DERIVATIONS ON SUBRINGS OF MATRIX RINGS

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Abstract. For a lower niltriangular matrix ring $A = NT_n(K)$ $(n \geq 3)$, we show that every derivation of $A$ is a sum of certain diagonal, trivial extension and strongly nilpotent derivation. Moreover, a strongly nilpotent derivation is a sum of an inner derivation and an uaz-derivation.

1. Introduction

Let $NT_n(K)(n \geq 3)$ be the ring of all (lower niltriangular) $n \times n$ matrices over an associative ring with identity $K$ which are all zeros on and above the main diagonal.

It is well-known (see [4], p.100) that if $F$ is a field, then any $F$-derivation of $M_n(F)$ is inner. Moreover, Amitsur [1] showed that any derivation of $M_n(K)$ is a sum of an inner derivation and a trivial extension and Nowicki [8] characterized derivations of special subrings of $M_n(K)$.

Dubish and Perlis [3] classified automorphisms on $NT_n(F)$ over a field $F$. Every automorphism on $NT_n(F)$ is equal to a product of certain diagonal automorphism, inner automorphism and nil automorphism. Moreover, Levchuk ([6], [7]) characterized automorphisms of $NT_n(K)$ and Kuzucuoglu and Levchuk [5] characterized automorphisms on $R_n(K, J) = NT_n(K) + M_n(J)$.

In this paper, we will characterize derivations of $NT_n(K)$. In section 2, we characterize ideals and characteristic ideals of $NT_n(K)$. In section 3, we show that for a derivation $\delta$ on $NT_n(K)$, $\delta = i_d + \bar{\sigma} + s_t$ where $i_d$ is a diagonal inner, $\bar{\sigma}$ is a trivial extension of $K$ and $s_t$ is a strongly nilpotent derivation. In section 4, we have that for a strongly...
nullpotent derivation $s_t$, $s_t = s_i + s_{uaz}$ where $s_i$ is an inner derivation and $s_{uaz}$ is an $uaz$-derivation. Moreover, we characterize the difference between $uaz$-derivations and $az$-derivations.

For a ring $R$, not necessarily contains 1, a derivation $\delta$ is an additive map on $R$ which satisfies

$$\delta(ab) = \delta(a)b + a\delta(b) \quad (a, b \in R).$$

We say that $\delta$ is an inner derivation if there exist $r \in R$ such that $\delta(x) = rx - xr$ for all $x \in R$.

For the convenience we have the followings:

1. $NT_n(K) \cong A_n \cong A$.
2. $e_{ij}$: matrix units of $M_n(K)$.
3. $A^k$: $k$-th product of $A$.
4. Any derivation $\sigma$ of $K$ can be extended to $A$ by putting

$$\tilde{\sigma}(\sum_{i,j} r_{ij}e_{ij}) = \sum_{i,j} \sigma(r_{ij})e_{ij} \quad (r_{ij} \in K).$$

It is easy to show that $\tilde{\sigma}$ is also a derivation of $A$. We call $\tilde{\sigma}$ a trivial extension of $\sigma$.

5. Let $B_n$ be the set of all matrices in $M_n(K)$ with zeros above the diagonal. Then each diagonal matrix $d = \sum d_i e_{ii} (d_i \in K)$ determines a derivation $i_d(x) = [d, x]$ of $B_n$ and the derivation $i_d$ induces on $A$. We call $i_d$ a diagonal derivation.

6. Since we can regard $A$ as a $K$-module, we define a $K$-derivation on $A$ by $\delta(rx) = r\delta(x) (r \in K, x \in A)$.

7. For all $x \in A$, we denote $\{x\}_{ij} = \pi_{ij}(x)$.

2. Ideals of $A$

The ideals of $NT_n(F)$ are characterized in Dubisch and Perlis [3], which are referred to “staircase open polygon”. Also, the ideals of $A$ can be regarded similarly. But we characterize ideals of $A$ another way. For any subset $H$ of $A$, trivially $\sum \pi_{ij}(H)e_{ij} \supseteq H$. If $H = \sum \pi_{ij}(H)e_{ij}$, we call $H$ a direct subset of $A$.

**Proposition 2.1.** Let $H$ be a subset of $A$. If $H$ is an ideal of $A$, then the followings hold;

1. $\pi_{ij}(H)$ is a subgroup of $K$. 
(2) For all \( s > i \), \( \pi_{sj}(H) \supseteq K \pi_{ij}(H) \).
(3) For all \( t < j \), \( \pi_{it}(H) \supseteq \pi_{ij}(H)K \).

Conversely, if \( H \) is a direct subset of \( A \) and satisfies above (1), (2) and (3), then \( H \) is an ideal of \( A \).

Proof. The proof of the first statement is obvious.

Conversely, by (2)

\[
\pi_{ij}(NT_n(K)H) = \sum_{\lambda=1}^{n} \pi_{i\lambda}(NT_n(K))\pi_{\lambda j}(H)
\]
\[
= \sum_{\lambda=j+1}^{i-1} \pi_{i\lambda}(NT_n(K))\pi_{\lambda j}(H)
\]
\[
= \sum_{\lambda=j+1}^{i-1} K\pi_{\lambda j}(H) = K\pi_{i-1,j}(H) \subseteq \pi_{ij}(H).
\]

Thus, by \( \sum \pi_{ij}(H)e_{ij} = H \), \( NT_n(K)H \subseteq H \), that is, \( AH \subseteq H \).

Similarly, by (3) and \( \sum \pi_{ij}(H)e_{ij} = H \), \( HA \subseteq H \). Therefore, \( H \) is an ideal of \( A \).

Next example shows that an ideal of \( A \) is not necessarily direct and for noetherian ring \( K \), \( A \) is not noetherian in general.

EXAMPLE 2.2. For the rational number field \( \mathbb{Q} \) and the ring of integers \( \mathbb{Z} \), let \( K = M_2(\mathbb{Q}) \) and \( A = NT_3(K) \). Denote \( f_{ij}(i, j = 1, 2) \) by matrix units of \( M_2(\mathbb{Q}) \). Set

\[
H_k = \left\{ \begin{pmatrix} 0 & 0 & 0 \\ \frac{n}{2\pi} f_{21} & 0 & 0 \\ T & \frac{n}{2\pi} f_{21} & 0 \end{pmatrix} \middle| n \in \mathbb{Z} \right\}, \quad k = 1, 2, \ldots
\]

where \( T = \mathbb{Q}f_{11} + \mathbb{Q}f_{21} + \mathbb{Q}f_{22} \). Then we have the following properties;

(1) \( H_k \) are ideals but not direct.

(2) \( T \) is not an ideal of \( K \).

(3) For a trivial extension \( \delta \) of an inner derivation of \( K \), \( H_k \) is not invariant in general.

(4) \( K \) is noetherian. But since \( H_1 \subseteq H_2 \subseteq H_3 \subseteq \cdots \), \( A \) is not noetherian.
DEFINITION 2.3. Let $C$ be a subring of a ring $R$. $C$ is called characteristic if every derivation $\delta$ on $R$ induces a derivation on $C$.

Obviously for $k (1 < k < n)$, the $k$-th powers $A^k$ are characteristic ideals of $A$.

For $x \in A$, it is important to find characteristic ideals of $A$ which contain $\delta(x)$. We introduce certain characteristic ideals of $A$ which contains matrix unit $e_{ij} (i > j)$.

Let $C_l$ be the totality of matrices in $A$ whose columns beyond the $l$-th are zero. Then $C_l$ is an ideal of $A$. Likewise, an ideal is given by the set $R_k$ of all matrices in $A$ whose first $k - 1$ rows are zero.

PROPOSITION 2.4 [3]. $C_l$ is the left annihilator of $A^l$ and $R_k$ is the right annihilator of $A^k$.

THEOREM 2.5. $C_l$ and $R_k$ are characteristic ideals. Moreover, for each derivation $\delta$ on $A$ and each matrix unit $e_{kl} \in A$, $\delta(e_{kl}) \in C_l \cap R_k$.

Proof. For arbitrary derivation $\delta$ of $A$, let $c \in C_l$ and $x \in A^l$. Then by Proposition 2.4

$$0 = \delta(cx) = \delta(c)x + c\delta(x).$$

Since $A^l$ is a characteristic ideal $\delta(x) \in A^l$ and $c\delta(x) = 0$. So, $\delta(c)x = 0$. This means $\delta(c)A^l = 0$. Thus $\delta(c) \in C_l$. Therefore $C_l$ is a characteristic ideal.

Similarly, $R_k$ is a characteristic ideal.

Moreover, $e_{kl} \in C_l \cap R_k$. So $\delta(e_{kl}) \in C_l \cap R_k$. \qed

From the Theorem 2.5, $\delta(e_{kl}) \in C_l \cap R_k$. So we have the following;

$$\delta(e_{k,k-1}) = \sum_{i \geq k} \sum_{j \leq k-1} \beta_{ij}^{(k)} e_{ij}, \quad \beta_{ij}^{(k)} \in K.$$  \hspace{1cm} (*)

Now we characterize the characteristic ideals of $A$

THEOREM 2.6. Let $H$ be a characteristic ideal of $A$. Then the followings hold;

1. $\pi_{ij}(H)$ is an ideal of $K$.
2. $H$ is direct.
3. $\pi_{ij}(H)$ is a characteristic ideal of $K$. 
Proof. (1) For \( r \in K \), let \( i_d \) be a diagonal derivation induced by \( d = re_{ii} \), that is, \( i_d(x) = dx - xd \) for all \( x \in A \). Then

\[
r \pi_{ij}(H) = \pi_{ij}(rH) = \pi_{ij}([d, H]) = \pi_{ij}(i_d(H)) \subseteq \pi_{ij}(H).
\]

So, \( \pi_{ij}(H) \) is a left ideal of \( K \).

To show that \( \pi_{ij}(H) \) is a right ideal, take \( d = -re_{jj} \). Then

\[
\pi_{ij}(H)r = \pi_{ij}(Hr) = \pi_{ij}([d, H]) = \pi_{ij}(i_d(H)) \subseteq \pi_{ij}(H).
\]

So, \( \pi_{ij}(H) \) is a right ideal of \( K \). Therefore, \( \pi_{ij}(H) \) is an ideal of \( K \).

(2) Since \( H \) is a characteristic ideal,

\[
[-e_{jj}, e_{ii}, H] = \pi_{ij}(H)e_{ij} \subseteq H.
\]

So, \( \sum \pi_{ij}(H)e_{ij} \subseteq H \), that is, \( \sum \pi_{ij}(H)e_{ij} = H \).

(3) Let \( \sigma \) be a derivation of \( K \). Then trivial extension \( \bar{\sigma} \) of \( \sigma \) is a derivation of \( A \). So, \( \bar{\sigma}(H) \subseteq H \). Thus, \( \sigma(\pi_{ij}(H)) \subseteq \pi_{ij}(H) \). Therefore, \( \pi_{ij}(H) \) is a characteristic ideal. \( \square \)

Corollary 2.7. Let \( H \) be a characteristic ideal of \( A \). If \( \pi_{ij}(H) = K \) or 0, then \( H \) is generated by \( C_{l_1} \cap R_{k_1}, C_{l_2} \cap R_{k_2}, \ldots, C_{l_t} \cap R_{k_t} \).

3. Characterizations of derivations

Since \( A \) is a free \( K \)-module with basis \( \{e_{ij}\} \), derivations of \( A \) highly depends on the image of \( e_{ij} \). Every \( K \)-module derivation of \( A \) is determined by the image of \( e_{ij} \), but in general every derivation of \( A \) is not determined by the image of \( e_{ij} \). However, we get a useful lemma which says that for any derivation \( \delta \) of \( \{\delta(e_{ij})\}_{ij} = 0 \), the coordinate function of \( \delta \) is also a derivation of \( K \).

Lemma 3.1. Suppose \( \delta \) is a derivation of \( A \) and \( \{\delta(e_{ij})\}_{ij} = 0 \) for all \( i > j \). Define the coordinate function \( \delta_{ij} : K \rightarrow K \) such that \( \delta_{ij}(r) = \delta(re_{ij})_{ij}(r \in K) \). Then \( \delta_{ij} = \delta_{21} \) and \( \delta_{ij} \) is a derivation of \( K \).

Proof. For \( r \in K \), we get

\[
\delta_{31}(r) = \{\delta(re_{31})\}_{31} = \{\delta(re_{32}e_{21})\}_{31}
\]

\[
= \{\delta(re_{32})e_{21} + re_{32}\delta(e_{21})\}_{31}
\]

\[
= \{\delta(re_{32})e_{21}\}_{31}
\]

\[
= \{\delta_{21}(r)e_{31}\}_{31} = \delta_{32}(r).
\]
On the other hand,

\[
\delta_{31}(r) = \{\delta(re_{31})\}_{31} = \{\delta(re_{32}e_{21})\}_{31} = \{\delta(e_{32}re_{21})\}_{31} \\
= \{\delta(e_{32})re_{21} + e_{32}\delta(re_{21})\}_{31} = \{\delta_{21}(r)e_{31}\}_{31} \\
= \delta_{21}(r).
\]

Hence, \(\delta_{32} = \delta_{21}\). Similarly, we can show that for \(4 \leq k \leq n\), \(\delta_{k,k-1} = \delta_{21}\).

If \(i - j \geq 2\),

\[
\delta_{ij}(r) = \{\delta(re_{ij})\}_{ij} = \{\delta(re_{i,i-1} \cdots e_{j+1,j})\}_{ij} \\
= \{\delta_{i,i-1}(r)e_{ij}\}_{ij} = \delta_{i,i-1}(r).
\]

Therefore, for all \(i > j\), \(\delta_{ij} = \delta_{21}(i > j)\).

Now we will show that \(\delta_{31}\) is a derivation of \(K\). For arbitrary \(r, r' \in K\),

\[
\delta_{31}(rr') = \{\delta(rr'e_{31})\}_{31} = \{\delta(re_{32}r'e_{21})\}_{31} \\
= \{\delta(re_{32})r'e_{21} + re_{32}\delta(r'e_{21})\}_{31} \\
= \{\delta(re_{32})\}_{32}r' + r\{\delta(r'e_{21})\}_{21} \\
= \delta_{32}(r)r' + r\delta_{21}(r') = \delta_{31}(r)r' + r\delta_{31}(r).
\]

So, \(\delta_{31}\) is a derivation of \(K\). This means \(\delta_{ij}\) is a derivation of \(K\). \(\square\)

**Corollary 3.2.** Suppose that \(\delta, \delta'\) are derivations of \(A\) satisfying \(\{\delta(e_{ij})\}_{ij} = \{\delta'(e_{ij})\}_{ij}\) for all \(i > j\). Then \(\delta_{ij} - \delta'_{ij} = \delta_{21} - \delta'_{21}\) and \(\delta_{ij} - \delta'_{ij}\) is a derivation of \(K\).

**Corollary 3.3.** Let \(\delta_{ij} : K \longrightarrow K(i > j)\) be derivations. Define \(\delta : A \longrightarrow A\) by \(\delta(\sum_{i > j} r_{ij}e_{ij}) = \sum_{i > j} \delta_{ij}(r_{ij})e_{ij}\). If \(\delta\) is a derivation of \(A\), then for all \(i > j\), \(\delta_{ij} = \delta_{21}\).

**Proof.** Since \(\delta(e_{ij}) = 0\), \(\{\delta(e_{ij})\}_{ij} = 0\). And for all \(r \in K\), \(\delta_{ij}(r) = \{\delta(re_{ij})\}_{ij}\). So, by Lemma 3.1, \(\delta_{ij} = \delta_{21}\). \(\square\)

**Lemma 3.4.** If \(\delta\) is a diagonal derivation of \(A\). Then
\[
(1) \quad \delta(e_{k,k-1}) = \alpha_k e_{k,k-1}, \text{ where } \alpha_k \in K \text{ and } 2 \leq k \leq n.
\]
\[
(2) \quad \delta(e_{kl}) = \alpha_{kl} e_{kl}, \quad \text{where } \alpha_{kl} = \alpha_k + \alpha_{k-1} + \cdots + \alpha_{l+1}.
\]
Conversely, if \(\delta\) is a derivation of \(A\) satisfying (1) and (2), then \((\delta - i_d)(e_{kl}) = 0\), where \(i_d\) is a diagonal derivation induced by \(d = \alpha_2 e_{22} + \cdots + (\alpha_2 + \cdots + \alpha_n)e_{nn}\).
Proof. The proof of the first statement is obvious. Conversely, for all \( k > l \),
\[
   i_d(e_{kl}) = de_{kl} - e_{kl}d \\
   = (\alpha_2 + \cdots + \alpha_k)e_{kl} - (\alpha_2 + \cdots + \alpha_l)e_{kl} \\
   = (\alpha_k + \cdots + \alpha_{l+1})e_{kl}.
\]
So, \( (\delta - i_d)(e_{kl}) = 0 \). \( \square \)

**Lemma 3.5.** Let \( \delta \) be a derivation on \( A \). Then there exists a diagonal derivation \( i_d \) such that \( \{(\delta - i_d)(e_{ij})\}_{ij} = 0 \).

**Proof.** The quantities \( \delta(e_{k,k-1}), \ldots, \delta(e_{l+1,l}) \) can be denoted as equations (*) in section 2 with corner coefficients \( \beta_{k,k-1}^{(k)} \equiv \alpha_k, \ldots, \beta_{l+1,l}^{(l+1)} \equiv \alpha_{l+1} \). By multiplying the equations, we find that
\[
   \delta(e_{kl}) = \sum_{i \geq k} \sum_{j \leq l} \beta_{ij}^{(kl)} e_{ij} \quad (\beta_{ij}^{(kl)} \in K)
\]
with corner coefficients \( \beta_{kl}^{(kl)} \equiv \alpha_{kl} = \alpha_k + \alpha_{k-1} + \cdots + \alpha_{l+1} \).

The quantities \( \alpha_l, \alpha_{kl} \) thus fulfill the conditions (1) and (2) of Lemma 3.4. So, the correspondence \( e_{kl} \mapsto \alpha_{kl}e_{kl} \) generates a diagonal derivation \( i_d \) such that \( \{(\delta(e_{ij}))\}_{ij} = \{i_d(e_{ij})\}_{ij} \), that is, \( \{(\delta - i_d)(e_{ij})\}_{ij} = 0 \). \( \square \)

**Definition 3.6.** Let \( s_t \) be a derivation on \( A \). \( s_t \) is called a strongly nilpotent derivation if for all \( x \in A^k, s_t(x) \in A^{k+1} \).

Obviously every strongly nilpotent derivation of \( A \) is nilpotent and every inner derivation of \( A \) is strongly nilpotent.

**Proposition 3.7.** Suppose that \( \delta \) is a derivation on \( A \) such that \( \{(\delta(e_{ij}))\}_{ij} = 0 \) for all \( i > j \). Then there exists a trivial extension \( \tilde{\sigma} \) of \( A \) such that \( \delta - \tilde{\sigma} \) is strongly nilpotent.

**Proof.** By Lemma 3.1, it is obvious. \( \square \)

**Theorem 3.8.** Let \( \delta \) be a derivation of \( A \). Then \( \delta = i_d + \tilde{\sigma} + s_t \) where \( i_d \) is a diagonal inner, \( \tilde{\sigma} \) is a trivial extension of \( K \) and \( s_t \) is a strongly nilpotent derivation.

**Proof.** By Lemma 3.5, there exists a diagonal derivation \( i_d \) such that \( \{(\delta - i_d)(e_{ij})\}_{ij} = 0 \). And by Proposition 3.7, there exists a trivial extension \( \tilde{\sigma} \) of \( A \) such that \( (\delta - i_d) - \tilde{\sigma} \) is strongly nilpotent. \( \square \)
4. *uaz*- derivations of $A$

Matrix units $e_{21}, e_{31}, \ldots, e_{n1}$ are left annihilators of $A$ and matrix units $e_{n1}, e_{n2}, \ldots, e_{n,n-1}$ are right annihilators of $A$. There exist derivations that the images of these matrix units are zero. These derivations are important to characterize strongly nilpotent derivations.

**Definition 4.1.** A strongly nilpotent derivation $\delta$ of $A$ is called a *uaz*-derivation if $\delta(u) = 0$ for every matrix unit $u$ which is an absolute left or right divisor of zero.

**Theorem 4.2.** Let $\delta$ be a strongly nilpotent derivation of $A$. Then $\delta$ is a *uaz*-derivation of $A$ if and only if $\delta(e_{k,k-1}) = \gamma_k e_{n1} (k = 2, \ldots, n)$ where $\gamma_2 = \gamma_n = 0$ and the remaining $\gamma_k$ are arbitrary scalars.

**Proof.** ($\Leftarrow$) By the hypothesis, $\delta(e_{21}) = \delta(e_{n,n-1}) = 0$. Since $\delta$ is a derivation of $A$, we can get

$$
\delta(e_{k,k-2}) = \delta(e_{k,k-1}e_{k-1,k-2}) \\
= \delta(e_{k,k-1})e_{k-1,k-2} + e_{k,k-1}\delta(e_{k-1,k-2}) \\
= \gamma_k e_{n1} e_{k-1,k-2} + e_{k,k-1}\gamma_{k-1} e_{n1} = 0.
$$

So, $\delta(e_{kj}) = 0$ for $j < k - 1$. Thus, $\delta$ is a *uaz*-derivation.

($\Rightarrow$) i) If $k = 2$ or $n$, then $\gamma_2 = \gamma_n = 0$ by hypothesis.

ii) Assume $2 < k < n$.

Since $\delta$ is a strongly nilpotent derivation, let $\delta(e_{k,k-1}) = t_k$ with $t_k \in A^2$ and $t_k \in C_{k-1} \cap R_k$ by the Theorem 2.5.

Now $0 = \delta(e_{k1}) = \delta(e_{k,k-1}e_{k-1,1}) = \delta(e_{k,k-1})e_{k-1,1} = t_k e_{k-1,1}$. So, $(k-1)$-th column of the matrix $t_k = 0$.

For $1 < j < k - 1$, $0 = \delta(e_{k,k-1}e_{j1}) = \delta(e_{k,k-1})e_{j1} = t_k e_{j1}$. So, the $j$-th column of $t_k = 0$ for all $1 < j < k - 1$.

Thus, the $j$-th column of $t_k = 0$ for all $1 < j \leq k - 1$.

On the other hand, $0 = \delta(e_{nk}e_{k,k-1}) = e_{nk}\delta(e_{k,k-1}) = e_{nk}t_k$. So, the $k$-th row of $t_k = 0$. Also, for $n > j > k$, $0 = \delta(e_{nj}e_{k,k-1}) = e_{nj}\delta(e_{k,k-1}) = e_{nj}t_k$. So, $j$-th row of $t_k = 0$, for all $k < j < n$.

Thus, $j$-th row of $t_k = 0$, for all $k \geq j < n$.

Therefore, $t_k = \gamma_k e_{n1}$.

**Theorem 4.3.** Let $s_t$ be a strongly nilpotent derivation of $A$. Then $s_t = s_i + s_{uaz}$ where $s_i$ is an inner derivation and $s_{uaz}$ is a *uaz*-derivation.
Proof. It is enough to show that for a strongly nilpotent derivation $s_t$ there exist an inner derivation $s_i$ such that $s_t - s_i$ is a uaz-derivation.

By (*) and the hypothesis, we can set

$$s_t(e_{k1}) = \sum_{p > k} \alpha_{pk} e_{p1} \quad (k = 2, \ldots, n - 1).$$

The scalars $\alpha_{pk}$ are thus defined for $n \geq p > k > 1$ and $s_t(e_{k1}) = (\sum_{p > k} \alpha_{pq} e_{pq}) e_{k1} = [a, e_{k1}]$, where $a = \sum_{p > q > 1} \alpha_{pq} e_{pq}$.

So, the inner derivation $s_a(x) = [a, x]$ has the property $s_1 \equiv s_t - s_a$ maps $e_{k1}$ to zero.

Since $s_1$ is strongly nilpotent, $s_1(e_{nk}) = \sum_{q < k} \beta_{kq} e_{nq}$. And since for all $p(1 < p < k), e_{nk} e_{p1} = 0$. So, we have $0 = s_1(e_{nk} e_{p1}) = s_1(e_{nk}) e_{p1} + e_{nk} s_1(e_{p1}) = s_1(e_{nk}) e_{p1} = (\sum_{q < k} \beta_{kq} e_{nq}) e_{p1} = \beta_{kp} e_{n1}$.

It follows that the coefficients $\beta_{kp}$ are zero except possibly for $\beta_{k1}(k = 2, \ldots, n - 1)$. So, $s_1(e_{nk}) = \beta_{k1} e_{n1} = e_{nk}(\sum_{j=2}^{n-1} \beta_{j1} e_{j1})$.

Let $-b = \sum_{j=2}^{n-1} \beta_{j1} e_{j1}$. Then the inner derivation $s_b(x) = [b, x]$ has the property $(s_1 - s_b)e_{nl} = 0(l = 1, \ldots, n - 1)$ and $(s_1 - s_b)e_{k1} = -s_b(e_{k1}) = 0(k = 2, \ldots, n)$.

Therefore, $s_1 - s_b = s_t - (s_a + s_b)$ is a uaz-derivation and $s_a + s_b$ is an inner derivation.

\[\Box\]

Corollary 4.4. Let $\delta$ be a derivation of $A$. Then $\delta = i_d + \delta + s_i + s_{uaz}$ where $i_d$ is a diagonal inner, $\delta$ is a trivial extension of $K$, $s_i$ is an inner derivation and $s_{uaz}$ is a uaz-derivation.

The left(right) annihilators of $A$ are the quantities $e_{21}, e_{31}, \ldots, e_{n1}$ ($e_{n1}, e_{n2}, \ldots, e_{n,n-1})$ and their linear combinations.

Definition 4.5. A strongly nilpotent derivation $\delta$ of $A$ is called an az-derivation (annihilator zero derivation) if $\delta(a) = 0$ for every absolute left or right divisor of zero $a$.

It is obvious that an az-derivation is a uaz-derivation. Moreover, for a $K$-derivation, an az-derivation is equal to a uaz-derivation.

In general, every derivation cannot be expressed as a sum of diagonal, trivial extension, inner and az-derivations. The derivation given in the next example is a uaz-derivation, but not an az-derivation.

Example 4.6. Let

$$A = \begin{pmatrix}
0 & 0 & 0 \\
\mathbb{Z}[X] & 0 & 0 \\
\mathbb{Z}[X] & \mathbb{Z}[X] & 0
\end{pmatrix}$$
where $\mathbb{Z}[X]$ is a polynomial ring over an integer $\mathbb{Z}$.

Define $\delta : A \rightarrow A$ by

$$
\delta(\sum f_{ij}e_{ij})(i > j) = \frac{d}{dx} f_{21} e_{31}.
$$

Then $\delta$ is strongly nilpotent and inner part of $\delta$ is 0, that is, $\delta$ is a \textit{uaz}-derivation. But $\delta \neq az$-derivation.

References


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