\[
= h(z) + \frac{1}{H(z) + 1}zh'(z).
\]

From (2.4), (2.5), and (2.6) we conclude that
\[
h(z) + \frac{1}{H(z) + 1}zh'(z) < q_{k,\gamma}(z).
\]

For letting \( A = 0 \) and \( B(z) = \frac{1}{H(z) + 1} \), we obtain
\[
\Re(B(z)) = \frac{1}{|1 + H(z)|^2} \Re(1 + H(z)) > 0.
\]

The above inequality satisfies the conditions required by Lemma B. Hence \( h(z) < q_{k,\gamma}(z) \) and so the proof is complete.

Using a similar argument we can prove

**Theorem 4.** If \( I^\alpha \in UQC(k, \gamma, \beta) \) then \( I^{\alpha+1} \in UQC(k, \gamma, \beta) \).

Now we examine the closure properties of the integral operator \( L_c \).

**Theorem 5.** Let \( c > \frac{-k}{k+1} \). If \( I^\alpha \in UST(k, \gamma) \) so is \( L_c(I^\alpha) \).

**Proof.** From definition of \( L_c(f) \) and the linearity of operator \( I^\alpha \) we have
\[
z(I^\alpha L_c(f))' = (c + 1)I^\alpha f(z) - cI^\alpha L_c(f).
\]

Substituting \( \frac{z(I^\alpha L_c(f))'}{I^\alpha L_c(f)} = p(z) \) in (2.7) we may write
\[
(2.8) \quad p(z) = (c + 1) \frac{I^\alpha f(z)}{I^\alpha L_c(f)} - c.
\]
Differentiating (2.8) gives
\[
\frac{z(I^\alpha f(z))'}{I^\alpha f(z)} = \frac{z(I^\alpha L_c(f))'}{I^\alpha L_c(f)} + \frac{zp'(z)}{p(z) + c} = p(z) + \frac{zp'(z)}{p(z) + c}.
\]
Therefore, the theorem follows by Lemma A, since \(\Re(q_{k,\gamma}(z) + c) > 0\).

A similar argument leads to

**Theorem 6.** Let \(c > \frac{(k+\gamma)}{k+1}\). If \(I^\alpha \in UCV(k, \gamma)\) so is \(L_c(I^\alpha)\).

**Theorem 7.** Let \(c > \frac{(k+\gamma)}{k+1}\). If \(I^\alpha \in UCC(k, \gamma, \beta)\) so is \(L_c(I^\alpha)\).

**Proof.** By definition, there exists a function
\[
k(z) = I^\alpha g(z) \in UST(k, \beta)
\]
so that
\[
z(I^\alpha f(z))' \prec q_{k,\gamma}(z) \quad (z \in U).
\]

(2.9)

Now from (2.7) we have
\[
\frac{z(I^\alpha f')}{I^\alpha g} = \frac{z(I^\alpha L_c(z f')) + cI^\alpha L_c(z f')}{z(I^\alpha L_c(g(z)))' + cI^\alpha L_c(g(z))}
\]
\[
= \frac{z(I^\alpha L_c(g(z)))'}{I^\alpha L_c(g(z))} \prec q_{k,\gamma}(z) \quad (z \in U).
\]

(2.10)

Since \(I^\alpha g \in UST(k, \beta)\), by Theorem(5), we have \(L_c(I^\alpha g) \in UST(k, \beta)\).

Letting \(\frac{z(I^\alpha L_c(g(z)))'}{I^\alpha L_c(g(z))} = H(z)\), we note that \(\Re(H(z)) > \frac{k+\beta}{k+1}\). Now for \(h(z) = \frac{z(I^\alpha L_c(f(z)))'}{I^\alpha L_c(g(z))}\) we obtain
\[
z(I^\alpha L_c(f(z)))' = h(z)I^\alpha L_c(g(z)).
\]

(2.11)

Differentiating both sides of (2.11) yields
\[
\frac{z(I^\alpha zL_c(f))'}{I^\alpha L_c(g)} = zh'(z) + h(z)\frac{z(I^\alpha L_c(g))'}{I^\alpha L_c(g)}
\]
\[
= zh'(z) + H(z)h(z).
\]

(2.12)
Therefore from (2.10) and (2.12) we obtain
\[
\frac{z(I^\alpha f(z))'}{I^\alpha g} = \frac{zh'(z) + h(z)H(z) + ch(z)}{H(z) + c}.
\]
This in conjunction with (2.9) leads to
\[
(2.13) \quad h(z) + \frac{zh'(z)}{H(z) + c} < q_{k,\gamma}(z).
\]
Letting \( B(z) = \frac{1}{H(z) + c} \) in (2.13) we note that \( \Re(B(z)) > 0 \) if \( c > -\frac{k+\beta}{k+1} \).
Now for \( A = 0 \) and \( B \) as described we conclude the proof since the required conditions of Lemma B are satisfied.
A similar argument yields

**Theorem 8.** Let \( c > -\frac{(k+\gamma)}{k+1} \). If \( I^\alpha \in UQC(k, \gamma, \beta) \), so is \( L_c(I^\alpha) \).

**References**


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