ON THE STABILITY OF A JENSEN TYPE
FUNCTIONAL EQUATION ON GROUPS

VALERIĬ A. FAĬZIEV AND PRASANNA K. SAHOO

ABSTRACT. In this paper we establish the stability of a Jensen type
functional equation, namely \( f(xy) - f(xy^{-1}) = 2f(y) \), on some
classes of groups. We prove that any group \( A \) can be embedded
into some group \( G \) such that the Jensen type functional equation is
stable on \( G \). We also prove that the Jensen type functional equation
is stable on any metabelian group, \( GL(n, \mathbb{C}) \), \( SL(n, \mathbb{C}) \), and \( T(n, \mathbb{C}) \).

1. Introduction

Given an operator \( T \) and a solution class \( \{u\} \) with the property that
\( T(u) = 0 \), when does \( \|T(v)\| \leq \varepsilon \) for an \( \varepsilon > 0 \) imply that \( \|u - v\| \leq \delta(\varepsilon) \)
for some \( u \) and for some \( \delta > 0 \)? This problem is called the stability
of the functional transformation [28]. It happened in 1940 that the
audience of the Mathematics Club of the University of Wisconsin had the
pleasure to listen to the talk of S.M. Ulam presenting a list of unsolved
problems. One of these problems can be considered as the starting point
of a new line of investigation: The stability problem. This problem can
be formulated as follows. If we replace a given functional equation by
a functional inequality, then under what conditions we can state that
the solutions of the inequality are close to the solutions of the equation.
For instance, given a group \( G_1 \), a metric group \( (G_2, d) \) and a positive
number \( \varepsilon \). The Ulam question is: does there exist a \( \delta > 0 \) such that
if \( f : G_1 \to G_2 \) satisfies \( d(f(xy), f(x)f(y)) < \delta \) for all \( x, y \in G_1 \), then
a homomorphism \( T : G_1 \to G_2 \) exists with \( d(f(x), T(x)) < \varepsilon \) for all
\( x, y \in G_1 \)?
In the case of a positive answer to this problem, we say that the homomorphisms $G_1 \to G_2$ are stable or that the Cauchy functional equation

$$f(x \cdot y) = f(x) \ast f(y)$$

is stable for the pair $(G_1, G_2)$.

See S. M. Ulam[27] for a discussion of such problems, as well as D. H. Hyers[11, 12], D. H. Hyers and S. M. Ulam[16, 17], J. Aczél and J. Dhombres[1]. The first affirmative answer was given by D. H. Hyers[11] in 1941. We present his result in theorem below.

**Theorem 1.1. (Hyers[11])** Let $E_1$ and $E_2$ be Banach spaces. If $f : E_1 \to E_2$ satisfies the inequality

$$(1.1) \quad \| f(x + y) - f(x) - f(y) \| < \varepsilon$$

for some $\varepsilon > 0$ and for all $x, y \in E_1$, then there exists a unique map $T : E_1 \to E_2$ such that

$$(1.2) \quad T(x + y) - T(x) - T(y) = 0 \text{ for all } x, y \in E_1$$

and

$$(1.3) \quad \| f(x) - T(x) \| < \varepsilon \text{ for all } x \in E_1.$$
when the domain of the function is replaced by an arbitrary group. The equation (1.5) was studied in the papers [2], [4] and [23]. The question of stability of equation (1.5) was investigated in [18]–[21] and [26]. In all these papers domain of $f$ is either an abelian group or some of its subsets. In [9], the present authors studied the stability of the equation (1.5) on arbitrary groups.

In the paper [24], the stability of the following Jensen type functional equation

$$2f\left(\frac{x-y}{2}\right) = f(x) - f(y)$$

(1.6)

was considered. Here again $f : \mathbb{R} \to \mathbb{R}$. Setting $\frac{1}{2}(x+y) = u$ and $\frac{1}{2}(x-y) = v$ we can rewrite the equation (1.6) as

$$2f(v) = f(u+v) - f(u-v).$$

The latter is equivalent to

$$f(xy) - f(xy^{-1}) = 2f(y)$$

(1.7)

and can be considered over an arbitrary group.

In the paper [24], the stability of equation (1.7) over a real normed space was considered. In the present paper we consider the stability of the Jensen type functional equation (1.7) over an arbitrary group.

2. Auxiliary results

Suppose that $G$ is an arbitrary group and $E$ is an arbitrary real Banach space.

**Definition 2.1.** We will say that a function $f : G \to E$ is a $(G;E)$-Jensen type function if for any $x, y \in G$ we have

$$f(xy) - f(xy^{-1}) - 2f(y) = 0.$$  

(2.1)

We denote the set of all $(G;E)$-Jensen type functions by $JT(G;E)$.

**Definition 2.2.** We will say that a function $f : G \to E$ is a $(G;E)$-quasisjensen type function if there is a $c > 0$ such that for any $x, y \in G$ we have

$$\|f(xy) - f(xy^{-1}) - 2f(y)\| \leq c.$$  

(2.2)
It is clear that the set of \((G; E)\)-quasijensen type functions is a real linear space. Denote it by \(KJT(G; E)\). From (2.2) we obtain
\[
\|f(y) - f(y^{-1}) - 2f(y)\| \leq c,
\]
therefore
\[(2.3) \quad \|f(y) + f(y^{-1})\| \leq c.
\]
Now letting \(y\) for \(x\) in (2.2), we get
\[
\|f(y^2) - f(1) - 2f(y)\| \leq c.
\]
Hence
\[(2.4) \quad \|f(x^2) - 2f(x)\| \leq c_2,
\]
where \(c_2 = c + \|f(1)\|\). Again substitution of \(x = y^2\) in (2.2) yields
\[
\|f(y^3) - f(y) - 2f(y)\| \leq c
\]
which is
\[(2.5) \quad \|f(y^3) - 3f(y)\| \leq c.
\]
Let \(c\) be as in (2.2) and define the set \(C\) as follows: \(C = \{ c_m \mid m \in \mathbb{N} \}\), where \(c_0 = 0, c_2 = c + \|f(1)\|\), and \(c_m = c + c_{m-2}\), if \(m > 2\).

**Lemma 2.3.** Let \(f \in KJT(G; E)\) such that
\[
\|f(xy) - f(xy^{-1}) - 2f(y)\| \leq c.
\]
Then for any \(x \in G\) and any \(m \in \mathbb{N}\) the following relation holds:
\[(2.6) \quad \|f(x^m) - mf(x)\| \leq c_m.
\]

**Proof.** The proof is by induction on \(m\). For \(m = 3\) the lemma is established. Suppose that for \(m\) the lemma has been already established and let us verify it for \(m + 1\). Letting \(x = y^m\) in (2.2), we have
\[
\|f(y^{m+1}) - f(y^{m-1}) - 2f(y)\| \leq c.
\]
By induction hypothesis, we have
\[
\|f(y^{m-1}) - (m - 1)f(y)\| \leq c_{m-1}
\]
and hence,
\[
\|f(y^{m+1}) - (m + 1)f(y)\| \leq c_{m+1} = c + c_{m-1}.
\]
Now the lemma is proved.
Lemma 2.4. Let \( f \in KJT(G; E) \). For any \( m > 1, k \in \mathbb{N} \) and \( x \in G \) we have
\[
\| f(x^{m^k}) - m^k f(x) \| \leq c_m (1 + m + \cdots + m^{k-1})
\]
and
\[
\left\| \frac{1}{m^k} f(x^{m^k}) - f(x) \right\| \leq c_m.
\]

Proof. The proof will be based on induction on \( k \). If \( k = 1 \), then (2.7) follows from (2.6). Suppose (2.7) is true for \( k \) and let us verify it for \( k + 1 \). Substituting \( x^m \) for \( x \) in (2.7) implies
\[
\| f(x^{m^{k+1}}) - m^k f(x^m) \| \leq c_m (1 + m + \cdots + m^{k-1}).
\]
Now using (2.6) we obtain
\[
\| m^k f(x^m) - m^{k+1} f(x) \| \leq c_m m^k
\]
and hence
\[
\| f(x^{m^{k+1}}) - m^{k+1} f(x) \| \leq c_m (1 + m + \cdots + m^k).
\]
The latter implies
\[
\left\| \frac{1}{m^{k+1}} f(x^{m^{k+1}}) - f(x) \right\| \leq c_m (1 + m + \cdots + m^k) \frac{1}{m^{k+1}} \leq c_m.
\]
This completes the proof of the lemma.

From (2.8) it follows that for any \( x \in G \) the set
\[
\left\{ \frac{1}{m^k} f(x^{m^k}) \right\}_{k \in \mathbb{N}}
\]
is bounded. Substituting \( x^{m^n} \) in place of \( x \) in (2.8), we obtain
\[
\left\| \frac{1}{m^k} f(x^{m^{n+k}}) - f(x^{m^n}) \right\| \leq c_m
\]
Thus
\[
\left\| \frac{1}{m^{n+k}} f(x^{m^{n+k}}) - \frac{1}{m^n} f(x^{m^n}) \right\| \leq \frac{c_m}{m^n} \rightarrow 0 \quad \text{as} \quad n \rightarrow \infty.
\]
From the latter, it follows that the sequence
\[
\left\{ \frac{1}{m^k} f(x^{m^k}) \right\}_{k \in \mathbb{N}}
\]
is a Cauchy sequence. Since the real Banach space \( E \) is complete, the above sequence has a limit and we denote it by \( \varphi_m(x) \). Thus

\[
\varphi_m(x) = \lim_{k \to \infty} \frac{1}{m^k} f(x^{mk}).
\]

From (2.8), it follows that

\[
\|\varphi_m(x) - f(x)\| \leq c_m, \quad \forall x \in G.
\]

**Lemma 2.5.** Let \( f \in KJT(G; E) \) such that

\[
\|f(xy) - f(xy^{-1}) - 2f(y)\| \leq c \quad \forall x, y \in G.
\]

Then for any \( m \in \mathbb{N} \), we have \( \varphi_m \in KJT(G; E) \).

**Proof.** Indeed, by (2.9)

\[
\|\varphi_m(xy) - \varphi_m(xy^{-1}) - 2\varphi_m(y)\|
\]

\[
= \|\varphi_m(xy) - f(xy) - \varphi_m(xy^{-1}) + f(xy^{-1}) - 2\varphi_m(y) + 2f(y)
\]

\[
+ f(xy) - f(xy^{-1}) - 2f(y)\|
\]

\[
\leq \|\varphi_m(xy) - f(xy)\| + \|\varphi_m(xy^{-1}) - f(xy^{-1})\|
\]

\[
+ 2\|\varphi_m(x) - f(x)\| + \|f(xy) - f(xy^{-1}) - 2f(y)\|
\]

\[
\leq 4c_m + c.
\]

This completes the proof of the lemma.

For any \( x \in G \) we have the relation

\[
\varphi_m(x^{mk}) = m^k \varphi_m(x).
\]

Indeed,

\[
\varphi_m(x^{mk}) = \lim_{\ell \to \infty} \frac{1}{m^\ell} f((x^{mk})^{m^\ell}) = \lim_{\ell \to \infty} \frac{m^k}{m^{k+\ell}} f(x^{mk+\ell})
\]

\[
= m^k \lim_{p \to \infty} \frac{1}{m^p} f(x^{mp}) = m^k \varphi_m(x).
\]

**Lemma 2.6.** If \( f \in KJT(G; E) \), then \( \varphi_2 = \varphi_m \) for any \( m \geq 2 \).

**Proof.** By Lemma 2.5, we have \( \varphi_2, \varphi_m \in KJT(G; E) \). Hence the function

\[
g(x) = \lim_{k \to \infty} \frac{1}{m^k} \varphi_2(x^{mk})
\]

is well-defined and is a \((G; E)\)-quasijensen type function.
It is clear that \( g(x^{m^k}) = m^k g(x) \) and \( g(x^{2^k}) = 2^k g(x) \) for any \( x \in G \) and any \( k \in \mathbb{N} \). From (2.9), it follows that there are \( d_1, d_2 \in \mathbb{R}_+ \) such that for all \( x \in G \)

\[
\| \varphi_2(x) - g(x) \| \leq d_1 \quad \text{and} \quad \| \varphi_m(x) - g(x) \| \leq d_2.
\]

Hence \( g \equiv \varphi_2 \) and \( g \equiv \varphi_m \) and we obtain \( \varphi_2 \equiv \varphi_m \).

**Definition 2.7.** By \((G; E)\)-pseudojensen type function we will mean a \((G; E)\)-quasijensen type function \( f \) such that \( f(x^n) = nf(x) \) for any \( x \in G \) and any \( n \in \mathbb{N} \).

The space of \((G; E)\)-pseudojensen type function will be denoted by \( PJT(G; E) \).

**Lemma 2.8.** For any \( f \in KJT(G; E) \), the function

\[
\hat{f}(x) = \lim_{k \to \infty} \frac{1}{2^k} f(x^{2^k})
\]

is well-defined and is a \((G; E)\)-pseudojensen function such that for any \( x \in G \)

\[
\| \hat{f}(x) - f(x) \| \leq c_2.
\]

**Proof.** By Lemma 2.5, \( \hat{f} \) is a \((G; E)\)-quasijensen type function. Now by Lemma 2.6, we have \( \hat{f}(x^m) = \varphi_m(x^m) = m \varphi_m(x) = m \hat{f}(x) \). Thus \( \varphi_m(x) = \hat{f}(x) \) and hence \( \varphi_2(x) = \hat{f}(x) \) by Lemma 2.6. From equality \( \hat{f} = \varphi_2 \) we have \( \| \hat{f}(x) - f(x) \| = \| \varphi_2(x) - f(x) \| \leq c_2 \).

**Remark 2.9.** If \( f \in PJT(G; E) \), then:

1. \( f(x^{-n}) = -nf(x) \) for any \( x \in G \) and \( n \in \mathbb{N} \);
2. if \( y \in G \) is an element of finite order then \( f(y) = 0 \);
3. if \( f \) is a bounded function on \( G \), then \( f \equiv 0 \).

**Proof.** Suppose for some \( c > 0 \) the following relation holds

\[
\| f(xy) - f(xy^{-1}) - 2f(y) \| \leq c.
\]

From (2.3) it follows that

\[
\| f(y^k) + f(y^{-k}) \| \leq c, \forall y \in G, \forall k \in \mathbb{N}.
\]

The last inequality is equivalent to \( k \| f(y) + f(y^{-1}) \| \leq c \) or \( \| f(y) + f(y^{-1}) \| \leq \frac{c}{k} \) for all \( y \in G \) and all \( k \in \mathbb{N} \). The latter implies \( f(y^{-1}) = -f(y) \). Thus for any \( n \in \mathbb{N} \), we have

\[
f(y^{-n}) = f((y^n)^{-1}) = -f(y^n) = -nf(y).
\]

Hence, the assertion 1 is established.

Similarly we verify the assertions 2 and 3.
We denote by $B(G; E)$ the space of all bounded mappings on a group $G$ that take values in $E$.

**Theorem 2.10.** For an arbitrary group $G$ the following decomposition holds

$$KJT(G; E) = PJT(G; E) \oplus B(G; E).$$

**Proof.** It is clear that $PJT(G; E)$ and $B(G; E)$ are subspaces of $KJT(G; E)$, and $PJT(G; E) \cap B(G; E) = \{0\}$. Hence the subspace of $KJT(G; E)$ generated by $PJT(G; E)$ and $B(G; E)$ is their direct sum. That is $PJT(G; E) \oplus B(G; E) \subseteq KJT(G; E)$. Let us verify that $KJT(G; E) \subseteq PJT(G; E) \oplus B(G; E)$. Indeed, if $f \in KJT(G; E)$, then by Lemma 2.8 we have $\hat{f} \in PJT(G; E)$ and $\hat{f} - f \in B(G; E)$.

**Definition 2.11.** Let $E$ be a Banach space and $G$ be a group. A mapping $f : G \to E$ is said to be a $(G; E)$-quasiadditive mapping of a group $G$ if the set $\{f(xy) - f(x) - f(y) \mid x, y \in G\}$ is bounded.

**Definition 2.12.** By a $(G; E)$-pseudoadditive mapping of a group $G$ we mean its $(G; E)$-quasiadditive mapping $f$ that satisfies $f(x^n) = nf(x)$ for all $x \in G$ and for all $n \in \mathbb{Z}$.

**Definition 2.13.** A quasicharacter of a group $G$ is a real-valued function $f$ on $G$ such that the set $\{f(xy) - f(x) - f(y) \mid x, y \in G\}$ is bounded.

**Definition 2.14.** By a pseudocharacter of a group $G$ we mean its quasicharacter $f$ that satisfies $f(x^n) = nf(x)$ for all $x \in G$ and all $n \in \mathbb{Z}$.

The set of all $(G; E)$-quasiadditive mappings is a vector space (with respect to the usual operations of addition of functions and their multiplication by numbers), which will be denoted by $KAM(G; E)$. The subspace of $KAM(G; E)$ consisting of $(G; E)$-pseudoadditive mappings will be denoted by $PAM(G; E)$ and the subspace consisting of additive mappings from $G$ to $E$ will be denoted by $\text{Hom}(G; E)$. We say that a $(G; E)$-pseudoadditive mapping $\varphi$ of the group $G$ is nontrivial if $\varphi \notin \text{Hom}(G; E)$.

The space of quasicharacters will be denoted by $KX(G)$, the space of pseudocharacters will be denoted by $PX(G)$, and the the space of real additive characters on $G$ will be denoted by $X(G)$.

**Remark 2.15.** If a group $G$ has nontrivial pseudocharacter, then for any Banach space $E$ there is nontrivial $(G; E)$-pseudoadditive mapping.
Proof. Let $f$ be a nontrivial pseudocharacter of the group $G$ and $e \in E$ such that $e \neq 0$. Consider a mapping $\varphi : G \to E$ such that $\varphi(x) = f(x) \cdot e$. It easy to see that $\varphi$ is nontrivial $(G; E)$-pseudoadditive mapping.

In [7] and [8], some classes of groups having nontrivial pseudocharacters are considered.

**Theorem 2.16.** For any group $G$ the following relations hold:

1. $KAM(G; E) \subseteq KJT(G; E), \ PAM(G; E) \subseteq PJT(G; E)$, and $\text{Hom}(G; E) \subseteq JT(G; E)$;

2. If $f \in PJT(G; E)$, and $f(xy) = f(yx)$ for any $x, y \in G$, then $f \in PAM(G; E)$.

3. If $f \in PJT(G; E)$, and for some $a, b \in G$ we have $ab = ba$, then $f(ab) = f(a) + f(b)$.

**Proof.** (1) Let $f \in KAM(G; E)$ and $c > 0$ such that $\|f(xy) - f(x) - f(y)\| \leq c$ for all $x, y \in G$. Then we have

$$
\|f(xy) - f(xy^{-1}) - 2f(y)\| \\
= \|f(xy) - f(x) - f(y) - f(xy^{-1}) + f(x) + f(y^{-1})\| \\
= \|f(xy) - f(x) - f(y) - (f(xy^{-1}) - f(x) - f(y^{-1}))\| \\
\leq \|f(xy) - f(x) - f(y)\| + \|f(xy^{-1}) - f(x) - f(y^{-1})\| \leq 2c,
$$

that is, $KAM(G; E) \subseteq KJT(G; E)$. Hence, $PAM(G; E) \subseteq PJT(G; E)$.

(2) Let $f \in PJT(G; E)$, $c > 0$ such that $\|f(xy) - f(xy^{-1}) - 2f(y)\| \leq c$ and $f(xy) = f(yx)$ for all $x, y \in G$. Then we have

$$
2\|f(xy) - f(x) - f(y)\| \\
= \|f(xy) - f(xy^{-1}) - 2f(y) + f(xy) - f(yx^{-1}) - 2f(x)\| \\
\leq \|f(xy) - f(xy^{-1}) - 2f(y)\| + \|f(yx) - f(yx^{-1}) - 2f(x)\| \leq 2c.
$$

Hence $\|f(xy) - f(x) - f(y)\| \leq c$ and $f \in PAM(G; E)$.

(3) Let $A$ be the subgroup of $G$ generated by elements $a$ and $b$. From the previous item we have $PJT(A; E) = PAM(A; E)$. Then for some $c > 0$ and for any $n \in \mathbb{N}$, we get

$$
n \| f(ab) - f(a) - f(b) \| = \| f((ab)^n) - f(a^n) - f(b^n) \| \\
= \| f(a^n b^n) - f(a^n) - f(b^n) \| \leq c.
$$

The latter is possible only if $f(ab) - f(a) - f(b) = 0$. 

Corollary 2.17. If \( G \) is an abelian group, then \( \text{PJT}(G; E) = \text{Hom}(G; E) \).

3. Stability

Suppose that \( G \) is a group and \( E \) is a real Banach space.

Definition 3.1. We shall say that the equation (2.1) is stable for the pair \((G; E)\) if for any \( f : G \to E \) satisfying functional inequality

\[
\|f(xy) - f(xy^{-1}) - 2f(y)\| \leq c \quad \forall x, y \in G
\]

for some \( c > 0 \) there is a solution \( j \) of the functional equation (2.1) such that the function \( j(x) - f(x) \) belongs to \( B(G; E) \).

It is clear that the equation (2.1) is stable on \( G \) if and only if \( \text{PJT}(G; E) = \text{JT}(G; E) \). From Corollary 2.17 it follows that the equation (2.1) is stable on any abelian group. We will say that a \((G; E)\)-pseudojensen function \( f \) is nontrivial if \( f \notin \text{JT}(G; E) \).

Theorem 3.2. Let \( E_1, E_2 \) be a Banach spaces over reals. Then the equation (2.1) is stable for the pair \((G; E_1)\) if and only if it is stable for the pair \((G; E_2)\).

Proof. Let \( E \) be a Banach space and \( \mathbb{R} \) be the set of reals. Suppose that the equation (2.1) is stable for the pair \((G; E)\). Suppose that (2.1) is not stable for the pair \((G, \mathbb{R})\), then there is a nontrivial real-valued pseudojensen type function \( f \) on \( G \). Now let \( e \in E \) and \( \|e\| = 1 \). Consider the function \( \varphi : G \to E \) given by the formula \( \varphi(x) = f(x) \cdot e \). It is clear that \( \varphi \) is a nontrivial pseudojensen type \( E \)-valued function, and we obtain a contradiction.

Now suppose that the equation (2.1) is stable for the pair \((G, \mathbb{R})\), that is, \( \text{PJT}(G, \mathbb{R}) = \text{JT}(G, \mathbb{R}) \). Denote by \( E^* \) the space of linear bounded functionals on \( E \) endowed by functional norm topology. It is clear that for any \( \psi \in \text{PJT}(G, E) \) and any \( \lambda \in E^* \) the function \( \lambda \circ \psi \) belongs to the space \( \text{PJT}(G, \mathbb{R}) \). Indeed, for some \( c > 0 \) and any \( x, y \in G \) we have

\[
\|\psi(xy) - \psi(xy^{-1}) - 2\psi(y)\| \leq c.
\]

Hence

\[
|\lambda \circ \psi(xy) - \lambda \circ \psi(xy^{-1}) - \lambda \circ 2\psi(y)| = |\lambda(\psi(xy) - \psi(xy^{-1}) - 2\psi(y))| \\
\leq c\|\lambda\|.
\]

Obviously, \( \lambda \circ \psi(x^n) = n\lambda \circ \psi(x) \) for any \( x \in G \) and for any \( n \in \mathbb{N} \). Hence the function \( \lambda \circ \psi \) belongs to the space \( \text{PJT}(G, \mathbb{R}) \). Let \( f : G \to H \) be a nontrivial pseudojensen type mapping. Then there are \( x, y \in G \) such
that \( f(xy) - f(xy^{-1}) - 2f(y) \neq 0 \). Hahn–Banach Theorem implies that there is a \( \ell \in E^* \) such that \( \ell(f(xy) - f(xy^{-1}) - 2f(y)) \neq 0 \), and we see that \( \ell \circ f \) is a nontrivial pseudojensen type real–valued function on \( G \). This contradiction proves the theorem.

In what follows the space \( KJT(G, \mathbb{R}) \) will be denoted by \( KJT(G) \), the space \( PJT(G, \mathbb{R}) \) will be denoted by \( PJT(G) \), the space \( JT(G, \mathbb{R}) \) will be denoted by \( JT(G) \).

**Corollary 3.3.** The equation (2.1) over a group \( G \) is stable if and only if \( PJT(G) = JT(G) \).

Due to the previous theorem we may simply say that the equation (2.1) is stable or not stable.

**Remark 3.4.** For any group \( G \) and any Banach space \( E \) the following relation \( PAM(G; E) \cap JT(G; E) = \text{Hom}(G; E) \) holds.

**Proof.** It is clear that \( \text{Hom}(G; E) \subseteq PAM(G; E) \cap JT(G; E) \).

Lemma 1 from [6] asserts that if \( f \in PAM(G; E) \), then for any \( x, y \in G \) we have \( f(xy) = f(yx) \).

Suppose that \( f \in PAM(G; E) \cap JT(G; E) \). Since \( f \in JT(G; E) \), the map \( f \) satisfies

\[
(3.1) \quad f(xy) - f(xy^{-1}) - 2f(y) = 0.
\]

Interchanging \( x \) with \( y \) in (3.1), we have

\[
f(yx) - f(yx^{-1}) - 2f(x) = 0.
\]

Taking into account the relations

\[
f(yx) = f(xy) \quad \text{and} \quad f(yx^{-1}) = -f(xy^{-1}),
\]

we get

\[
(3.2) \quad f(xy) + f(xy^{-1}) - 2f(x) = 0.
\]

Adding (3.1) and (3.2), we obtain \( 2f(xy) - 2f(x) - 2f(y) = 0 \). Hence \( f(xy) = f(x) + f(y) \) and \( f \in \text{Hom}(G; E) \), so

\[
(3.3) \quad PAM(G; E) \cap JT(G; E) = \text{Hom}(G; E).
\]

**Remark 3.5.** If a group \( G \) has nontrivial pseudocharacter, then the equation (2.1) is not stable on \( G \).

**Proof.** Let \( \varphi \) be a nontrivial pseudocharacter of \( G \). Suppose that there is \( j \in JT(G) \) such that the function \( \varphi - j \) is bounded. Then there is a \( c > 0 \) such that \( |\varphi(x) - j(x)| \leq c \) for any \( x \in G \). Hence for any \( n \in \mathbb{N} \) we have \( c \geq |\varphi(x^n) - j(x^n)| = n|\varphi(x) - j(x)| \) and we see that
the latter is possible if $\varphi(x) = j(x)$. So, $\varphi \in PX(G) \cap JT(G)$. Hence, $\varphi \in X(G)$ and we come to a contradiction with the assumption about $\varphi$.

Let $G$ be an arbitrary group. For $a, b, c \in G$, we set $[a, b] = a^{-1}b^{-1}ab$ and $[a, b, c] = [[a, b], c]$.

**Definition 3.6.** We shall say that $G$ is *metabelian* if for any $x, y, z \in G$ we have $[[x, y], z] = 1$.

It is clear that if $[x, y] = 1$, then $[[x, y], z] = 1$, and hence any abelian group is metabelian.

Our next goal is to prove a stability theorem for any metabelian group. Consider the group $H$ over two generators $a, b$ and the following defining relations:

$$[b, a]a = a[b, a], \quad b[b, a] = [b, a]b.$$  

If we set $c = [b, a]$ we get the following representation of $H$ in terms of generators and defining relations:

$$(3.4) \quad H = \langle a, b, c \mid c = [b, a], \quad [c, a] = [c, b] = 1 \rangle.$$ 

It is well known that each element of $H$ can be uniquely represented as $g = a^mb^n c^k$, where $m, n, k \in \mathbb{Z}$. The mapping

$$g = a^mb^n c^k \rightarrow \begin{bmatrix} 1 & n & k \\ 0 & 1 & m \\ 0 & 0 & 1 \end{bmatrix}$$ 

is an isomorphism between $H$ and $UT(3, \mathbb{Z})$.

**Lemma 3.7.** Let $f \in PJT(H)$ and $f(c) = 0$, then $f \in X(H)$.

**Proof.** Let $x = a^mb^n c^k$ and $y = a^{m_1}b^{n_1} c^{k_1}$ be two elements from $H$, then from the representation (3.4) it follows

$$xy = a^{m+m_1}b^{n+n_1}c^{m_1n+k+k_1}, \quad yx = a^{m+m_1}b^{n+n_1}c^{mn_1+k+k_1}.$$  

Hence by Theorem 2.16 we have

$$f(xy) = f(a^{m+m_1}b^{n+n_1}) + f(c^{m_1n+k+k_1}) = f(a^{m+m_1}b^{n+n_1}),$$

$$f(yx) = f(a^{m+m_1}b^{n+n_1}) + f(c^{mn_1+k+k_1}) = f(a^{m+m_1}b^{n+n_1}).$$

Thus $f(xy) = f(yx)$ for any $x, y \in H$. By Theorem 2.16 we obtain that $f \in PX(H)$. From the representation (3.4) it follows that the subgroup of $H$ generated by element $c$ is the commutator subgroup of $H$. Lemma 2 from [6] establishes that if $G$ is a group and $\varphi \in PX(G)$ such that $\varphi|_{G'} \equiv 0$, then $\varphi \in X(G)$. Here $G'$ is the commutator subgroup of $G$. Hence, $f \in X(H)$. 

LEMMA 3.8. Let \( f \in PJT(H) \), then \( f(c) = 0 \).

Proof. Let \( x = a^{m}b^{n}c^{k} \), \( y = a^{m_{1}}b^{n_{1}}c^{k_{1}} \), then
\[
x y^{-1} = a^{m}b^{n}c^{k} c^{-k_{1}} b^{-n_{1}} a^{-m_{1}} = a^{m-m_{1}}b^{n-n_{1}}c^{m_{1}n_{1}-m_{1}n+k-k_{1}}.
\]

Hence by Theorem 2.16, we obtain
\[
f(xy) - f(xy^{-1}) - 2f(y) = f(a^{m+m_{1}}b^{n+n_{1}}c^{nm_{1}+k+k_{1}})
- f(a^{m-m_{1}}b^{n-n_{1}}c^{m_{1}n_{1}-nm_{1}+k-k_{1}}) - 2f(a^{m_{1}}b^{n_{1}}c^{k_{1}})
= f(a^{m+m_{1}}b^{n+n_{1}}) + f(c^{nm_{1}+k+k_{1}})
- f(a^{m-m_{1}}b^{n-n_{1}}) - f(c^{m_{1}n_{1}-nm_{1}+k-k_{1}})
- 2f(a^{m_{1}}b^{n_{1}}) - 2f(c^{k_{1}})
= f(a^{m+m_{1}}b^{n+n_{1}}) - f(a^{m-m_{1}}b^{n-n_{1}}) - 2f(a^{m_{1}}b^{n_{1}})
+ f(c^{nm_{1}+k+k_{1}}) - f(c^{m_{1}n_{1}-nm_{1}+k-k_{1}}) - 2f(c^{k_{1}})
= f(a^{m+m_{1}}b^{n+n_{1}}) - f(a^{m-m_{1}}b^{n-n_{1}}) - 2f(a^{m_{1}}b^{n_{1}})
+ f(c^{nm_{1}+k+k_{1}-m_{1}n_{1}+nm_{1}+k-k_{1}-2k_{1}})
+ f(c^{2nm_{1}-m_{1}n_{1}})
\]

Hence the set
\[
M = \left\{ f(a^{m+m_{1}}b^{n+n_{1}}) - f(a^{m-m_{1}}b^{n-n_{1}}) - 2f(a^{m_{1}}b^{n_{1}})
+ f(c^{2nm_{1}-m_{1}n_{1}}) \mid m, n, k, m_{1}, n_{1} \in \mathbb{Z} \right\}
\]
is bounded. Let us set \( n_{1} = n = 2l \), then for some \( \Delta \), we have
\[
|f(a^{m+m_{1}}b^{n+n_{1}}) - f(a^{m_{1}}b^{n_{1}}) - 2f(b^{l})| \leq \Delta, \tag{3.5}
\]
\[
|f(a^{m_{1}}b^{n_{1}}) - f(a^{m_{1}}) - 2f(b^{l})| \leq \Delta. \tag{3.6}
\]

Taking into account these two relations, we see that the set
\[
M_{1} = \left\{ f(a^{m+m_{1}}b^{n+n_{1}}) - f(a^{m-m_{1}}b^{n-n_{1}})
- 2f(a^{m_{1}}b^{n_{1}}) \mid m, n, k, m_{1}, n_{1} \in \mathbb{Z} \right\}
\]
is bounded. Now from boundedness of the sets \( M \) and \( M_1 \) it follows that the set
\[
\left\{ f(c^{2nm_1-m_1n}) = f(c^{nm_1}) \, | \, n, m_1 \in \mathbb{Z} \right\}
\]
is bounded too. But it is possible only if \( f(c) = 0 \).

**Lemma 3.9.** \( PJT(H) = X(H) \).

**Proof.** The proof follows from Lemma 3.7 and Lemma 3.8.

**Theorem 3.10.** The equation (2.1) is stable on any metabelian group.

**Proof.** Let \( G \) be a metabelian group and \( f \in PJT(G) \). If \( x, y \in G \), then there is a homomorphism \( \tau \) of \( H \) into \( G \) such that \( \tau(a) = x \) and \( \tau(b) = y \). Obviously, the function \( f^*(g) = f(\tau(g)) \) belongs to \( PJT(H) \).

Now if \( f(xy) - f(xy^{-1}) - 2f(y) \neq 0 \), then \( f^*(ab) - f^*(ab^{-1}) - 2f^*(b) \neq 0 \) and we arrive at a contradiction with the previous Lemma 3.9. Thus \( f \in JT(G) \) and \( PJT(G) = JT(G) \). Therefore the equation (2.1) is stable on \( G \).

4. Some classic groups \( GL(n, \mathbb{C}), SL(n, \mathbb{C}), T(n, \mathbb{C}) \)

For any group \( G \) denote by \( G^2 \) its subset \( \{ x^2 \mid x \in G \} \).

**Theorem 4.1.** Let \( G \) be a group such that \( G = G^2 \), then \( PJT(G) = PX(G) \).

**Proof.** Let \( f \in PJT(G) \). For some \( c > 0 \) and any \( x, y \in G \), we have
\[
|f(xy^2) - f(xy^{-1}) - 2f(y)| \leq c,
\]
hence
\[
|f(xy^2) - f(x) - 2f(y)| = |f(xy^2) - f(x) - f(y^2)| \leq c.
\]

Let \( x, z \) be an arbitrary elements from \( G \), then for some \( y \in G \) we have \( z = y^2 \). Now from (4.1) it follows \( |f(xz) - f(x) - f(y)| = |f(xy^2) - f(x) - f(y^2)| \leq c \). Hence \( f \in PX(G) \).

**Theorem 4.2.** Let \( G \) denote the group \( GL(n, \mathbb{C}), SL(n, \mathbb{C}) \) or \( T(n, \mathbb{C}) \). Then the equation (2.1) is stable over \( G \).

**Proof.** Let \( G \) be one of the groups \( GL(n, \mathbb{C}), SL(n, \mathbb{C}) \) or \( T(n, \mathbb{C}) \). For any \( x \in G \) there is \( y \in G \) such that \( y^2 = x \). By Theorem 4.1, we have \( PJT(G) = PX(G) \). Let us show that \( PX(G) = X(G) \). The group \( T(n, \mathbb{C}) \) is solvable, hence by Theorem 1 from [6] we have \( PJT(T(n, \mathbb{C})) = \)
Proof. Let $G$ be one of the groups $GL(n, \mathbb{C}), SL(n, \mathbb{C})$ or $T(n, \mathbb{C})$. For any $x \in G$ there is $y \in G$ such that $y^2 = x$. By Theorem 4.1, we have $PJT(G) = PX(G)$. Let us show that $PX(G) = X(G)$. The group $T(n, \mathbb{C})$ is solvable, hence by Theorem 1 from [6] we have $PJT(T(n, \mathbb{C})) = PX(T(n, \mathbb{C})) = X(T(n, \mathbb{C}))$. Consider the group $SL(n, \mathbb{C})$. It is well known that the group $SL(n, \mathbb{C})$ is generated by the set of elementary matrices, and that every elementary matrix is conjugate with its inverse. Hence, if $f \in PX(SL(n, \mathbb{C}))$ and $x$ an elementary matrix, then $f(x) = 0$. It is well known that for any $n \in \mathbb{N}$ there exists $k(n)$ such that every element from $SL(n, \mathbb{C})$ can be represented as product no more then $k(n)$ elementary matrices.

If $|f(xy) - f(x) - f(y)| \leq c$ for all $x, y \in SL(n, \mathbb{C})$, then for any $g \in SL(n, \mathbb{C})$ we have $|f(g)| \leq k(n)c$, and we see that $f$ is a bounded function of $SL(n, \mathbb{C})$. Therefore $f \equiv 0$ and $PX(SL(n, \mathbb{C})) = 0$. It is well known that $SL(n, \mathbb{C})$ is commutator subgroup of $GL(n, \mathbb{C})$. By Lemma 2 from [6] it follows that if a pseudocharacter of a group $G$ is zero on its commutator subgroup $G'$ then this pseudocharacter is a character of $G$. Hence, we get $PX(GL(n, \mathbb{C})) = X(GL(n, \mathbb{C}))$. So in any cases we have $PX(G) = X(G)$ and the equation (2.1) is stable over $G$.

Remark 4.3. Note that the Jensen type functional equation (2.1) is not stable on the group $G$ if $G$ is either $GL(2, \mathbb{Z})$ or $SL(2, \mathbb{Z})$. This is due to the fact that $SL(2, \mathbb{Z})$ has a nontrivial pseudocharacter (see Remark 3.5). Thus, in general, the equation (2.1) is not stable on groups $GL(n, \mathbb{Z})$ and $SL(n, \mathbb{Z})$.

5. The theorem of embedding

Definition 5.1. Let $G$ be a group, $f \in PJT(G; E)$, and $b$ an automorphism of $G$. We will say that $f$ is invariant relative to $b$ if for any $x \in G$ the relation $f(x^b) = f(x)$ holds. If the latter relation is valid for any $b \in B$, where $B$ is a group of automorphism of $G$, then we will say that $f$ is invariant relative to $B$.

From now on, the set of pseudojensen type functions on $G$ invariant relative to $B$ will be denoted by $PJT(G, B; E)$ and if $E = R$, then the space $PJT(G, B; R)$ will be denoted $PJT(G, B)$.

Theorem 5.2. Let $H$ and $A$ be a groups such that $A$ is an abelian group and $H = H^2, A = A^2$. Let $Q = A \cdot H$ be a semidirect product of groups $A$ and $H$, $A$ acts by automorphism on $H$, and $H < Q$. Then $PJT(Q) = PX(Q) = X(A) \oplus PX(H, A)$ and $X(Q) = X(A) \oplus X(H, A)$. 

Proof. Suppose that \( f \in PJT(Q) \) and for some \( c > 0 \) and for any \( x, y \in Q \), we have

\[
|f(xy) - f(xy^{-1}) - 2f(y)| \leq c.
\]

We can assume that \( f|_A \equiv 0 \). Indeed, the restriction of \( f \) to \( A \) is an element of the space \( PJT(A) \). Hence by Corollary 2.17 it is an element of the space \( X(A) \). Let \( \varphi = f \circ \tau \), where \( \tau : Q \to A \) a natural epimorphism with \( \ker \tau = H \). It is clear \( \varphi \in X(Q) \). Hence in order to show that \( f \in PX(Q) \) it is necessary and sufficient to show that \( \pi = f - \varphi \in PX(Q) \). But it is clear that \( \pi|_A \equiv 0 \). So we can assume \( f|_A \equiv 0 \).

Let \( a, b \in A , u, v \in H \). Then we have

\[
|f(uaa) - f(uaa^{-1}) - 2f(a)| \leq c.
\]

Hence

\[
|f(ua^2) - f(u)| \leq c.
\]

Since \( A = A^2 \) we get that for any \( a \in A \) the following relation

\[
|f(ua) - f(u)| \leq c,
\]

or

\[
|f(au^a) - f(u)| \leq c.
\]

It follows that

\[
|f(au) - f(ua^{-1})| \leq c.
\]

For any \( b \in A \) and \( v \in H \), we have \( 2f(bv) = f((bv)^2) = f(b^2v^b v) \).

Taking into account (5.2), we get

\[
|f(b^2v^b v) - f(v^{b^2}v^b)| \leq c
\]

or

\[
|2f(bv) - f(v^{b^{-1}}v^{b^{-2}})| \leq c.
\]

From the latter inequality and (5.2), we get

\[
|2f(v^{b^{-1}}) - f(v^{b^{-1}}v^{b^{-2}})| \leq 3c.
\]

Now by Theorem 4.1, we obtain

\[
|2f(v^{b^{-1}}) - f(v^{b^{-1}}) - f(v^{b^{-2}})| \leq 4c
\]

or

\[
|f(v^{b^{-1}}) - f(v^{b^{-2}})| \leq 4c.
\]
Let us put \( d = b^{-1} \) and \( w = v^d \). Now from (5.4) it follows that for any \( d \in A \) and any \( w \in H \) the following relation

(5.5) \[ |f(w) - f(w^d)| \leq 4c \]

holds. Changing \( w \) by \( w^n \) in the last relation we get

\[ |f(w^n) - f((w^n)^d)| \leq 4c. \]

Hence

\[ |f(w) - f(w^d)| = \frac{1}{n} |f(w^n) - f((w^n)^d)| \leq \frac{1}{n} 4c. \]

And we see that for any \( d \in A \) and any \( w \in H \) the following relation

(5.6) \[ f(w) = f(w^d) \]

holds. Now from (5.2), we get

(5.7) \[ |f(au) - f(u)| \leq c. \]

Let \( \psi = f|_H \), then from the relation (5.6) we get \( \psi^d = \psi \). Or \( \psi \in PX(H, A) \).

Now let us show that \( f \in PX(Q) \). Let \( x = au \) and \( y = bv \). Taking into account (5.6) and (5.7) we have

\[ |f(xy) - f(x) - f(y)| = |f(abu^b v) - f(au) - f(bv)| \]

\[ = |f(abu^b v) - f(u^b v) - f(au) + f(u) - f(bv)| \]

\[ + f(v) + f(u^b v) - f(u) - f(v)| \]

\[ \leq |f(abu^b v) - f(u^b v)| + |f(au) + f(u)| \]

\[ + |f(bv) + f(v)| + |f(u^b v) - f(u) - f(v)| \]

\[ \leq 4c. \]

So, \( f \in PX(Q) \) and \( PJT(Q) = PX(Q) \). Now by Theorem 2 from [5] we obtain \( PX(Q) = X(A) \oplus PX(H, A) \). The latter relation implies \( X(Q) = X(A) \oplus X(H, A) \).

Let \( A \) and \( B \) be an arbitrary groups. For each \( b \in B \) denote by \( A(b) \) a group that is isomorphic to \( A \) under isomorphism \( a \rightarrow a(b) \). Denote by \( D = A^{(B)} = \prod_{b \in B} A(b) \) the direct product of groups \( A(b) \). It is clear that if \( a_1(b_1)a_2(b_2) \cdots a_k(b_k) \) is an element of \( D \), then for any \( b \in B \), the mapping

\[ b^* : a_1(b_1)a_2(b_2) \cdots a_k(b_k) \rightarrow a_1(1b_1)a_2(b_2b) \cdots a_k(b_kb) \]

is an automorphism of \( D \) and \( b \rightarrow b^* \) is an embedding of \( B \) into \( \text{Aut} \ D \).

Hence, we can form a semidirect product \( G = B \cdot D \). This group is called the wreath product of the groups \( A \) and \( B \), and will be denoted
by $G = A \cap B$. We will identify the group $A$ with subgroup $A(1)$ of $D$, where $1 \in B$. Hence, we can assume that $A$ is a subgroup of $D$.

**Lemma 5.3.** Any group $G$ can be embedded into a group $H$ such that $H^2 = H$.

**Proof.** The group $H$ can be constructed by using amalgamated free product (see [22]) or by using wreath product (see [3]).

**Theorem 5.4.** Let $G$ be an arbitrary group. Then $G$ can be embedded into a group $Q$ such that $PJ(Q) = JT(Q) = X(Q)$. Hence the equation (2.1) stable over $Q$.

**Proof.** Let us fix an arbitrary infinite Abelian group $A$ such that $A = A^2$. Let us choose a group $H$ satisfying Lemma 5.3.

Let us verify that the equation (2.1) is stable on $Q = H \cap A$. Denote by $D$ the subgroup of $Q$ generated by $H(b), b \in A$. The group $D$ satisfies condition $D^2 = D$. By Theorem 5.2 we have $PJ(Q) = PX(Q) = X(A) \oplus PX(D, A)$.

Let us verify that $PX(D, A) = X(D, A)$. Suppose that $f \in PX(D, A)$ Let $b_i$ for $i \in \mathbb{N}$ be distinct elements from $A$. Let $a, \alpha \in H$. Consider elements $u_k = a(b_1)a(b_2)\cdots a(b_k)$ and $v_k = \alpha(b_1)\alpha(b_2)\cdots \alpha(b_k)$. Then by Corollary 2.17, for any $k \in \mathbb{N}$, we have

$$|f(u_kv_k) - f(u_k) - f(v_k)| = \left| \sum_{i=1}^{k} [f(a\alpha(b_i)) - f(a(b_i)) - f(\alpha(b_i))] \right|.$$  

By formula (5.6), we have $f(d(b_i)) = f(d(b_i)(1)) = f((1))$ for any $d \in A$ and for any $i \in \mathbb{N}$. Let $r = f(a\alpha) - f(a) - f(\alpha)$. Hence $r = f(a\alpha(b_i)) - f(a(b_i)) - f(\alpha(b_i))$ for any $i \in N$. Therefore

$$|f(u_kv_k) - f(u_k) - f(v_k)| = \left| \sum_{i=1}^{k} [f(a\alpha(b_i)) - f(a(b_i)) - f(\alpha(b_i))] \right|.$$  

$$= |k|f(a\alpha(1)) - f(a(1)) - f(\alpha(1))|.$$  

$$= k|r|.$$  

Further we have

$$|f(u_kv_k) - f(u_k) - f(v_k)| \leq c.$$  

Hence

$$k|r| < c.$$
and
\[ |r| \leq c \frac{1}{k} \quad \forall k \in \mathbb{N}. \]

The latter is possible only if \( r = 0 \). Thus \( f(a\alpha) - f(a) - f(\alpha) = 0 \) and \( f \in X(D, A) \). Hence \( PJT(Q) = X(A) \oplus X(D, A) \). And we see that \( PJT(Q) = X(Q) \). So the equation is stable (2.1) on the group \( Q \).

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References


VALERIY A. FAIZIEV, TVER STATE AGRICULTURAL ACADEMY, TVER SAKHAROVO, RUSSIA

Current address: Zheleznodorozhnikov Street, 31/1-13, Tver 170043, Russia

E-mail: vfaiz@tvcom.ru

PRASANNA K. SAHOO, DEPARTMENT OF MATHEMATICS, UNIVERSITY OF LOUISVILLE, LOUISVILLE, KENTUCKY 40292, U.S.A

E-mail: sahoo@louisville.edu