NOTES ON A NON-ASSOCIATIVE ALGEBRA WITH
EXPONENTIAL FUNCTIONS II

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ABSTRACT. For the evaluation algebra $F[e^{\pm x}]_M$, if $M = \{\partial\}$, then
\[ \text{Der}_{non}(F[e^{\pm x}]_M) \]
of the evaluation algebra $F[e^{\pm x}]_M$ is found in the paper [15]. For $M = \{\partial, \partial^2\}$, we find $\text{Der}_{non}(F[e^{\pm x}]_M)$ of the evaluation algebra $F[e^{\pm x}]_M$ in this paper. We show that there is a non-associative algebra which is the direct sum of derivation invariant subspaces.

1. Preliminaries

Let $F$ be a field of characteristic zero (not necessarily algebraically closed). Throughout this paper, $\mathbb{N}$ and $\mathbb{Z}$ will denote the non-negative integers and the integers, respectively. Let $A$ be an associative algebra and $M = \{\delta | \delta \text{ is a mapping from } A \text{ to itself}\}$. The evaluation algebra $A_M = \{a\delta | a \in A, \delta \in M\}$ with the obvious addition and the multiplication $*$ is defined as follows:
\[ a_1 \delta_1 * a_2 \delta_2 = a_1 \delta_1(a_2) \delta_2 \]
for any $a_1 \delta_1, a_2 \delta_2 \in A_M$ [1], [2], [3], [10]. For $A_M$, if $M = \{id\}$, then the ring $A_M = A$ where id is the identity map of $A$. Note that $A_M = \langle A_M, +, * \rangle$ is not an associative ring generally. Using the commutator $[,]$ of $A_M$, we can define the semi-Lie ring $A_M[,] = \langle A_M, +, [\cdot, \cdot] \rangle$ [1]. If the Jacobi identity holds in $A_M[,]$, then $A_M[,]$ is a Lie ring [13]. Generally, $A_M$ is not a Lie ring, because of the Jacobi identity. Let $F[e^{\pm x_1}, e^{\pm x_2}, \ldots, e^{\pm x_n}]$ be a ring in the formal power series ring $F[[x_1, x_2, \ldots, x_n]]$ [5], [6]. If we take the subalgebra $F[e^{\pm x}]$ in $F[[x_1, x_2, \ldots, x_n]]$ and the map $M = \{\partial, \partial^2\}$, we have the simple evaluation algebra $F[e^{\pm x}]_M$ [4], [8], [9], [11], [12]. We know that $F[e^{\pm x}]_{\langle \partial, \partial^2 \rangle} = F[e^{\pm x}]_{\langle \partial \rangle} \oplus F[e^{\pm x}]_{\langle \partial^2 \rangle}$ such that $F[e^{\pm x}]_{\langle \partial \rangle}$ and $F[e^{\pm x}]_{\langle \partial^2 \rangle}$ are simple [1], [2], [15].
2. Derivations of $\mathbf{F}[e^{\pm x}]_M$

From now on, $M$ denotes the set $\{\partial, \partial^2\}$. For any element $e^{ax}\partial^n$ of $\mathbf{F}[e^{\pm x}]_M$, we define degree $\text{deg}(e^{ax}\partial^n)$ as $a$ where $u \in \{1, 2\}$. Throughout out the paper, using the degree, for any element $l$ in $\mathbf{F}[e^{\pm x}]_M$, the element $l$ can be written as follows:

$$
\begin{align*}
    l &= C(a_1, 1)e^{a_1x}\partial + C(a_2, 1)e^{a_2x}\partial + \cdots + C(a_r, 1)e^{a_rx}\partial \\
    &\quad + C(b_1, 2)e^{b_1x}\partial^2 + C(b_2, 2)e^{b_2x}\partial^2 + \cdots + C(b_s, 2)e^{b_sx}\partial^2
\end{align*}
$$

such that $a_1 > \cdots > a_r$ and $b_1 > \cdots > b_s$ with appropriate scalars. Thus we can define the order of elements of $\mathbf{F}[e^{\pm x}]_M$ obviously.

**Note 1.** Let $I = \{1, 2\}$. For any basis element $e^{mx}\partial^i$, $i \in I$, of the non-associative algebra $\mathbf{F}[e^{\pm x}]_M$, if we define $\mathbf{F}$-additive linear map $D_c$ of the non-associative algebra $\mathbf{F}[e^{\pm x}]_M$ as follows:

$$
D_c(e^{mx}\partial^i) = cm^i e^{mx}\partial^i
$$

then $D_c$ can be linearly extended to a derivation of the non-associative algebra $\mathbf{F}[e^{\pm x}]_M$ where $c \in \mathbf{F}$.

**Lemma 2.1.** Let $D$ be a derivation of $\mathbf{F}[e^{\pm x}]_M$. $D(\partial) = 0$ if and only if $D(\partial^2) = 0$.

**Proof.** Let $D$ be the derivation of $\mathbf{F}[e^{\pm x}]_M$. Let us assume that $D(\partial) = 0$. Since $\partial$ is the left (multiplicative) identity of $e^{x}\partial$, we have that $\partial * D(e^{x}\partial) = D(e^{x}\partial)$. Assume that $D(e^{x}\partial) = \sum c_{0, 1} e^{ax}\partial + \sum c_{b, 2} e^{bx}\partial^2$, where $c_{0, 1}, c_{b, 2} \in \mathbf{F}$. Since $\partial$ is the left (multiplicative) identity of $e^{x}\partial$, we prove that $D(e^{x}\partial) = c_{1, 1} e^{x}\partial + c_{1, 2} e^{x}\partial^2$. Since $D(e^{x}\partial * e^{x}\partial) = 0$, we have that $D(e^{x}\partial) = -c_{1, 1} e^{-x}\partial - c_{1, 2} e^{-x}\partial^2$. Since $\partial^2$ is the left (multiplicative) identity of $e^{x}\partial$, we have that $D(\partial^2) = 0$.

Conversely, let us assume that $D(\partial^2) = 0$. Similarly as $D(\partial) = 0$ case, we can prove that if $D(\partial^2) = 0$, then $D(\partial) = 0$ easily. This completes the proof of the lemma. \hfill \Box

**Lemma 2.2.** For any $D \in \text{Der}_\text{non}(\mathbf{F}[e^{\pm x}]_M)$, $D$ is the derivation $D_c$ in Note 1 for appropriate scalar.

**Proof.** Let $D$ be a derivation of $\mathbf{F}[e^{\pm x}]_M$. Since $\partial$ is in the right annihilator of $\partial^2$, we have that $D(\partial) = c_1 \partial + c_2 \partial^2$ for $c_1, c_2 \in \mathbf{F}$. Similarly, we have that $D(\partial^2) = c_3 \partial + c_4 \partial^2$ for $c_3, c_4 \in \mathbf{F}$. Since $\partial$ is a left (multiplicative) identity of $e^{x}\partial$, we put $D(e^{x}\partial)$ as follows:

$$
D(e^{x}\partial) = C(a_1, 1)e^{a_1x}\partial + \#_1 + C(b_1, 2)e^{b_1x}\partial^2 + \#_2,
$$

where $e^{a_1x}\partial$ and $e^{b_1x}\partial^2$ are appropriate maximal elements of $D(e^{x}\partial)$, $\#_1$ and $\#_2$ are the sums of its remaining terms with non-zero coefficients respectively.
Since \( \partial \) is a left (multiplicative) identity of \( e^x \partial \), we have that
\[
c_1e^x \partial + c_2e^x \partial + a_1C(a_1, 1)e^{a_1x} \partial + \#_3 + b_1C(b_1, 2)e^{b_1x} \partial^2 + \#_4
\]
(3) \[= C(a_1, 1)e^{a_1x} \partial + \#_1 + C(b_1, 2)e^{b_1x} \partial^2 + \#_2,
\]
where \( \#_3 \) and \( \#_4 \) are the sums of the remaining terms of the left side. This implies that \( b_1 = 1 \) and we have the following two cases, \( a_1 \neq 1 \) and \( a_1 = 1 \). If \( a_1 \neq 1 \), then the equality (3) does not hold. Thus we have that \( a_1 = b_1 = 1 \) and \( c_1 = -c_2 \), i.e., \( D(\partial) = c_1 \partial - c_1 \partial^2 \). Similarly, by considering the minimal elements of \( D(e^x \partial) \) with \( \partial \) and \( \partial^2 \), we can prove that \( D(e^x \partial) = C(1, 1)e^x \partial + C(1, 2)e^x \partial^2 \) and \( D(\partial^2) = c_3 \partial - c_3 \partial^2 \). By \( D(e^x \partial e^{-x} \partial) = -D(\partial) = c_1 \partial^2 - c_1 \partial \), we have that
\[
\{C(1, 1)e^x \partial + C(1, 2)e^x \partial^2\} \ast e^{-x} \partial + e^x \partial \ast D(e^{-x} \partial) = c_1 \partial^2 - c_1 \partial.
\]
This implies that \( e^x \partial \ast D(e^{-x} \partial) = c_1 \partial^2 - c_1 \partial + C(1, 1)\partial - C(1, 2)\partial \). So we have that
(4) \[D(e^{-x} \partial) = -c_1e^{-x} \partial^2 + c_1e^{-x} \partial - C(1, 1)e^{-x} \partial + C(1, 2)e^{-x} \partial.\]
By (4), we can prove that
\[
D(e^{-x} \partial) \ast e^x \partial
\]
(5) \[= \{-c_1e^{-x} \partial^2 + c_1e^{-x} \partial - C(1, 1)e^{-x} \partial + C(1, 2)e^{-x} \partial\} \ast e^x \partial
\]
\[= -c_1 \partial + c_1 \partial - C(1, 1)\partial + C(1, 2)\partial
\]
\[= -C(1, 1)\partial + C(1, 2)\partial.
\]
On the other hand, by \( D(e^{-x} \partial \ast e^x \partial) = D(\partial) = c_1 \partial - c_1 \partial^2 \), we have that \( D(e^{-x} \partial) \ast e^x \partial + e^{-x} \partial \ast \{C(1, 1)e^x \partial + C(1, 2)e^x \partial^2\} = c_1 \partial - c_1 \partial^2 \). This implies that
(6) \[D(e^{-x} \partial) \ast e^x \partial = c_1 \partial - c_1 \partial^2 - C(1, 1)e^x \partial - C(1, 2)e^x \partial^2.
\]
By comparing (5) with (6), we have that \( c_1 = 0 \) and \( C(1, 2) = 0 \). This implies that
(7) \[D(\partial) = 0,
\]
(8) \[D(e^x \partial) = C(1, 1)e^x \partial,
\]
(9) \[D(e^{-x} \partial) = -C(1, 1)e^{-x} \partial.
\]
Thus by induction \( p \in \mathbb{Z} \) of \( e^{px} \partial \), we can prove that \( D(e^{px} \partial) = pC(1, 1)e^{px} \partial \). Since \( \partial \) is the left multiplicative identity of \( e^x \partial^2 \), we have that \( D(e^x \partial^2) = c_5e^x \partial + c_6e^x \partial^2 \) for \( c_5, c_6 \in \mathbb{F} \). By \( D(e^x \partial \ast e^x \partial^2) = D(\partial^2) \), we prove that \( c_3 = c_5 \) and \( c_6 = C(1, 1) - c_3 \), i.e., \( D(e^x \partial^2) = c_3e^x \partial + (C(1, 1) - c_3)e^x \partial^2 \). By (9), we have that \( D(e^{-2x} \partial) = -2C(1, 1)e^{-2x} \partial \). By \( D(e^{-2x} \partial \ast e^x \partial^2) = D(e^{-x} \partial^2) \), we also have that \( D(e^{-x} \partial^2) = -C(1, 1)e^{-x} \partial^2 \). By \( D(e^x \partial^2 \ast e^{-x} \partial^2) = D(\partial^2) \), we have that \( c_3 = 0 \). This implies that
(10) \[D(\partial^2) = 0,
\]
(11) \[ D(e^x \partial^2) = C(1,1)e^x \partial^2, \]
(12) \[ D(e^{-x} \partial^2) = -C(1,1)e^{-x} \partial^2. \]
Thus by induction \( p \in \mathbb{Z} \) of \( e^{px} \partial^2 \), we can prove that \( D(e^{px} \partial^2) = pC(1,1)e^{px} \partial^2 \). Thus \( D \) can be linearly extended to the derivation \( D_c \) which is defined in Note 1 by putting \( c = C(1,1) \). Therefore we have proven the lemma.

**Theorem 2.1.** \( \text{Der}_{non}(\mathbf{F}[e^{\pm x}]_M) \) is generated by \( D_c \) in Note 1.

**Proof.** The proof of the theorem is straightforward by Lemma 2.2 and Note 1, so omit its details.

**Corollary 2.1.** \( \dim(\text{Der}_{non}(\mathbf{F}[e^{\pm x}]_M)) = 1. \)

**Proof.** The proof of the corollary is straightforward by Theorem 2.1.

**Corollary 2.2.** For any \( D \) in \( \text{Der}_{non}(\mathbf{F}[e^{\pm x}]_M), D(\mathbf{F}[e^{\pm x}]_{\{\partial^u\}}) \subset \mathbf{F}[e^{\pm x}]_{\{\partial^v\}}, \)
\( 1 \leq u \leq 2, \) i.e., \( \mathbf{F}[e^{\pm x}]_{\{\partial^u\}} \) is derivation invariant.

**Proof.** The proof of the corollary is straightforward by Theorem 2.1.

**Proposition 2.1.** There is no isomorphism from \( \mathbf{F}[e^{\pm x}]_{\{\partial\}} \) to \( \mathbf{F}[e^{\pm x}]_{\{\partial^2\}} \) as non-associative algebras.

**Proof.** Let us assume that there is an isomorphism \( \theta \) from \( \mathbf{F}[e^{\pm x}]_{\{\partial\}} \) to \( \mathbf{F}[e^{\pm x}]_{\{\partial^2\}} \) as non-associative algebras. We can prove that \( \theta(\partial) = c\partial^2 \) for \( c \in \mathbf{F}^* \) easily. Since \( \partial \) is the left identity of \( e^x \partial \), we have that \( c\partial^2 * \theta(e^x \partial) = \theta(e^x \partial) \). Since \( \mathbf{F} \) is a field of characteristic zero, we have that \( c = 1 \) and either \( \theta(e^x \partial) = d_1 e^x \partial^2 \) or \( \theta(e^x \partial) = d_2 e^{-x} \partial^2 \) for \( d_1, d_2 \in \mathbf{F}^* \). Let us assume that \( \theta(e^x \partial) = d_1 e^x \partial^2 \). By \( (e^{-x} \partial * e^x \partial) = \partial^2 \), we also have that \( \theta(e^{-x} \partial) = d_1^{-1} e^{-x} \partial^2 \). From the inequality \( \theta(e^x \partial * e^{-x} \partial) \neq -\partial^2 \) we can derive a contradiction. In the same way, if we assume that \( \theta(e^x \partial) = d_2 e^{-x} \partial^2 \), we can also derive a contradiction. These contradictions show that there is no algebra isomorphism from \( \mathbf{F}[e^{\pm x}]_{\{\partial\}} \) to \( \mathbf{F}[e^{\pm x}]_{\{\partial^2\}} \). We have proven the proposition.

**Corollary 2.3.**

\[ \text{Hom}_{non}(\mathbf{F}[e^{\pm x}]_{\{\partial\}}, \mathbf{F}[e^{\pm x}]_{\{\partial^2\}}) = \{0\}, \]

where \( \text{Hom}_{non}(\mathbf{F}[e^{\pm x}]_{\{\partial\}}, \mathbf{F}[e^{\pm x}]_{\{\partial^2\}}) \) is the set of all non-associative algebra homomorphisms from \( \mathbf{F}[e^{\pm x}]_{\{\partial\}} \) to \( \mathbf{F}[e^{\pm x}]_{\{\partial^2\}} \) and \( 0 \) is the zero map.

**Proof.** Since \( \mathbf{F}[e^{\pm x}]_{\{\partial\}} \) and \( \mathbf{F}[e^{\pm x}]_{\{\partial^2\}} \) are simple, the proof of the corollary is straightforward by Schur’s lemma and Proposition 2.1.

**Corollary 2.4.** There is no automorphism \( \theta \) of \( \mathbf{F}[e^{\pm x}]_M \) such that \( \theta(\partial) = c\partial^2 \) for any \( c \in \mathbf{F}^* \).

**Proof.** The proof of the corollary is straightforward by Proposition 2.1 [7].
Note 2. If $M = \{\partial, \ldots, \partial^n\}$, then we have the following decomposition of subalgebras (not just as vector subspaces) of $F[e^{\pm x}]_M$
\[
F[e^{\pm x}]_M = \bigoplus_{1 \leq u \leq n} F[e^{\pm x}]_{\{\partial^u\}},
\]
where $F[e^{\pm x}]_{\{\partial^u\}}$, $1 \leq u \leq n$, are simple [1], [2], [15]. By Theorem 2.1, we know that each subalgebra $F[e^{\pm x}]_{\{\partial^u\}}$, $1 \leq u \leq n$, of $F[e^{\pm x}]_M$ is derivation invariant.

Proposition 2.2. For $n > 1$, the matrix ring $M_n(F)$ is not embedded in the non-associative algebra $F[e^{\pm x}]_M$.

Proof. The proof of the proposition is standard, so let us omit it. \qed

Question. For any $D \in \text{Der}_{\text{non}}(F[e^{\pm x}]_M)$ (resp. $\theta \in \text{Aut}_{\text{non}}(F[e^{\pm x}]_M)$), does $D(F[e^{\pm x}]_{\{\partial^u\}}) \subset F[e^{\pm x}]_{\{\partial^u\}}$ (resp. $\theta(F[e^{\pm x}]_{\{\partial^u\}}) \subset F[e^{\pm x}]_{\{\partial^u\}}$) hold where $1 \leq u \leq n$?

References

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