ON A $q$-FOCK SPACE AND ITS UNITARY DECOMPOSITION

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ABSTRACT. A Fock representation of $q$-commutation relation is studied by constructing a $q$-Fock space as the space of the representation, the $q$-creation and $q$-annihilation operators ($-1 < q < 1$). In the case of $0 < q < 1$, the $q$-Fock space is interpolated between the Boson Fock space and the full Fock space. Also, a unitary decomposition of the $q$-Fock space ($q \neq 0$) is studied.

1. Introduction

The existence of a Fock representation of $q$-commutation relation (introduced by Greenberg [6], Bożejko and Speicher [3]) was first studied in [4] by constructing a $q$-Fock space as the space of representation, see also [2]. A representation of the $q$-commutation relation ($-1 \leq q \leq 1$) is given as the form:

$$a(\zeta)a^*(\eta) - qa^*(\eta)a(\zeta) = \langle \zeta, \eta \rangle \cdot 1, \quad \zeta, \eta \in H,$$

where $H$ is a Hilbert space and $a$ is an operator (on a Hilbert space $K$)-valued linear map. Here the Hilbert space $K$ is called the space of the representation of the relation. The $q$-commutation relation ($-1 < q < 1$) provides an interpolation between the fermionic and bosonic commutation relations which correspond to $q = -1$ and $q = 1$, respectively. The spaces of the representation of the fermionic and bosonic commutation relations are called the Fermion and Boson Fock spaces, respectively. Also, the full Fock space corresponds to $p = 0$. It seems natural to construct a $q$-Fock space as the space of the representation of the $q$-commutation relation ($0 < q < 1$) which is interpolated between the full Fock space and the Boson Fock space.

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Main purpose of this paper is to construct a $q$-Fock space as the space of the representation of the $q$-commutation relation such that for $0 < q < 1$, the $q$-Fock space is interpolated between the full Fock space and the Boson Fock space. Then we study a unitary decomposition of the $q$-Fock space. We hope that the decomposition will be useful for the study of quantum martingales. We refer to [7, 10], for the case of Boson.

The paper is organized as follows. In Section 2 we study a Fock representation of $q$-commutation relation as constructing a $q$-Fock space. In Section 3 we prove a unitary decomposition of the $q$-Fock space.

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2. Fock representation of $q$-commutation relation

Let $\Gamma_0(H)$ be the full Fock space over a complex Hilbert space $H$ with the inner product $\langle \cdot, \cdot \rangle_0$. Let $\Gamma_{0, \text{finite}}^\text{finite}(H)$ be the linear span of vectors of the forms $\xi_1 \otimes \cdots \otimes \xi_n \in H^{\otimes n}$, $n = 0, 1, 2, \ldots$, where $H^{\otimes 0} = \mathbb{C}\Omega$ for the vacuum vector $\Omega \in \Gamma_0(H)$.

Let $q \in (-1, 1)$ be fixed. For each $n = 0, 1, 2, \ldots$, we put

$$[n]_q = 1 + q + \cdots + q^{n-1}, \quad [0]_q = 0.$$ 

The $q$-factorial is defined as

$$[n]_q! = [1]_q[2]_q \cdots [n]_q, \quad [0]_q! = 1.$$ 

Let $S_n$ denote the symmetric group of all permutations on $\{1, \ldots, n\}$ and $I(\sigma)$ denote the number of inversions of the permutation $\sigma \in S_n$ defined by

$$I(\sigma) = \#\{(i, j) | 1 \leq i < j \leq n, \sigma(i) > \sigma(j)\}.$$ 

The operator $P_q$ is defined on $\Gamma_{0, \text{finite}}^\text{finite}(H)$ by a linear extension of

$$P_q\Omega = \Omega;$$

$$P_q(\xi_1 \otimes \cdots \otimes \xi_n) = \sum_{\sigma \in S_n} q^{I(\sigma)}\xi_{\sigma(1)} \otimes \cdots \otimes \xi_{\sigma(n)}.$$ 

Put

$$\xi_1 \otimes_q \cdots \otimes_q \xi_n := P_q(\xi_1 \otimes \cdots \otimes \xi_n), \quad \xi_i \in H, \quad i = 1, \ldots, n.$$
Then we have

\begin{equation}
(2.1) \quad \xi_1 \otimes_q \cdots \otimes_q \xi_n = \sum_{i=1}^{n} q^{i-1} \xi_i \otimes (\xi_1 \otimes_q \cdots \otimes_q \xi_i \otimes_q \cdots \otimes_q \xi_n).
\end{equation}

Let \( \Gamma^\text{finite}_q(H) \) be the linear span of vectors of the forms \( \xi_1 \otimes_q \cdots \otimes_q \xi_n \in H^{\otimes n}, n = 0, 1, 2, \ldots \). Now, we consider the sesquilinear form \( \langle \cdot, \cdot \rangle_q \) defined on \( \Gamma^\text{finite}_q(H) \) by a sesquilinear extension of

\begin{equation}
(2.2) \quad \langle [\xi_1 \otimes_q \cdots \otimes_q \xi_n, \eta_1 \otimes_q \cdots \otimes_q \eta_m] \rangle_q := \delta_{nm} \langle [\xi_1 \otimes_q \cdots \otimes_q \xi_n, \eta_1 \otimes_q \cdots \otimes_q \eta_m] \rangle_0.
\end{equation}

Then by applying Theorem 2.2 in [4], we can easily see that the sesquilinear form \( \langle \cdot, \cdot \rangle_q \) is the strictly positive, i.e., \( \langle \xi, \xi \rangle_q > 0 \) for \( 0 \neq \xi \in \Gamma^\text{finite}_q(H) \).

**Lemma 2.1.** (1) For any \( \sigma \in S_n \), we have \( I(\sigma) = I(\sigma^{-1}) \);

(2) For any \( \xi_i, \eta_i \in H, i = 1, \ldots, n \), we have

\[
\langle [\xi_1 \otimes_q \cdots \otimes_q \xi_n, \eta_1 \otimes_q \cdots \otimes_q \eta_n] \rangle_0 = \langle [\xi_1 \otimes_q \cdots \otimes_q \xi_n, \eta_1 \otimes_q \cdots \otimes_q \eta_n] \rangle_0.
\]

**Proof.** The proof is straightforward. \( \square \)

The completion of \( \Gamma^\text{finite}_q(H) \) with respect to \( \langle \cdot, \cdot \rangle_q \) is called the \textit{q-Fock space} and denoted by \( \Gamma_q(H) \).

For each \( \zeta \in H \), we define the \textit{q-creation operator} \( a^*(\zeta) \) and the \textit{q-annihilation operator} \( a(\zeta) \) on the dense subspace \( \Gamma^\text{finite}_0(H) \) of the \textit{q-Fock space} \( \Gamma_q(H) \) as follows:

\begin{equation}
(2.3) \quad a^*(\zeta) \Omega = \zeta;
\end{equation}

\begin{equation}
(2.4) \quad a(\zeta) \xi_1 \otimes_q \cdots \otimes_q \xi_n = \frac{1}{\sqrt{[n+1]_q}} \zeta \otimes_q \xi_1 \otimes_q \cdots \otimes_q \xi_n
\end{equation}

and

\begin{equation}
(2.5) \quad \zeta \otimes^1 (\xi_1 \otimes_q \cdots \otimes_q \xi_n) = \sum_{i=1}^{n} q^{i-1} \langle \zeta, \xi_i \rangle \xi_1 \otimes_q \cdots \otimes_q \xi_i \otimes_q \cdots \otimes_q \xi_n,
\end{equation}

where \( f \otimes^1 g \) is the left 1-contraction of \( f \in H \) and \( g \in H^{\otimes m} \), see [8]. From (2.1), we have

\[
(2.5) \quad \zeta \otimes^1 (\xi_1 \otimes_q \cdots \otimes_q \xi_n) = \sum_{i=1}^{n} q^{i-1} \langle \zeta, \xi_i \rangle \xi_1 \otimes_q \cdots \otimes_q \xi_i \otimes_q \cdots \otimes_q \xi_n,
\]
where the symbol $\xi_i$ means that $\xi_i$ has to be deleted in the tensor product and $\langle \cdot, \cdot \rangle$ denotes the inner product on $H$.

**Theorem 2.2.** Let $\zeta \in H$.

1. The operators $a^*(\zeta)$ and $a(\zeta)$ are adjoints of each other on $\Gamma_q^{\text{finite}}(H)$ with respect to $\langle \cdot, \cdot \rangle_q$.
2. The operators $a^*(\zeta)$ and $a(\zeta)$ are bounded on $\Gamma_q(H)$.

**Proof.** (1) From (2.3), (2.2) and (1) in Lemma 2.1, for any $\xi_i, \eta_j \in H$, $i = 1, \ldots, n - 1$, $j = 1, \ldots, n$, we have

\[
\langle a^*(\zeta)\xi_1 \otimes_q \cdots \otimes_q \xi_{n-1}, \eta_1 \otimes_q \cdots \otimes_q \eta_n \rangle_q
\]

\[
= \frac{[n_q]!}{[n_q]} \sum_{\sigma \in S_n} q^{I(\sigma)} \langle f_{\sigma(1)}, \eta_1 \rangle \cdots \langle f_{\sigma(n)}, \eta_n \rangle
\]

\[
= \sqrt{[n_q]} [n_q - 1]q! \sum_{\sigma \in S_n} q^{I(\sigma^{-1})} \langle f_1, \eta_{\sigma^{-1}(1)} \rangle \cdots \langle f_n, \eta_{\sigma^{-1}(n)} \rangle
\]

\[
= \sqrt{[n_q]} [n_q - 1]q! \sum_{\tau \in S_n} q^{I(\tau)} \langle \zeta, \eta_{\tau(1)} \rangle \langle \xi_1, \eta_{\tau(2)} \rangle \cdots \langle \xi_{n-1}, \eta_{\tau(n)} \rangle
\]

\[
= \sqrt{[n_q]} [n_q - 1]q! \langle \langle \xi_1 \otimes \cdots \otimes \xi_{n-1}, \zeta \otimes 1 \eta_1 \otimes_q \cdots \otimes_q \eta_n \rangle \rangle_0,
\]

where $f_1 = \zeta$ and $f_i = \xi_{i-1}$, $i = 2, \ldots, n$. For convenience, we put $\vec{\xi} = \xi_1 \otimes \cdots \otimes \xi_{n-1}$, $\vec{\eta} = \eta_1 \otimes_q \cdots \otimes_q \eta_n$.

Then by (2.5) and (2) in Lemma 2.1 we have

\[
\langle \langle \vec{\xi}, \zeta \otimes 1 \eta_1 \otimes_q \cdots \otimes_q \eta_n \rangle \rangle_0
\]

\[
= \langle \langle \vec{\xi}, P_q \left( \sum_{i=1}^n q^{i-1} \langle \zeta, \eta_i \rangle \eta_1 \otimes \cdots \otimes \eta_i \otimes \cdots \otimes \eta_n \right) \rangle \rangle_0
\]

\[
= \langle \langle \vec{\xi}, q \sum_{i=1}^n q^{i-1} \langle \zeta, \eta_i \rangle \eta_1 \otimes \cdots \otimes \eta_i \otimes \cdots \otimes \eta_n \rangle \rangle_0
\]

\[
= \frac{1}{[n_q]} \langle \langle \vec{\xi}, q \sum_{i=1}^n q^{i-1} \langle \zeta, \eta_i \rangle \eta_1 \otimes_q \cdots \otimes_q \eta_i \otimes_q \cdots \otimes_q \eta_n \rangle \rangle_q.
\]

Therefore, by (2.6), (2.7) and (2.4) we have

\[
\langle \langle a^*(\zeta)\xi_1 \otimes_q \cdots \otimes_q \xi_{n-1}, \eta_1 \otimes_q \cdots \otimes_q \eta_n \rangle \rangle_q
\]

\[
= \langle \langle \xi_1 \otimes_q \cdots \otimes_q \xi_{n-1}, a(\zeta)\eta_1 \otimes_q \cdots \otimes_q \eta_n \rangle \rangle_q.
\]
which follows the proof.

(2) By a simple modification of the proof of (ii) in Theorem 6 in [1], we prove that for any $\zeta \in H$ and $\xi_i \in H$, $i = 1, \ldots, n$,

$$
\langle P_q(\zeta \otimes \xi_1 \otimes \cdots \otimes \xi_n), \zeta \otimes \xi_1 \otimes \cdots \otimes \xi_n \rangle_0 \\
\leq \frac{1}{1 - q} |\zeta|_0^2 \langle P_q(\xi_1 \otimes \cdots \otimes \xi_n), \xi_1 \otimes \cdots \otimes \xi_n \rangle_0.
$$

Therefore, for any $\zeta \in H$ and $\xi_i \in H$, $i = 1, \ldots, n$, we obtain that

$$
\|a^*(\zeta)\xi_1 \otimes_q \cdots \otimes_q \xi_n\|^2_q \\
= [n]_q! \langle P_q(\zeta \otimes \xi_1 \otimes \cdots \otimes \xi_n), \zeta \otimes \xi_1 \otimes \cdots \otimes \xi_n \rangle_0 \\
\leq \frac{1}{1 - q} |\zeta|_0^2 [n]_q! \langle P_q(\xi_1 \otimes \cdots \otimes \xi_n), \xi_1 \otimes \cdots \otimes \xi_n \rangle_0 \\
= \frac{1}{1 - q} |\zeta|_0^2 \|\xi_1 \otimes_q \cdots \otimes_q \xi_n\|_q^2.
$$

Hence we prove that $\|a^*(\zeta)\|_{\text{OP}} \leq 1/\sqrt{1 - q} |\zeta|_0$ which proves (2). □

**Theorem 2.3.** The $q$-creation and $q$-annihilation operators fulfill the $q$-commutation relation, i.e.,

$$(2.8) \quad a(\zeta)a^*(\eta) - qa^*(\eta)a(\zeta) = \langle \zeta, \eta \rangle \cdot 1, \quad \zeta, \eta \in H.$$

**Proof.** By (2.5), for any $\xi_i \in H$, $i = 1, \ldots, n$ we have

$$(2.9) \quad a(\zeta)a^*(\eta)\xi_1 \otimes_q \cdots \otimes_q \xi_n \\
= \frac{1}{\sqrt{[n+1]_q}}a(\zeta)\eta \otimes_q \xi_1 \otimes_q \cdots \otimes_q \xi_n \\
= \langle \zeta, \eta \rangle \xi_1 \otimes_q \cdots \otimes_q \xi_n \\
+ \sum_{i=1}^n q^i \langle \zeta, \xi_i \rangle \eta \otimes_q \xi_1 \otimes_q \cdots \otimes_q \xi_i \otimes_q \cdots \otimes_q \xi_n.$$

On the other hand, we have

$$(2.10) \quad \sum_{i=1}^n q^i \langle \zeta, \xi_i \rangle \eta \otimes_q \xi_1 \otimes_q \cdots \otimes_q \xi_i \otimes_q \cdots \otimes_q \xi_n \\
= q\sqrt{[n]_q} a^*(\eta) \left( \sum_{i=1}^n q^{i-1} \langle \zeta, \xi_i \rangle \xi_1 \otimes_q \cdots \otimes_q \xi_i \otimes_q \cdots \otimes_q \xi_n \right) \\
= qa^*(\eta)a(\zeta)\xi_1 \otimes_q \cdots \otimes_q \xi_n.$$
Therefore, by (2.9) and (2.10) we have
\[
a(\zeta)a^*(\eta)\xi_1 \otimes_q \cdots \otimes_q \xi_n \\
= \langle \zeta, \eta \rangle \xi_1 \otimes_q \cdots \otimes_q \xi_n + qa^*(\eta)a(\zeta)\xi_1 \otimes_q \cdots \otimes_q \xi_n
\]
which proves (2.8). \qed

The **Boson Fock space** is defined by
\[
\Gamma_1(H) = \bigoplus_{n=0}^{\infty} H^{\otimes n}
\]
\[
= \left\{ \phi = (f_n)_{n=0}^{\infty} | f_n \in H^{\otimes n}, n = 0, 1, \ldots, \| \phi \|_1 < \infty \right\},
\]
where \(H^{\otimes n}\) is the symmetric \(n\)-tensor product and \(\| \phi \|^2_1 = \sum_{n=0}^{\infty} |f_n|^2\).
Then we have the following.

**Theorem 2.4.** For any \(0 \leq q \leq 1\) we have the following continuous inclusions:
\[
\Gamma_1(H) \subset \Gamma_q(H) \subset \Gamma_0(H).
\]
In particular, \(\Gamma_1(H)\) is isometrically embedded into \(\Gamma_q(H)\) and the second inclusion is contraction.

**Proof.** For any \(\xi_i \in H, i = 1, \ldots, n, \xi_1 \otimes \cdots \otimes \xi_n\) denotes the symmetric \(n\) tensor product of \(\xi_i, i = 1, \ldots, n\), i.e.,
\[
\xi_1 \otimes \cdots \otimes \xi_n = \frac{1}{n!} \sum_{\sigma \in S_n} \xi_{\sigma(1)} \otimes \cdots \otimes \xi_{\sigma(n)}.
\]
Then we have
\[
P_q(\xi_1 \otimes \cdots \otimes \xi_n) = [n]_q!\xi_1 \otimes \cdots \otimes \xi_n
\]
and
\[
\| \xi_1 \otimes \cdots \otimes \xi_n \|_q^2 = \frac{1}{[n]_q!^2} \| P_q(\xi_1 \otimes \cdots \otimes \xi_n) \|_q^2
\]
\[
= \frac{1}{[n]_q!} \langle P_q(\xi_1 \otimes \cdots \otimes \xi_n), \xi_1 \otimes \cdots \otimes \xi_n \rangle_0
\]
\[
= \langle \langle \xi_1 \otimes \cdots \otimes \xi_n, \xi_1 \otimes \cdots \otimes \xi_n \rangle_0
\]
\[
= \| \xi_1 \otimes \cdots \otimes \xi_n \|_1^2.
\]
Hence \(\Gamma_1(H)\) is isometrically embedded into \(\Gamma_q(H)\).

To prove the second inclusion, we first note that for \(0 \leq q < 1\)
\[
P_q^{[n]} > 0 \quad \text{and} \quad \| P_q^{[n]} \|_{\text{op}} \leq [n]_q!,
\]
see Theorem 2 in [1], where \( P_q^{[n]} \) is the restriction of \( P_q \) to \( H^\otimes n \). Therefore, for \( 0 \leq q < 1 \) we have

\[
\left( P_q^{[n]} \right)^2 \leq \| P_q^{[n]} \|_{\text{op}} P_q^{[n]} \leq [n]_q! P_q^{[n]},
\]

see Proposition 2.2.13 in [5]. Hence for any \( \xi_i \in H, i = 1, \ldots, n \) we have

\[
\| \xi_1 \otimes_q \cdots \otimes_q \xi_n \|_0^2 = \langle P_q(\xi_1 \otimes \cdots \otimes \xi_n), P_q(\xi_1 \otimes \cdots \otimes \xi_n) \rangle_0 \\
\leq [n]_q! \langle P_q(\xi_1 \otimes \cdots \otimes \xi_n), \xi_1 \otimes \cdots \otimes \xi_n \rangle_0 \\
= \| \xi_1 \otimes_q \cdots \otimes_q \xi_n \|_q^2
\]

which proves the second inclusion is contraction. \( \Box \)

3. Unitary decomposition of \( q \)-Fock space

Let \( S_{m,n} \) denote the symmetric group of all permutations on \( \{m, m+1, \ldots, n\} \) for \( n \geq m \).

**Lemma 3.1.** Let \( \sigma \in S_{1,m+n} \) with \( \sigma = \tau \cup \lambda \) for some \( \tau \in S_{1,m} \) and \( \lambda \in S_{m+1,m+n} \). Then we have

\[
I(\sigma) = I(\tau) + I(\lambda).
\]

The proof is obvious.

**Theorem 3.2.** Let \( H_1, H_2 \) be Hilbert spaces and let \( H = H_1 \oplus H_2 \). Suppose that \( q \neq 0 \). There exists a unique unitary isomorphism between

\[
U : \Gamma_q(H_1 \oplus H_2) \longrightarrow \Gamma_q(H_1) \otimes \Gamma_q(H_2)
\]

satisfying the relations:

\[
U \left( \frac{1}{\sqrt{|m+n|}_q!} P_q(\xi_1 \otimes \cdots \otimes \xi_m \otimes \eta_1 \otimes \cdots \otimes \eta_n) \right) = \frac{1}{\sqrt{|m|}_q! |n|_q!} P_q(\xi_1 \otimes \cdots \otimes \xi_m) \otimes P_q(\eta_1 \otimes \cdots \otimes \eta_n)
\]

for all \( \xi_i \in H_1, \eta_j \in H_2, 1 \leq i \leq m, 1 \leq j \leq n, m = 1, 2, \ldots, n = 1, 2, \ldots \) and

\[
U \Omega = \Omega_1 \otimes \Omega_2,
\]

where \( \Omega_1 \) and \( \Omega_2 \) are vacuum vectors in \( \Gamma_q(H_1) \) and \( \Gamma_q(H_2) \), respectively.
Proof. The proof is a simple modification of the proof of Proposition 19.7 in [9]. We may assume without loss of generality that \( H_1 \) and \( H_2 \) are mutually orthogonal subspaces of \( H \). Put

\[
S = \{ \Omega, \frac{1}{\sqrt{|m+n|qq!}} P_q(\xi_1 \otimes \cdots \otimes \xi_m \otimes \eta_1 \otimes \cdots \otimes \eta_n) | \xi_i \in H_1, \\
\eta_j \in H_2, 1 \leq i \leq m, 1 \leq j \leq n, m = 1, 2, \ldots, n = 1, 2, \ldots \}
\]

\[
S_1 = \{ \Omega_1, \frac{1}{\sqrt{|m|qq!}} P_q(\xi_1 \otimes \cdots \otimes \xi_m) | \xi_i \in H_1, 1 \leq i \leq m, m = 1, 2, \ldots \}
\]

\[
S_2 = \{ \Omega_2, \frac{1}{\sqrt{|n|qq!}} P_q(\eta_1 \otimes \cdots \otimes \eta_n) | \eta_j \in H_2, 1 \leq j \leq n, n = 1, 2, \ldots \}.
\]

Then \( S, S_1, S_2 \) are total in \( \Gamma_q(H), \Gamma_q(H_1), \Gamma_q(H_2) \), respectively. Also, the set \( \{ \xi \otimes \eta | \xi \in S_1, \eta \in S_2 \} \) is total in \( \Gamma_q(H) \). Thus it is enough to show that the map defined by (3.1) preserves the scalar product. Let \( \xi_i, \xi_k \in H_1, 1 \leq i \leq m, 1 \leq k \leq m', \) and \( \eta_j, \eta_l \in H_2, 1 \leq j \leq n, 1 \leq l \leq n' \). We now consider the following three different cases:

**Case 1:** \( m + n \neq m' + n' \):

By (2.2), we see that \( P_q(\xi_1 \otimes \cdots \otimes \xi_m \otimes \eta_1 \otimes \cdots \otimes \eta_n) \) and \( P_q(\xi'_1 \otimes \cdots \otimes \xi'_m \otimes \eta'_1 \otimes \cdots \otimes \eta'_n) \) are orthogonal.

**Case 2:** \( m + n = m' + n' \) and \( m \neq m' \):

Without loss of generality let \( m < m' \) and \( n > n' \). Put

\[
(\xi_1, \ldots, \xi_m, \eta_1, \ldots, \eta_n) = (\xi_1, \ldots, \xi_{m+n})
\]

and

\[
(\xi'_1, \ldots, \xi'_{m'}, \eta'_1, \ldots, \eta'_{n'}) = (\xi'_1, \ldots, \xi'_{m+n}).
\]

Note that for any \( \sigma \in S_{m+n} \), \( \{ \xi_{\sigma(1)}, \ldots, \xi_{\sigma(m')} \} \) contains at least one \( \eta_j \in H_2 \). Since \( H_1 \) and \( H_2 \) are orthogonal, by (2) in Lemma 2.1 we have

\[
\langle P_q(\xi'_1 \otimes \cdots \otimes \xi'_{m+n}), P_q(\xi_1 \otimes \cdots \otimes \xi_{m+n}) \rangle_q
\]

\[
= \frac{1}{|m+n|qq!} \langle P_q(\xi'_1 \otimes \cdots \otimes \xi'_{m+n}), \xi_1 \otimes \cdots \otimes \xi_{m+n} \rangle_0
\]

\[
= \frac{1}{|m+n|qq!} \sum_{\sigma \in S_{m+n}} q^l(\sigma) \langle \xi'_1, \xi_{\sigma(1)} \rangle \cdots \langle \xi'_{m'}, \xi_{\sigma(m')} \rangle
\]

\[
\times \langle \eta'_1, \eta_{\sigma(m'+1)} \rangle \cdots \langle \eta'_{n'}, \eta_{\sigma(n'+1)} \rangle
\]

\[
= 0.
\]

**Case 3:** \( m = m' \) and \( n = n' \):

Define \( (\xi_1, \ldots, \xi_{m+n}) \) and \( (\xi'_1, \ldots, \xi'_{m+n}) \) as in (3.2) and (3.3), respectively. Note that if there exists \( 1 \leq i \leq m \) such that \( \sigma(i) \in \{ m +
1, \ldots, m + n} \text{ for } \sigma \in S_{m+n}, \text{ then }

\left\langle \zeta_1, \zeta_{\sigma(1)}' \right\rangle \cdots \left\langle \zeta_{m+n}, \zeta_{\sigma(m+n)}' \right\rangle = 0.

Therefore, for any \sigma such that \sigma = \tau \cup \lambda \text{ with some } \tau \in S_{1:m} \text{ and } \lambda \in S_{m+1:m+n},

\left\langle \zeta_1, \zeta_{\sigma(1)}' \right\rangle \cdots \left\langle \zeta_{m+n}, \zeta_{\sigma(m+n)}' \right\rangle

can be a non-zero value. Hence we have

\begin{align*}
\left\langle \left( P_q(\zeta_1' \otimes \cdots \otimes \zeta_{m+n}'), P_q(\zeta_1 \otimes \cdots \otimes \zeta_{m+n}) \right) \right\rangle_q \\
= [m + n]_q! \left\langle \left( \zeta_1' \otimes \cdots \otimes \zeta_{m+n}', P_q(\zeta_1 \otimes \cdots \otimes \zeta_{m+n}) \right) \right\rangle_0 \\
= [m + n]_q! \sum_{\sigma \in S_{m+n}} q^{I(\sigma)} \left\langle \zeta_1', \zeta_{\sigma(1)} \right\rangle \cdots \left\langle \zeta_{m'}, \zeta_{\sigma(m')} \right\rangle \\
\times \left\langle \eta_1', \zeta_{\sigma(m'+1)} \right\rangle \cdots \left\langle \eta_{m'}, \zeta_{\sigma(m+n)} \right\rangle \\
= [m + n]_q! \sum_{\sigma = \tau \cup \lambda; \tau \in S_{1:m}, \lambda \in S_{m+1:m+n}} q^{I(\sigma)} \\
\times \left\langle \zeta_1', \zeta_{\sigma(1)} \right\rangle \cdots \left\langle \zeta_{m'}, \zeta_{\sigma(m')} \right\rangle \\
\times \left\langle \eta_1', \zeta_{\lambda(m'+1)} \right\rangle \cdots \left\langle \eta_{m'}, \zeta_{\lambda(m+n)} \right\rangle.
\end{align*}

Therefore, by Lemma 3.1 we have

\begin{align*}
\left\langle \left( P_q(\zeta_1' \otimes \cdots \otimes \zeta_{m+n}'), P_q(\zeta_1 \otimes \cdots \otimes \zeta_{m+n}) \right) \right\rangle_q \\
= [m + n]_q! \sum_{\tau \in S_{1:m}, \lambda \in S_{m+1:m+n}} q^{I(\tau)} q^{I(\lambda)} \\
\times \left\langle \zeta_1', \xi_{\tau(1)} \right\rangle \cdots \left\langle \zeta_{m'}, \xi_{\tau(m)} \right\rangle \\
\times \left\langle \eta_1', \zeta_{\lambda(m+1)} \right\rangle \cdots \left\langle \eta_{m'}, \zeta_{\lambda(m+n)} \right\rangle \\
= [m + n]_q! \sum_{\tau \in S_m, \lambda \in S_n} q^{I(\tau)} q^{I(\lambda)} \left\langle \zeta_1', \xi_{\tau(1)} \right\rangle \cdots \left\langle \zeta_{m'}, \xi_{\tau(m)} \right\rangle \\
\times \left\langle \eta_1', \eta_{\lambda(1)} \right\rangle \cdots \left\langle \eta_{m'}, \eta_{\lambda(n)} \right\rangle \\
= \frac{[m + n]_q!}{[m]_q! [n]_q!} \left\langle \left( P_q(\zeta_1' \otimes \cdots \otimes \zeta_m'), P_q(\zeta_1 \otimes \cdots \otimes \zeta_m) \right) \right\rangle_q \\
\times \left\langle \left( P_q(\eta_1' \otimes \cdots \otimes \eta_m'), P_q(\eta_1 \otimes \cdots \otimes \eta_m) \right) \right\rangle_q
\end{align*}

which completes the proof. \qed}
References


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