A GENERALIZATION OF
INSERTION-OF-FACTORS-PROPERTY

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ABSTRACT. We in this note introduce the concept of g-IFP rings which is a generalization of IFP rings. We show that from any IFP ring there can be constructed a right g-IFP ring but not IFP. We also study the basic properties of right g-IFP rings, constructing suitable examples to the situations raised naturally in the process.

1. Introduction

Throughout this paper all rings are associative with identity unless otherwise stated. Given a ring $R$ we use $J(R)$, $N_*(R)$, and $N(R)$ to represent the Jacobson radical, the prime radical (i.e., lower nilradical), and the set of all nilpotent elements in $R$, respectively; and $r_R(-)$ ($l_R(-)$) is used for the right (left) annihilator over $R$, i.e., $r_R(S) = \{ a \in R \mid sa = 0 \text{ for all } s \in S \}$ ($l_R(S) = \{ b \in R \mid bs = 0 \text{ for all } s \in S \}$), where $S \subseteq R$ or $S$ is a subset of a right (left) $R$-module. If $S = \{ a \}$ then we write $r_R(a)$ ($l_R(a)$) in place of $r_R(\{a\})$ ($l_R(\{a\})$). $a \in R$ is said to be right (left) regular if $r_R(a) = 0$ ($l_R(a) = 0$). $a \in R$ is called a left (right) zero-divisor if $r_R(a) \neq 0$ ($l_R(a) \neq 0$). A zero-divisor means an element that is neither right nor left regular.

In a commutative ring the set of nilpotent elements forms an ideal that coincides with the prime radical with the help of [7, Proposition 3.2.1]. This property is also possessed by certain noncommutative rings, which are called 2-primal. Shin [11, Proposition 1.11] proved that given a ring $R$, $N_*(R) = N(R)$ if and only if every minimal prime ideal $P$ of $R$ is completely prime (i.e., $R/P$ is a domain): Birkenmeier et al. [2] called such rings 2-primal; while Hirano [5] used the term N-ring for the concept.

A well-known property between “commutative” and “2-primal” is the insertion-of-factors-property (or simply IFP) due to Bell [1]; A right (or left) ideal $I$ of a ring $R$ is said to have the IFP if $ab \in I$ implies $aRb \subseteq I$ for $a,b \in R$. A ring $R$ is called IFP if the zero ideal of $R$ has the IFP. Shin [11] used the term SF for the IFP; while Habeb [4] used the term zero insertive (or simply zi) for it, in the study of QF-3 rings. IFP rings are also known as semicommutative in

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Narbonne’s paper [10]. Shin proved that IFP rings are 2-primal [11, Theorem 1.5]. A ring is called reduced if it has no nonzero nilpotent elements. It is trivial to check that reduced rings are IFP, whence the IFP condition is also between “reduced” and “2-primal”. It is trivial that subrings of IFP rings are also IFP, so we use this fact freely in this note.

We in this note introduce another generalization of the IFP condition that is different from the 2-primal condition. We in this note call a ring \( R \) right generalized IFP (or simply, right g-IFP) provided that there is \( 0 \neq b' \in R \) with \( aRb' = 0 \) whenever \( ab = 0 \) for \( a, b \in R \) with \( b \neq 0 \). The left g-IFP ring can be defined symmetrically. A ring is called g-IFP if it is both left and right g-IFP.

2. Basic structure and examples of right g-IFP rings

In this section we observe the ring-theoretic properties of g-IFP rings, and relationship between g-IFP rings and concerned concepts. Denote the set of all left (right) zero-divisors in a ring \( R \) by \( zd_l(R) \) \((zd_r(R))\). We start with the following lemma.

**Lemma 2.1.** For a ring \( R \) the following conditions are equivalent:

1. \( R \) is right g-IFP;
2. \( r_R(a) \) contains a nonzero ideal of \( R \) for each \( a \in zd_l(R) \);
3. \( r_R(aR) \neq 0 \) for each \( a \in zd_l(R) \).

**Proof.** (1)\(\Rightarrow\)(2): \( a \in zd_l(R) \) implies \( r_R(a) \neq 0 \), so \( r_R(a) \) contains a nonzero left ideal of \( R \), say \( Rb \), if \( R \) is right g-IFP. Thus \( 0 = aRb = aRbR \) and \( RbR \subseteq r_R(a) \).

(2)\(\Rightarrow\)(3): If \( r_R(a) \) contains a nonzero ideal \( I \) of \( R \) then \( 0 = aI = aRI \) and \( I \subseteq r_R(aR) \).

(3)\(\Rightarrow\)(1): Let \( r_R(a) \neq 0 \) for \( a \in R \). Then we get \( r_R(aR) \neq 0 \) by the condition, so \( R \) is right g-IFP. \( \square \)

IFP rings are clearly g-IFP but the converse need not hold by the following. The example below also shows that the g-IFP condition is not left-right symmetric. Given a ring \( R \) we use \( R[[x]] \) \((R[[x]])\) to denote the polynomial (power series) ring with an indeterminate \( x \) over \( R \).

**Example 2.2.** Let \( D \) be a division ring and let \( T = D[x]/(x^2) \), where \( (x^2) \) is the ideal of \( D[x] \) generated by \( x^2 \). Write \( \delta = x + (x^2) \). Then \( T = D \oplus D\delta \) with \( \delta^2 = 0 \) and each element of the form \( a + b\delta \) is invertible when \( a \) is nonzero. Now consider the ring \( R = \begin{pmatrix} T/D\delta & T/D\delta \\ 0 & T \end{pmatrix} \). Notice that all nonzero proper ideals of \( R \) are

\[
I_1 = \begin{pmatrix} 0 & T/D\delta \\ 0 & D\delta \end{pmatrix}, \quad I_2 = \begin{pmatrix} 0 & T/D\delta \\ 0 & T \end{pmatrix}, \quad I_3 = \begin{pmatrix} 0 & T/D\delta \\ 0 & 0 \end{pmatrix}, \quad I_4 = \begin{pmatrix} 0 & 0 \\ 0 & D\delta \end{pmatrix}, \quad I_5 = \begin{pmatrix} T/D\delta & T/D\delta \\ 0 & 0 \end{pmatrix} \text{ and } I_6 = \begin{pmatrix} T/D\delta & T/D\delta \\ 0 & D\delta \end{pmatrix}.
\]
It is easily checked that each $I_i$ is the set of zero divisors for all $1 \leq i \leq 6$. Note that every left zero-divisor of $R$ is contained in $I_k$ for some $k \in \{1, 2, \ldots, 6\}$, and that $r_R(I_j) \neq 0$ for all $j$. Thus $R$ is right $g$-IFP by Lemma 2.1.

Next we show that $R$ is not left $g$-IFP. Note \[
\begin{pmatrix}
1 & 0 \\
0 & 0
\end{pmatrix}
\begin{pmatrix}
0 & 0 \\
0 & 1
\end{pmatrix} = 0
\]
and \[
R
\begin{pmatrix}
0 & 0 \\
0 & 1
\end{pmatrix} = \begin{pmatrix}
0 & T/D\delta \\
0 & T
\end{pmatrix}.
\]
Let \[
\begin{pmatrix}
a & b \\
b & c + d\delta
\end{pmatrix} \in l_R
\begin{pmatrix}
0 & T/D\delta \\
0 & T
\end{pmatrix}.
\]
From \[
\begin{pmatrix}
a & b \\
b & c + d\delta
\end{pmatrix}
\begin{pmatrix}
0 & 0 \\
0 & 1
\end{pmatrix} = 0,
\]
we get $b = 0$ and $c + d\delta = 0$; hence \[
\begin{pmatrix}
a & 0 \\
0 & 0
\end{pmatrix}
\begin{pmatrix}
0 & 0 \\
0 & 1
\end{pmatrix} = 0
\]
forges $a = 0$. Thus \[
l_R
\begin{pmatrix}
0 & T/D\delta \\
0 & T
\end{pmatrix}
\]
is not left $g$-IFP by Lemma 2.1.

A ring is called abelian if every idempotent is central. It is trivial to check that IFP rings are abelian, but right or left $g$-IFP rings need not be abelian by Example 2.2.

A ring $R$ is called directly finite if $xy = 1$ implies $yx = 1$ for $x, y \in R$. It is trivial to check that abelian rings are directly finite, and 2-primal rings are also directly finite by [2, Proposition 2.10]. A ring $R$ is called von Neumann regular if for each $a \in R$ there exists $x \in R$ such that $a = axa$. Abelian von Neumann regular rings are reduced (hence $g$-IFP) by [3, Theorem 3.2].

**Lemma 2.3.** (1) Right or left $g$-IFP rings are directly finite.

(2) Direct sums (possibly without identity) and direct products of right $g$-IFP rings are also right $g$-IFP.

**Proof.** (1) Let $R$ be a right $g$-IFP ring. Assume that $xy = 1$ but $yx \neq 1$ for some $x, y \in R$. Then $yx$ is a non-identity idempotent and $yx(1 - yx) = 0$ with $1 - yx \neq 0$. Since $R$ is right $g$-IFP, we have $yxRb = 0$ for some nonzero $b \in R$; but $xRb = xyxRb = 0$ implies $0 \neq b = xyb \in xRb = 0$, a contradiction. Thus $R$ is directly finite. The proof of left case is similar.

(2) Suppose that $R_i$ $(i \in I)$ are right $g$-IFP rings, and let $R = \prod_{i \in I} R_i$ be the direct product of $R_i$’s. Set $ab = 0$ for $a = (a_i), b = (b_i) \in R$. Then $a_i b_i = 0$ for all $i \in I$. Since each $R_i$ is right $g$-IFP, we get $a_i R_i b'_i = 0$ for some $0 \neq b'_i \in R_i$. Let $b' = (b'_i) \in R$, then we have $aRb' = (a_i R_i b'_i) = 0$ for some $0 \neq b' \in R$, showing that $R$ is right $g$-IFP. The case of direct sums is similar.

**Remark.** As a byproduct of Lemma 2.3(1) we get that von Neumann regular rings need not be one-sided $g$-IFP. Let $F$ be a field and $R$ be the column finite infinite matrix ring over $F$. Note that $R$ is von Neumann regular. Let $a \in R$ be the matrix with $(i, i + 1)$-entry 1 and zero elsewhere, and $b \in R$ be the matrix with $(i + 1, i)$-entry 1 and zero elsewhere, where $i = 1, 2, \ldots$. Then $ab = 1$ but
ba \neq 1; hence R is not directly finite. Thus R is neither right nor left g-IFP by Lemma 2.3(1).

Note that IFP rings are both g-IFP and 2-primal, however one of the classes of g-IFP rings and 2-primal rings need not contain the other as can be seen by the following.

**Example 2.4.** (1) There is a g-IFP ring that is not 2-primal. Let $K$ be a field and $D_n = K\{x_n\}$ with relation $x_n^{n+2} = 0$, where $n$ is any nonnegative integer and $K\{x_n\}$ is the free algebra generated by $x_n$ over $K$. Note $D_n \cong K[x]/\langle x^{n+2} \rangle$ where $\langle x^{n+2} \rangle$ is the ideal of $K[x]$ generated by $x^{n+2}$. We use the ring in [6, Example 1.6]. Define $R_n = \begin{pmatrix} D_n & D_n x_n \\ D_n x_n & D_n \end{pmatrix}$. Notice that $J(R_n) = \begin{pmatrix} D_n x_n & D_n x_n \\ D_n x_n & D_n x_n \end{pmatrix}$ and $\frac{R_n}{J(R_n)} \cong \begin{pmatrix} K & 0 \\ 0 & K \end{pmatrix}$; hence $(f_1, f_2) \in R_n$ is invertible when the constants of $f_1$ and $f_4$ are both nonzero.

Now we will show that $R_n$ is g-IFP. Let $0 \neq \begin{pmatrix} f_1 & f_2 \\ f_3 & f_4 \end{pmatrix} \in R_n$ with $f_i \in D_n x_n$ for all $i$, and say that the smallest degree of nonzero $f_i$'s is $k$ for some $k$ with $1 \leq k < n + 2$. Then $\begin{pmatrix} f_1 & f_2 \\ f_3 & f_4 \end{pmatrix} R_n \begin{pmatrix} x_n^{n+2-k} & 0 \\ 0 & x_n^{n+2-k} \end{pmatrix} = 0$ with

$\begin{pmatrix} x_n^{n+2-k} & 0 \\ 0 & x_n^{n+2-k} \end{pmatrix} \neq 0$. Let $0 \neq \begin{pmatrix} f_1 & f_2 \\ f_3 & f_4 \end{pmatrix} \in R_n$ with $f_1 \notin D_n x_n$ and $f_i \in D_n x_n$ for $i \in \{2, 3, 4\}$. Then $\begin{pmatrix} f_1 & f_2 \\ f_3 & f_4 \end{pmatrix} R_n \begin{pmatrix} 0 & 0 \\ 0 & x_n^{n+1} \end{pmatrix} = 0$ because each matrix in $\begin{pmatrix} f_1 & f_2 \\ f_3 & f_4 \end{pmatrix} R_n$ is of the form $\begin{pmatrix} f & h \\ g & k \end{pmatrix}$ with $h, k \in D_n x_n$. Next let $0 \neq \begin{pmatrix} f_1 & f_2 \\ f_3 & f_4 \end{pmatrix} \in R_n$ with $f_4 \notin D_n x_n$ and $f_i \in D_n x_n$ for $i \in \{1, 2, 3\}$. Then we have $\begin{pmatrix} f_1 & f_2 \\ f_3 & f_4 \end{pmatrix} R_n \begin{pmatrix} x_n^{n+1} & 0 \\ 0 & 0 \end{pmatrix} = 0$ because each matrix in $\begin{pmatrix} f_1 & f_2 \\ f_3 & f_4 \end{pmatrix} R_n$ is of the form $\begin{pmatrix} f & h \\ g & k \end{pmatrix}$ with $f, g \in D_n x_n$.

Thus $R_n$ is right g-IFP. We can also show that $R_n$ is left g-IFP by a similar method.

Next let $R = \prod_{n=0}^{\infty} R_n$. Then $R$ is g-IFP by Lemma 2.3(2) since every $R_n$ is g-IFP. Consider two sequences $(a_n), (b_n) \in R$ such that $a_n = \begin{pmatrix} 0 & x_n \\ 0 & 0 \end{pmatrix}$ and $b_n = \begin{pmatrix} 0 & 0 \\ x_n & 0 \end{pmatrix}$ for all $n$. Then $(a_n), (b_n) \in N(R)$ since $(a_n)^2 = 0 = (b_n)^2$; but $(a_n) \notin N_+(R)$ or $(b_n) \notin N_+(R)$ since each component of $(a_n) + (b_n)$ is $\begin{pmatrix} 0 & x_n \\ x_n & 0 \end{pmatrix}$ and $(a_n) + (b_n)$ is not nilpotent. Thus $R$ is not 2-primal.
(2) There is a 2-primal ring that is neither right nor left g-IFP. Consider the 2 by 2 upper triangular ring \( R = \begin{pmatrix} \mathbb{Z}_2 & \mathbb{Z}_2 \\ 0 & \mathbb{Z}_2 \end{pmatrix} \) over the field \( \mathbb{Z}_2 \) of integers modulo 2. Consider two proper ideals \( I = R \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \) and \( J = R \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \) of \( R \), where \( \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \in zd_l(R) \) and \( \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \in zd_r(R) \). Then every element of \( I \) and \( J \) is a zero-divisor. However there cannot exist nonzero elements \( x, y \in R \) such that \( Ix = 0 \) and \( yJ = 0 \). Thus \( R \) is neither right nor left g-IFP by Lemma 2.1. But \( R \) is 2-primal by [2, Proposition 2.5].

Remark. (1) Subrings of right g-IFP rings need not be right g-IFP. Consider the right g-IFP ring \( R = \begin{pmatrix} T/D\delta & T/D\delta \\ 0 & T \end{pmatrix} \) with \( T = D \oplus D\delta \) in Example 2.2. Set \( D = \mathbb{Z}_2 \), the field of integers modulo 2. Then \( \begin{pmatrix} \mathbb{Z}_2 & \mathbb{Z}_2 \\ 0 & \mathbb{Z}_2 \end{pmatrix} \) is a subring of \( R \). But \( \begin{pmatrix} \mathbb{Z}_2 & \mathbb{Z}_2 \\ 0 & \mathbb{Z}_2 \end{pmatrix} \) is not right g-IFP by Example 2.4(2).

(2) Direct products of 2-primal rings need not be 2-primal by Marks [9] and [8, Example 1.7]. We here can obtain this result as a byproduct of Example 2.4(1). In fact \( J(R_n) = P(R_n) \) since \( J(R_n) \) is nilpotent, so \( R_n \) is 2-primal. But \( \prod_{i=0}^{\infty} R_n \) is not 2-primal by Example 2.4(1).

Denote the \( n \) by \( n \) matrix ring over a ring \( R \) by \( \text{Mat}_n(R) \) for a positive integer \( n \). Let \( R \) be a simple ring and \( S = \text{Mat}_2(R) \). Take \( a = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \in S \).

Then \( a \) is nilpotent, but \( SaS = S \) and \( r_S(SaS) = 0 \), \( l_S(SaS) = 0 \). Thus \( S \) is neither right nor left g-IFP by Lemma 2.1. By a similar manner, \( \text{Mat}_n(R) \) cannot be neither right nor left g-IFP for all \( n \geq 2 \).

In the following we find a kind of subring of \( n \) by \( n \) matrix ring that can be right or left g-IFP. Given a ring \( R \) we consider a ring extension

\[
R_n = \begin{pmatrix} a & a_{12} & a_{13} & \cdots & a_{1n} \\ 0 & a & a_{23} & \cdots & a_{2n} \\ 0 & 0 & a & \cdots & a_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & a \end{pmatrix} \quad | \quad a, a_{ij} \in R
\]

integer. About \( R_n \) we have the following useful results:

(i) \( R_n \) is IFP for \( n \leq 3 \) by [6, Proposition 1.2] when \( R \) is a reduced ring;

(ii) \( R_n \) need not be IFP for \( n \geq 2 \) by [6, Example 1.3] when \( R \) is an IFP ring;

(iii) \( R_n \) cannot be IFP for \( n \geq 4 \) by [6, Example 1.3] over any ring \( R \).

With the help of these results and the following proposition, we can say that from any IFP ring there can be constructed a right g-IFP ring but not IFP.
\textbf{Proposition 2.5.} A ring $R$ is right g-IFP if and only if $R_n$ over $R$ is right g-IFP for any $n$.

\textit{Proof.} Suppose that $R$ is right g-IFP and let
\[
\begin{pmatrix}
a & a_{12} & \cdots & a_{1n} \\
0 & a & \cdots & a_{2n} \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & a
\end{pmatrix}
\begin{pmatrix}
b & b_{12} & \cdots & b_{1n} \\
0 & b & \cdots & b_{2n} \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & b
\end{pmatrix}
= 0 \text{ with }
\begin{pmatrix}
b & b_{12} & \cdots & b_{1n} \\
0 & b & \cdots & b_{2n} \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & b
\end{pmatrix} \neq 0
\]
in $R_n$. If $a = 0$ then we have
\[
\begin{pmatrix}
0 & a_{12} & \cdots & a_{1n} \\
0 & 0 & \cdots & a_{2n} \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & 0
\end{pmatrix}
R_n
\begin{pmatrix}
0 & 0 & \cdots & 1 \\
0 & 0 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & 0
\end{pmatrix}
= 0.
\]
Assume $a \neq 0$. By the condition $aRb' = 0$ for some nonzero $b' \in R$ since $ab = 0$. So
\[
\begin{pmatrix}
a & a_{12} & \cdots & a_{1n} \\
0 & a & \cdots & a_{2n} \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & a
\end{pmatrix}
R_n
\begin{pmatrix}
0 & 0 & \cdots & b' \\
0 & 0 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & 0
\end{pmatrix}
= \begin{pmatrix}
0 & 0 & \cdots & aRb' \\
0 & 0 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & 0
\end{pmatrix} = 0.
\]
Thus $R_n$ is right g-IFP.

Conversely assume that $R_n$ is right g-IFP and let $ab = 0$ for $a, b \in R$.
\[
\begin{pmatrix}
a & 0 & \cdots & 0 \\
0 & a & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & a
\end{pmatrix}
\begin{pmatrix}
b & 0 & \cdots & 0 \\
0 & b & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & b
\end{pmatrix}
= 0 \text{ and so we have } AR_nB = 0 \text{ for some nonzero } B \in R_n \text{ by the condition, where } A = \begin{pmatrix}
a & 0 & \cdots & 0 \\
0 & a & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & a
\end{pmatrix}.
\]
Say
\[
B = \begin{pmatrix}
b & b_{12} & \cdots & b_{1n} \\
0 & b & \cdots & b_{2n} \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & b
\end{pmatrix} \text{. If } b \neq 0 \text{ then } aRb = 0. \text{ Next set } b = 0 \text{ and say that } j \text{ and } k \text{ are smallest with respect to the property } b_{jk} \neq 0. \text{ Then since } AR_nB \text{ contains }
\begin{pmatrix}
0 & \cdots & 0 & arb_{jk} & \cdots & arb_{jn} \\
0 & \cdots & 0 & \cdots & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
0 & \cdots & 0 & \cdots & 0
\end{pmatrix}
\]
for all \( r \in R \), we also get \( aRb_{j,k} = 0 \) from \( AR_nB = 0 \). Thus \( R \) is right g-IFP. \( \square \)

We also have that \( R_n \) is 2-primal for any \( n \) if and only if \( R \) is a 2-primal ring, with the help of [2, Propositions 2.2 and 2.5].

**Proposition 2.6.** Let \( R \) be a local ring with nilpotent \( J(R) \). Then \( R \) is g-IFP.

*Proof.* Let \( k \) be smallest with respect to \( J(R)^k = 0 \). Put \( ab = 0 \) for \( a, b \in R \) with \( b \neq 0 \). Then \( a \in J(R) \) and we get \( aJ(R)^{k-1} = 0 \), concluding that \( R \) is right g-IFP. Similarly \( R \) is left g-IFP. \( \square \)

**Proposition 2.7.** Let \( R \) be a semiprime right (resp. left) g-IFP ring. Then every left (resp. right) zero-divisor is a zero-divisor.

*Proof.* Let \( r_R(a) \neq 0 \) for \( a \in R \). Since \( R \) is right g-IFP, there is a nonzero ideal \( I \) of \( R \) such that \( aI = 0 \) by Lemma 2.1. Then \( (IaR)^2 = IaRIaR = IaIaR = 0 \), but \( R \) is semiprime and so \( Ia = 0 \), showing \( r_R(a) \neq 0 \). The proof of the other case is similar. \( \square \)

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