ON STABILITY OF THE FUNCTIONAL EQUATIONS HAVING RELATION WITH A MULTIPLICATIVE DERIVATION

EUN HWI LEE, ICK-SOON CHANG, AND YONG-SOO JUNG

ABSTRACT. In this paper we study the Hyers-Ulam-Rassias stability of the functional equations related to a multiplicative derivation.

1. Introduction

In 1940, the stability problem of functional equations has originally been stated by S. M. Ulam [26]. As an answer to the problem of Ulam, D. H. Hyers has proved the stability of the linear functional equation [8] in 1941, which states that if $\delta > 0$ and $f : X \to Y$ is mapping with $X, Y$ Banach spaces, such that

$$\|f(x + y) - f(x) - f(y)\| \leq \delta$$

(1.1)

for all $x, y \in X$, then there exists a unique additive mapping $T : X \to Y$ such that

$$\|f(x) - T(x)\| \leq \delta$$

for all $x, y \in X$.

In such a case, the additive functional equation $f(x + y) = f(x) + f(y)$ is said to have the Hyers-Ulam stability property on $(X, Y)$. This terminology is applied to all kinds of functional equations which have been studied by many authors (for instance, [9]-[11], [17]-[23]).

In 1978, Th. M. Rassias [17] succeeded in generalizing the Hyers’ result by weakening the condition for the bound of the left side of the inequality (1.1). Due to the fact, the additive functional equation $f(x + y) = f(x) + f(y)$ is said to have the Hyers-Ulam-Rassias stability property on $(X, Y)$. Since then, a number of results concerning the stability of different functional equations can be found in [3, 4, 5, 7, 9, 11, 14, 17].

Received September 20, 2006.
2000 Mathematics Subject Classification. 39B52, 39B72.
Key words and phrases. Hyers-Ulam-Rassias stability, multiplicative (Jordan) derivation.
We now consider functional equations which define multiplicative derivations and multiplicative Jordan derivations in algebras:

\begin{align}
(1.2) & \quad d(xy) = xd(y) + yd(x), \\
(1.3) & \quad g(x^2) = 2xg(x). \\
\end{align}

It is immediate to observed that the real-valued function \( f(x) = x \ln x \) is a solution of the functional equations (1.2) and (1.3).

During the 34-th International Symposium on Functional Equations, Gy. Maksa [1] posed the Hyers-Ulam stability problem for the functional equation (1.2) on the interval (0,1]. The first result concerning the superstability of this equation for functions between operator algebras was obtained by P. Šemrl [24]. On the other hand, Zs. Páles [16] remarked that the functional equation (1.2) for real-valued functions on \([1, \infty)\) is stable in the sense of Hyers and Ulam. In 1997, C. Borelli [2] demonstrated the stability of the equation (1.2). In particular, J. Tabor gave an answer to the question of Maksa in [25].

Here we introduce the next functional equation due to the functional equation (1.3):

\begin{align}
(1.4) & \quad h(rx^2 + 2x) = 2rxh(x) + 2h(x),
\end{align}

where \( r \) is a nonzero real number, and consider the following functional equation motivated by the functional equation (1.2):

\begin{align}
(1.5) & \quad h(x + y + rxy) = h(x) + h(y) + rxh(y) + ryh(x),
\end{align}

where \( r \) is a nonzero real number.

The purpose of this paper is to solve the functional equation (1.4), (1.5) and investigate the Hyers-Ulam-Rassias stability of the functional equation (1.4), (1.5), respectively.

2. Stability of Eq. (1.4) and Eq. (1.5)

It is easy to see that the real-valued function \( f(x) = (rx+1) \ln(rx+1) \), where \( r \) is a nonzero real number, is a solution of the functional equation (1.4) on the interval. Now we are ready to find out the general solution of the functional equation (1.4).

**Theorem 2.1.** Let \( X \) be a real (complex) vector space and \( r > 0 \). A function \( h : (-\frac{1}{r}, \infty) \rightarrow X \) satisfies the functional equation (1.4) for all \( x \in (-\frac{1}{r}, \infty) \) if and only if there exists a solution \( G : (0, \infty) \rightarrow X \) of the functional equation (1.3) such that

\[ h(x) = G(rx + 1) \]

for all \( x \in (-\frac{1}{r}, \infty) \).

**Proof.** Assume that a function \( h : (-\frac{1}{r}, \infty) \rightarrow X \) satisfies (1.4) for all \( x \in (-\frac{1}{r}, \infty) \). Then we can define the mapping \( G : (0, \infty) \rightarrow X \) by \( G(x) = h(\frac{x-1}{r}) \).
So we get
\[ G(x^2) = h\left(\frac{x^2 - 1}{r}\right) = h\left(r \left(\frac{x - 1}{r}\right)^2 + 2\left(\frac{x - 1}{r}\right)\right) \]
\[ = 2r\left(\frac{x - 1}{r}\right)h\left(\frac{x - 1}{r}\right) + 2h\left(\frac{x - 1}{r}\right) \]
\[ = 2xG(x) \]
for all \( x \in (0, \infty) \). Therefore \( G \) is a solution of the functional equation (1.3), as desired, and \( h(x) = G(rx + 1) \) for all \( x \in \left(-\frac{1}{r}, \infty\right) \).

The converse is obvious. \( \square \)

We here present the general solution of the functional equation (1.5).

**Theorem 2.2.** Let \( X \) be a real (complex) vector space and \( r > 0 \). A function \( h : (-\frac{1}{r}, \infty) \to X \) satisfies the functional equation (1.5) for all \( x \in (-\frac{1}{r}, \infty) \) if and only if there exists a solution \( D : (0, \infty) \to X \) of the functional equation (1.2) such that
\[ h(x) = D(rx + 1) \]
for all \( x \in (-\frac{1}{r}, \infty) \).

**Proof.** The arguments used in Theorem 2.1 carry over almost verbatim. \( \square \)

In particular, the previous two theorems hold for the case \( r < 0 \). Throughout this paper, \( \mathbb{R}^+ \) denotes the set of all nonnegative real numbers and \( X \) a real Banach space with the norm \( |\cdot| \).

**Theorem 2.3.** [15, Theorem 2.1] Let \( f : [c, \infty) \to X \) be a given function for some \( c \geq 1 \) and let \( \varphi : [c, \infty) \to \mathbb{R}^+ \) be a function such that
\[ |f(x^2) - 2xf(x)| \leq \varphi(x) \]
for all \( x \in [c, \infty) \). If the series \( \sum_{i=1}^{\infty} 2^{-i} \varphi(x^{2^{i-1}}) \) converges, then there exists a unique solution \( g : [c, \infty) \to X \) of equation (1.3) such that
\[ |f(x) - g(x)| \leq \sum_{i=1}^{\infty} 2^{-i} \varphi(x^{2^{i-1}}) \]
for all \( x \in [c, \infty) \).

**Theorem 2.4.** Let \( f : [0, \infty) \to X \) be a given function and \( r > 0 \). Assume that \( \varphi : [0, \infty) \to \mathbb{R}^+ \) is a function such that
\[ |f(rx^2 + 2x) - 2rxf(x) - 2f(x)| \leq \varphi(x) \]
for all \( x \in [0, \infty) \). If the series \( \sum_{i=1}^{\infty} 2^{-i} \varphi((rx+1)^{2^{i-1}}) \) converges, then there exists a unique solution \( h : [0, \infty) \to X \) of equation (1.4) such that
\[ |f(x) - h(x)| \leq \sum_{i=1}^{\infty} 2^{-i} \varphi((rx+1)^{2^{i-1}}) \]
for all \( x \in [0, \infty) \).
Proof. Now put \( x = \frac{t-1}{r} \) in (2.3) to obtain
\[
\left| f\left(\frac{t^2-1}{r}\right) - 2tf\left(\frac{t-1}{r}\right) \right| \leq \varphi\left(\frac{t-1}{r}\right).
\]

Let us define functions \( e, \psi : [1, \infty) \to X \) by
\[
e(t) = f\left(\frac{t-1}{r}\right), \quad \psi(t) = \varphi\left(\frac{t-1}{r}\right).
\]

Then, by Theorem 2.3, there exists a unique solution \( g : [1, \infty) \to X \) of equation (1.3) such that
\[
|e(t) - g(t)| \leq \sum_{i=1}^{\infty} 2^{-i}\psi(t^{2^{i-1}})
\]
for all \( t \in [1, \infty) \). Since \( t = rx + 1 \), we have
\[
|f(x) - g(rx + 1)| \leq \sum_{i=1}^{\infty} 2^{-i}\varphi\left(\frac{(rx + 1)^{2^{i-1}} - 1}{r}\right).
\]

Hence we can define a function \( h : [0, \infty) \to X \) by \( h(x) = g(rx + 1) \), and so
\[
h(rx^2 + 2x) = g((rx + 1)^2) = 2(rx + 1)g(rx + 1) = 2rxg(rx + 1) + 2g(rx + 1) = 2rxh(x) + 2h(x).
\]

The proof of the theorem is complete. \( \square \)

The following two corollaries are immediate consequences of Theorem 2.1.

**Corollary 2.5.** Let \( f : [0, \infty) \to \mathbb{R} \) be a given function and \( r > 0 \). Assume that \( \Delta : [0, \infty)^2 \to \mathbb{R}^+ \) is a function such that for any \( x, y \in [0, \infty) \),
\[
|f(x + y + rxy) - f(x) - f(y) - rxf(y) - ryf(x)| \leq \Delta(x, y).
\]

If the series
\[
\sum_{i=1}^{\infty} 2^{-i}\Delta\left(\frac{(rx + 1)^{2^{i-1}} - 1}{r}, \frac{(rx + 1)^{2^{i-1}} - 1}{r}\right)
\]
converges and
\[
2^{-n}\Delta\left(\frac{(rx + 1)^{2^n} - 1}{r}, \frac{(ry + 1)^{2^n} - 1}{r}\right)
\]
converges to zero for all \( x \in [0, \infty) \) then there exists a unique solution \( h : [0, \infty) \to \mathbb{R} \) of equation (1.5) such that
\[
|f(x) - h(x)| \leq \sum_{i=1}^{\infty} 2^{-i}\Delta\left(\frac{(rx + 1)^{2^{i-1}} - 1}{r}, \frac{(rx + 1)^{2^{i-1}} - 1}{r}\right)
\]
for all \( x \in [0, \infty) \).
Proof. For \( x = y \) in (2.5), we have
\[
|f(rx^2 + 2x) - 2rx f(x) - 2f(x)| \leq \Delta(x, x).
\]
Putting \( \varphi(x) = \Delta(x, x) \) and applying Theorem 2.4, one obtains
\[
h(x) = g(rx + 1) = \lim_{n \to \infty} \frac{f\left(\frac{(rx+1)^{2^n} - 1}{r}\right)}{2^n(rx + 1)^{2^n-1}}
\]
satisfying (2.6). We claim that \( h \) satisfies
\[
h(x + y + rxy) = h(x) + h(y) + rxh(y) + ryh(x).
\]

Note that
\[
(2.7) \quad f\left(\frac{(rx+1)^{2^n} (ry + 1)^{2^n} - 1}{r}\right) = f\left(\frac{(rx+1)^{2^n} - 1}{r}\right) + (\frac{ry + 1}{r})^{2^n-1} + r \cdot (\frac{rx+1}{r})^{2^n-1} \cdot (\frac{ry + 1}{r})^{2^n-1}.
\]

In the inequality (2.5), replace \( x \) by \( \frac{(rx+1)^{2^n} - 1}{r} \), \( y \) by \( \frac{(ry+1)^{2^n} - 1}{r} \) and consider the equality (2.7) to find that
\[
(2.8) \quad \left| f\left(\frac{(rx+1)^{2^n} (ry + 1)^{2^n} - 1}{r}\right) - (\frac{rx+1}{r})^{2^n-1} f\left(\frac{(rx+1)^{2^n} - 1}{r}\right) 
- (rx+1)^2 \cdot f\left(\frac{(ry + 1)^{2^n} - 1}{r}\right) - r x (rx+1)^2 \cdot f\left(\frac{(ry + 1)^{2^n} - 1}{r}\right)
- r y (ry+1)^2 \cdot f\left(\frac{(rx+1)^{2^n} - 1}{r}\right) \right| \leq \Delta\left(\frac{(rx+1)^{2^n} - 1}{r}, \frac{(ry + 1)^{2^n} - 1}{r}\right).
\]

Now if we divide the inequality (2.8) by \( 2^n (rx + 1)^{2^n-1} (ry + 1)^{2^n-1} \), then, since
\[
\frac{1}{2^n (rx + 1)^{2^n-1} (ry + 1)^{2^n-1}} \leq 1,
\]
we get
\[
\frac{1}{2^n (rx + 1)^{2^n-1} (ry + 1)^{2^n-1}} f\left(\frac{(rx+1)^{2^n} (ry + 1)^{2^n} - 1}{r}\right) 
- \frac{1}{2^n (rx + 1)^{2^n-1}} f\left(\frac{(rx+1)^{2^n} - 1}{r}\right) - \frac{1}{2^n (ry + 1)^{2^n-1}} \leq 1.
\]
\[ f\left(\frac{(ry + 1)2^n - 1}{r}\right) - \frac{rx}{2^n(ry + 1)^{2^n - 1}} f\left(\frac{(ry + 1)^{2^n} - 1}{r}\right) - \frac{ry}{2^n(rx + 1)^{2^n - 1}} f\left(\frac{(rx + 1)^{2^n} - 1}{r}\right) \leq 2^{-n} \Delta \left(\frac{(rx + 1)^{2^n} - 1}{r}, \frac{(ry + 1)^{2^n} - 1}{r}\right). \]

Taking the limit in the last inequality as \( n \to \infty \), we have
\[ h(x + y + rxy) - h(x) - h(y) - rxf(y) - ryf(x) = 0. \]

The proof of the corollary is complete. \( \square \)

**Corollary 2.6.** Let \( f : [0, \infty) \to X \) be a given function such that for some \( r > 0 \), \( \theta \geq 0 \) and \( p, q \leq 0 \),
\[ |f(x + y + rxy) - f(x) - f(y) - rxf(y) - ryf(x)| \leq \theta(x^p + y^q) \]
for all \( x, y \in [0, \infty) \). Then there exists a unique solution \( h : [0, \infty) \to X \) of equation (1.5) such that
\[ |f(x) - h(x)| \leq \sum_{i=1}^{\infty} 2^{-i\theta} \left[ \left(\frac{(rx + 1)^{2^i-1} - 1}{r}\right)^p + \left(\frac{(rx + 1)^{2^i-1} - 1}{r}\right)^q \right] \]
for all \( x \in [0, \infty) \).

**Proof.** Setting \( \Delta(x, y) = \theta(x^p + y^q) \) in the previous Corollary 2.5, we can obtain the desired result. \( \square \)

**Theorem 2.7.** [15, Theorem 2.5] Let \( f : (0, 1] \to X \) be a given function and let \( \varphi : (0, 1] \to \mathbb{R}^+ \) be a function satisfying
\[ |f(x^2) - 2xf(x)| \leq \varphi(x) \]
for all \( x \in (0, 1] \). If the series \( \sum_{i=0}^{\infty} 2^i \varphi(x^{2^{-i-1}}) \) converges, then there exists a unique solution \( h : (0, 1] \to X \) of the equation (1.3) such that
\[ |f(x) - h(x)| \leq \sum_{i=0}^{\infty} 2^i \varphi(x^{2^{-i-1}}) \]
for all \( x \in (0, 1] \).

**Theorem 2.8.** Let \( f : (-1/r, 0] \to X \) be a given function and let \( \varphi : (-1/r, 0] \to \mathbb{R}^+ \) be a function satisfying for some \( r > 0 \),
\[ |f(rx^2 + 2x) - 2rx^2f(x) - 2f(x)| \leq \varphi(x) \]
for all \( x \in (-1/r, 0] \). If the series \( \sum_{i=1}^{\infty} 2^i \varphi\left(\frac{(rx+1)^{2^i-1} - 1}{r}\right) \) converges, then there exists a unique solution \( h : (-1/r, 0] \to X \) of equation (1.4) such that
\[ |f(x) - h(x)| \leq \sum_{i=0}^{\infty} 2^i \varphi\left(\frac{(rx+1)^{2^i-1} - 1}{r}\right) \]
for all \(x \in (-1/r, 0]\).

Proof. As the proof of Theorem 2.4, if we set \(t = rx + 1\) in (2.12), then we have

\[
|f\left(\frac{t^2 - 1}{r}\right) - 2tf\left(\frac{t - 1}{r}\right)| \leq \varphi\left(\frac{t - 1}{r}\right).
\]

Define \(e, \psi : (0, 1] \to X\) by

\[
e(t) = f\left(\frac{t - 1}{r}\right), \quad \psi(t) = \varphi\left(\frac{t - 1}{r}\right).
\]

Then, by Theorem 2.7, there exists a unique solution \(d : (0, 1] \to X\) of the equation (1.3) such that

\[
|e(t) - d(t)| \leq \sum_{i=0}^{\infty} 2^i \varphi\left(\frac{t^{2^{-i-1}} - 1}{r}\right),
\]

where

\[
d(t) = \lim_{n \to \infty} 2^nt^{1-2^{-n}}f\left(\frac{t^{2^{-n}} - 1}{r}\right).
\]

Since \(e(t) = f\left(\frac{t-1}{r}\right)\) and \(t = rx + 1\),

\[
|f(x) - d(rx + 1)| \leq \sum_{i=0}^{\infty} 2^i \varphi\left(\frac{(rx + 1)^{2^{-i}} - 1}{r}\right).
\]

Now we can define \(h : (-1/r, 0] \to X\) by \(h(x) = d(rx + 1)\). Then

\[
\begin{align*}
    h(rx^2 + 2x) &= d((rx + 1)^2) = 2(rx + 1)d(rx + 1) \\
    &= 2rxd(rx + 1) + 2d(rx + 1) = 2rxh(x) + 2h(x),
\end{align*}
\]

which completes the proof. \(\square\)

Corollary 2.9. Let \(f : (-1/r, 0] \to \mathbb{R}\) be a given function and let \(\Delta : (-1/r, 0]^2 \to \mathbb{R}^+\) be a function satisfying for some \(r > 0\),

(2.14) \[|f(x + y + rxy) - f(x) - f(y) - rxf(y) - ryf(x)| \leq \Delta(x, y)\]

for all \(x, y \in (-1/r, 0]\). If the series

\[
\sum_{i=1}^{\infty} 2^i \Delta\left(\frac{(rx + 1)^{2^{-i-1}} - 1}{r}, \frac{(ry + 1)^{2^{-i-1}} - 1}{r}\right)
\]

converges and

\[
2^n \Delta\left(\frac{(rx + 1)^{2^{-i}} - 1}{r}, \frac{(ry + 1)^{2^{-i}} - 1}{r}\right)
\]

converges to zero, then there exists a unique solution \(h : (-1/r, 0] \to \mathbb{R}\) of equation (1.5) such that

(2.15) \[|f(x) - h(x)| \leq \sum_{i=1}^{\infty} 2^i \Delta\left(\frac{(rx + 1)^{2^{-i-1}} - 1}{r}, \frac{(ry + 1)^{2^{-i-1}} - 1}{r}\right)\]

for all \(x \in (-1/r, 0].\)
Proof. For $y = x$ in (2.14), we have

$$|f(rx^2 + 2x) - 2rx f(x) - 2f(x)| \leq \Delta(x, x).$$

Putting $\varphi(x) = \Delta(x, x)$ and applying Theorem 2.8, one obtains

$$h(x) = \lim_{n \to \infty} 2^n (rx + 1)^{1-2^{-n}} f\left(\frac{(rx + 1)^{2^{-n}} - 1}{r}\right),$$

which satisfies (2.15). We claim that $h$ satisfies

$$h(x + y + rxy) = h(x) + h(y) + r x h(y) + ry h(x).$$

Observed that

$$f\left(\frac{(rx + 1)^{2^{-n}} (ry + 1)^{2^{-n}} - 1}{r}\right) = f\left(\frac{(rx + 1)^{2^{-n}} - 1}{r}\right)$$

$$+ \frac{(ry + 1)^{2^{-n}} - 1}{r} + r \cdot \frac{(rx + 1)^{2^{-n}} - 1}{r} \cdot \frac{(ry + 1)^{2^{-n}} - 1}{r}.$$}

Now replacing $x$ and $y$ by $\frac{(rx + 1)^{2^{-n}} - 1}{r}$ and $\frac{(ry + 1)^{2^{-n}} - 1}{r}$ in (2.14), then

$$|f\left(\frac{(rx + 1)^{2^{-n}} (ry + 1)^{2^{-n}} - 1}{r}\right) - (rx + 1)^{2^{-n}}|.$$}

$$f\left(\frac{(ry + 1)^{2^{-n}} - 1}{r}\right) - (ry + 1)^{2^{-n}} f\left(\frac{(rx + 1)^{2^{-n}} - 1}{r}\right)$$

$$- r x (rx + 1)^{2^{-n}} - 1 f\left(\frac{(ry + 1)^{2^{-n}} - 1}{r}\right) - ry (ry + 1)^{2^{-n}} - 1.$$}

$$f\left(\frac{(rx + 1)^{2^{-n}} - 1}{r}\right) \leq \Delta\left(\frac{(rx + 1)^{2^{-n}} - 1}{r}, \frac{(ry + 1)^{2^{-n}} - 1}{r}\right).$$

Multiplying in the last inequality by $2^n (rx + 1)^{1-2^{-n}} (ry + 1)^{1-2^{-n}} (\leq 1)$, we have

$$2^n (rx + 1)^{1-2^{-n}} (ry + 1)^{1-2^{-n}} f\left(\frac{(rx + 1)^{2^{-n}} (ry + 1)^{2^{-n}} - 1}{r}\right)$$

$$- 2^n (ry + 1)^{1-2^{-n}} f\left(\frac{(ry + 1)^{2^{-n}} - 1}{r}\right) - 2^n (rx + 1)^{1-2^{-n}}.$$}

$$f\left(\frac{(rx + 1)^{2^{-n}} - 1}{r}\right) - 2^n r x (ry + 1)^{1-2^{-n}} f\left(\frac{(ry + 1)^{2^{-n}} - 1}{r}\right)$$

$$- 2^n r y (rx + 1)^{1-2^{-n}} f\left(\frac{(rx + 1)^{2^{-n}} - 1}{r}\right)\big|$$

$$\leq 2^n \Delta\left(\frac{(rx + 1)^{2^{-n}} - 1}{r}, \frac{(ry + 1)^{2^{-n}} - 1}{r}\right).$$

Taking the limit in the last inequality as $n \to \infty$, one obtains

$$h(x + y + r xy) - h(x) - h(y) - r x h(y) - r y h(x) = 0.$$
This completes the proof of the theorem. □

**Example 1.** For some \( \theta, p \leq 0 \), let
\[
f(x) = (rx + 1) \ln(rx + 1) + \theta(rx + 1)^{p-1}, \quad x \leq 0, \quad r > 0.
\]

Note that
\[
|f(rx^2 + 2x) - 2rf(x) - 2f(x)| = \theta |2(rx + 1)^p - (rx + 1)^{2(p-1)}|.
\]

In Theorem 2.4 setting \( \varphi(x) = \theta |2(rx + 1)^p - (rx + 1)^{2(p-1)}| \), we obtain the desired mapping \( h(x) = (rx + 1) \ln(rx + 1) \) satisfying (1.4).

**Example 2.** Consider
\[
f(x) = (rx + 1) \ln(rx + 1) + (\ln(rx + 1))^2, \quad -\frac{1}{r} < x \leq 0, \quad r > 0.
\]

Then
\[
|f(rx^2 + 2x) - 2rf(x) - 2f(x)| = 2(\ln(rx + 1))^2 - 2rx(\ln(rx + 1))^2.
\]

Taking \( \varphi(x) = 2(\ln(rx + 1))^2 - 2rx(\ln(rx + 1))^2 \) in Theorem 2.8, we have the desired mapping \( h(x) = (rx + 1) \ln(rx + 1) \) satisfying (1.4).

**Acknowledgement.** The authors would like to thank referees for their valuable comments. The second author dedicates this paper to his late father.

**References**


EUN HWI LEE  
DEPARTMENT OF MATHEMATICS  
JEONJU UNIVERSITY  
JEONJU 302-729, KOREA  
E-mail address: ehhj@jju.ac.kr

ICK-SOON CHANG  
DEPARTMENT OF MATHEMATICS  
MOKWON UNIVERSITY  
TAEJON 302-729, KOREA  
E-mail address: ischang@mokwon.ac.kr

YONG-SOO JUNG  
DEPARTMENT OF MATHEMATICS  
CHUNGNAM NATIONAL UNIVERSITY  
TAEJON 305-764, KOREA  
E-mail address: ysjung@math.cnu.ac.kr