RICCI CURVATURE OF INTEGRAL SUBMANIFOLDS OF AN $S$-SPACE FORM

JEONG-SIK KIM, MOHIT KUMAR DWIVEDI, AND MUKUT MANI TRIPATHI

ABSTRACT. Involving the Ricci curvature and the squared mean curvature, we obtain a basic inequality for an integral submanifold of an $S$-space form. By polarization, we get a basic inequality for Ricci tensor also. Equality cases are also discussed. By giving a very simple proof we show that if an integral submanifold of maximum dimension of an $S$-space form satisfies the equality case, then it must be minimal. These results are applied to get corresponding results for $C$-totally real submanifolds of a Sasakian space form and for totally real submanifolds of a complex space form.

1. Introduction

One of the most fundamental problems in submanifold theory is the following: Establish simple relationships between the main extrinsic invariants and the main intrinsic invariants of a submanifold. In [7], B.-Y. Chen established a sharp relationship between the Ricci curvature and the squared mean curvature for a submanifold in a Riemannian space form with arbitrary codimension. In [8], he gave the corresponding version of this inequality for totally real submanifolds in a complex space form. We find corresponding results for $C$-totally real submanifolds of a Sasakian space form in [10], [11] and [12].

The concept of framed metric structure unifies the concepts of almost Hermitian and almost contact metric structures. In particular, an $S$-structure generalizes Kaehler and Sasakian structure. In [1], D. Blair discusses principal toroidal bundles and generalizes the Hopf fibration to give a canonical example of an $S$-manifold playing the role of complex projective space in Kaehler geometry and the odd-dimensional sphere in Sasakian geometry. An $S$-manifold

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of constant $f$-sectional curvature $c$ is called an $S$-space form $\tilde{M}(c)$ [5], which generalizes the complex space form and Sasakian space form.

Motivated by the result of Chen in [7], recently in [9], a general basic inequality involving the Ricci curvature and the squared mean curvature of a submanifold in any Riemannian manifold is established and its several applications are presented. Using this inequality, in the present paper, we find a basic inequality for integral submanifolds of an $S$-space form $\tilde{M}(c)$ and apply this to recover the already known inequalities for totally real submanifolds in complex space forms and $C$-totally real submanifolds in Sasakian space forms. The paper is organized as follows. In section 2, we recall a brief account of Ricci curvature, $k$-Ricci curvature, scalar curvature in a Riemannian manifold and basic formulas and definitions for a submanifold. Then, we recall the result of [9] giving a general basic inequality involving the Ricci curvature and the squared mean curvature of a submanifold in any Riemannian manifold. Section 3 presents a brief account of framed metric manifold leading to $S$-space forms. In section 4, we give a very simple way to present a basic inequality for integral submanifolds of an $S$-space form $\tilde{M}(c)$. Then, the already known inequalities for totally real submanifolds in complex space forms and $C$-totally real submanifolds in Sasakian space forms become direct consequences. In section 5, we mainly prove that an integral submanifold of maximum dimension of an $S$-space form $\tilde{M}(c)$ satisfying the equality case becomes minimal. Then, we derive the same conclusion for Lagrangian submanifold of a complex space form and $C$-totally real submanifold of maximum dimension of a Sasakian space form.

2. Ricci curvature of submanifolds

Let $M$ be an $n$-dimensional Riemannian manifold. Let $\{e_1, \ldots, e_k\}$, $2 \leq k \leq n$, be an orthonormal basis of a $k$-plane section $\Pi_k$ of $T_pM$. If $k = n$ then $\Pi_n = T_pM$; and if $k = 2$ then $\Pi_2$ is a plane section of $T_pM$. For a fixed $i \in \{1, \ldots, k\}$, a $k$-Ricci curvature of $\Pi_k$ at $e_i$, denoted $\text{Ric}_{\Pi_k}(e_i)$, is defined by [7]

$$
\text{Ric}_{\Pi_k}(e_i) = \sum_{j \neq i}^k K_{ij},
$$

where $K_{ij}$ is the sectional curvature of the plane section spanned by $e_i$ and $e_j$. An $n$-Ricci curvature $\text{Ric}_{T_pM}(e_i)$ is the usual Ricci curvature of $e_i$, denoted $\text{Ric}(e_i)$. Thus for any orthonormal basis $\{e_1, \ldots, e_n\}$ for $T_pM$ and for a fixed $i \in \{1, \ldots, n\}$, we have

$$
\text{Ric}_{T_pM}(e_i) \equiv \text{Ric}(e_i) = \sum_{j \neq i}^n K_{ij}.
$$
The scalar curvature $\tau(\Pi_k)$ of the $k$-plane section $\Pi_k$ is given by

$$
\tau(\Pi_k) = \sum_{1 \leq i < j \leq k} K_{ij}.
$$

Geometrically, $\tau(\Pi_k)$ is the scalar curvature of the image $\exp_p(\Pi_k)$ of $\Pi_k$ at $p$ under the exponential map at $p$. The scalar curvature $\tau(p)$ of $M$ at $p$ is identical with the scalar curvature of the tangent space $T_pM$ of $M$ at $p$, that is, $\tau(p) = \tau(T_pM)$.

Let $M$ be an $n$-dimensional submanifold of an $m$-dimensional Riemannian manifold $\tilde{M}$ equipped with a Riemannian metric $\tilde{g}$. We use the inner product notation $\langle , \rangle$ for both the metrics $\tilde{g}$ of $\tilde{M}$ and the induced metric $g$ on the submanifold $M$. The Gauss and Weingarten formulas are given respectively by

$$
\tilde{\nabla}_X Y = \nabla_X Y + \sigma(X,Y) \quad \text{and} \quad \tilde{\nabla}_X N = -A_N X + \nabla^\perp_X N
$$

for all $X,Y \in TM$ and $N \in T^\perp M$, where $\tilde{\nabla}$, $\nabla$ and $\nabla^\perp$ are respectively the Riemannian, induced Riemannian and induced normal connections in $\tilde{M}$, $M$ and the normal bundle $T^\perp M$ of $M$ respectively, and $\sigma$ is the second fundamental form related to the shape operator $A$ by $\langle \sigma(X,Y), N \rangle = \langle A_N X, Y \rangle$. The equation of Gauss is given by

$$
R(X,Y,Z,W) = \tilde{R}(X,Y,Z,W) + \langle \sigma(X,W), \sigma(Y,Z) \rangle - \langle \sigma(X,Z), \sigma(Y,W) \rangle
$$

for all $X,Y,Z,W \in TM$, where $\tilde{R}$ and $R$ are the Riemann curvature tensors of $\tilde{M}$ and $M$ respectively. The curvature tensor $R^\perp$ of the normal bundle of $M$ is defined by

$$
R^\perp(X,Y)N = \nabla^\perp_X \nabla^\perp_Y N - \nabla^\perp_Y \nabla^\perp_X N - \nabla^\perp_{[X,Y]} N
$$

for all $X,Y \in TM$ and $N \in T^\perp M$. If $R^\perp = 0$, then the normal connection $\nabla^\perp$ of $M$ is said to be flat.

The mean curvature vector $H$ is given by $H = \frac{1}{n} \text{trace}(\sigma)$. The submanifold $M$ is totally geodesic in $\tilde{M}$ if $\sigma = 0$, and minimal if $H = 0$. If $\sigma(X,Y) = g(X,Y)H$ for all $X,Y \in TM$, then $M$ is totally umbilical.

The relative null space of $M$ at $p$ is defined by [7]

$$
N_p = \{ X \in T_pM \mid \sigma(X,Y) = 0 \text{ for all } Y \in T_pM \}
$$

which is also known as the kernel of the second fundamental form at $p$ [8].

Now, let $\{e_1, \ldots, e_n\}$ be an orthonormal basis of the tangent space $T_pM$ and $e_r$ belongs to an orthonormal basis $\{e_{n+1}, \ldots, e_m\}$ of the normal space $T^\perp pM$. We put

$$
\sigma^r_{ij} = \langle \sigma(e_i,e_j), e_r \rangle \quad \text{and} \quad \|\sigma\|^2 = \sum_{i,j=1}^n \langle \sigma(e_i,e_j), \sigma(e_i,e_j) \rangle.
$$
Let $K_{ij}$ and $\tilde{K}_{ij}$ denote the sectional curvature of the plane section spanned by $e_i$ and $e_j$ at $p$ in the submanifold $M$ and in the ambient manifold $\tilde{M}$ respectively. Thus, we can say that $K_{ij}$ and $\tilde{K}_{ij}$ are the "intrinsic" and "extrinsic" sectional curvature of the $\text{Span}\{e_i, e_j\}$ at $p$. In view of (3), we get
\begin{equation}
K_{ij} = \tilde{K}_{ij} + \sum_{\tau=n+1}^{m} (\sigma_{ii}^\tau \sigma_{jj}^\tau - (\sigma_{ij}^\tau)^2).
\end{equation}

From (4) it follows that
\begin{equation}
2\tau(p) = 2\tau(T_pM) + n^2 \|H\|^2 - \|\sigma\|^2,
\end{equation}
where $\tau(T_pM)$ denotes the scalar curvature of the $n$-plane section $T_pM$ in the ambient manifold $\tilde{M}$. Thus, we can say that $\tau(p)$ and $\tau(T_pM)$ are the "intrinsic" and "extrinsic" scalar curvature of the submanifold at $p$ respectively.

We denote the set of unit vectors in $T_pM$ by $T_p^1M$; thus
\[ T_p^1M = \{X \in T_pM \mid \langle X, X \rangle = 1\}. \]

Now, we recall the following result from [9].

**Theorem 2.1.** Let $M$ be an $n$-dimensional submanifold of a Riemannian manifold $\tilde{M}$. Then the following statements are true.

(a) For $X \in T_p^1M$ we have
\begin{equation}
\text{Ric}(X) \leq \frac{n^2}{4} \|H\|^2 + \overline{\text{Ric}}(T_pM)(X),
\end{equation}
where $\overline{\text{Ric}}(T_pM)(X)$ is the $n$-Ricci curvature of $T_pM$ at $X \in T_p^1M$ with respect to the ambient manifold $\tilde{M}$.

(b) The equality case of (6) is satisfied by $X \in T_p^1M$ if and only if
\begin{equation}
\sigma(X, X) = \frac{n}{2} H(p) \quad \text{and} \quad \sigma(X, Y) = 0
\end{equation}
for all $Y \in T_pM$ such that $\langle X, Y \rangle = 0$.

(c) The equality case of (6) holds for all $X \in T_p^1M$ if and only if either (1) $p$ is a totally geodesic point or (2) $n = 2$ and $p$ is a totally umbilical point.

From Theorem 2.1, we immediately have the following

**Corollary 2.2.** Let $M$ be an $n$-dimensional submanifold of a Riemannian manifold. For $X \in T_p^1M$ any two of the following three statements imply the remaining one.

(a) The mean curvature vector $H(p)$ vanishes.
(b) The unit vector $X$ belongs to the relative null space $N_p$.
(c) The unit vector $X$ satisfies the equality case of (6), namely
\begin{equation}
\text{Ric}(X) = \frac{1}{4} n^2 \|H\|^2 + \overline{\text{Ric}}(T_pM)(X).
\end{equation}
3. S-space forms

Let $\tilde{M}$ be a $(2m+s)$-dimensional framed metric manifold [17] (also known as framed f-manifold [13] or almost r-contact metric manifold [15]) with a framed metric structure $(f, \xi_\alpha, \eta^\alpha, \tilde{g})$, $\alpha \in \{1, \ldots, s\}$, that is, $f$ is a $(1,1)$ tensor field defining an $f$-structure of rank $2m$; $\xi_1, \ldots, \xi_s$ are vector fields; $\eta^1, \ldots, \eta^s$ are 1-forms and $\tilde{g}$ is a Riemannian metric on $\tilde{M}$ such that for all $X, Y \in TM$ and $\alpha, \beta \in \{1, \ldots, s\}$

\begin{equation}
    f^2 = -I + \eta^\alpha \otimes \xi_\alpha, \quad \eta^\alpha (\xi_\beta) = \delta^\alpha_\beta, \quad f (\xi_\alpha) = 0, \quad \eta^\alpha \circ f = 0,
\end{equation}

\begin{equation}
    \langle fX, fY \rangle = \langle X, Y \rangle - \sum_\alpha \eta^\alpha (X) \eta^\alpha (Y),
\end{equation}

\begin{equation}
    \Omega (X, Y) \equiv \langle X, fY \rangle = -\Omega (Y, X), \quad \langle X, \xi_\alpha \rangle = \eta^\alpha (X),
\end{equation}

where $\langle , \rangle$ denotes the inner product of the metric $\tilde{g}$. A framed metric structure is an S-structure [1] if the Nijenhuis tensor of $f$ equals $-2d\eta^\alpha \otimes \xi_\alpha$ and $\Omega = d\eta^\alpha$ for all $\alpha \in \{1, \ldots, s\}$.

When $s = 1$, a framed metric structure is an almost contact metric structure, while an S-structure is a Sasakian structure. When $s = 0$, a framed metric structure is an almost Hermitian structure, while an S-structure is a Kaehler structure. If a framed metric structure on $\tilde{M}$ is an S-structure then it is known [1] that

\begin{equation}
    (\tilde{\nabla}_X f)Y = \sum_\alpha \left( \langle fX, fY \rangle \xi_\alpha + \eta^\alpha (Y) f^2 X \right),
\end{equation}

\begin{equation}
    \tilde{\nabla} \xi_\alpha = -f, \quad \alpha \in \{1, \ldots, s\}.
\end{equation}

The converse may also be proved. In case of Sasakian structure (that is, $s = 1$), (12) implies (13). In Kaehler case (that is, $s = 0$), we get $\tilde{\nabla} f = 0$. For $s > 1$, examples of S-structures are given in [1], [2] and [4]. Thus, the bundle space of a principal toroidal bundles over a Kaehler manifold with certain conditions is an S-manifold. Thus, a generalization of the Hopf fibration $\pi : S^{2m+1} \to PC^m$ is a canonical example of an S-manifold playing the role of complex projective space in Kaehler geometry and the odd-dimensional sphere in Sasakian geometry.

A plane section in $T_p\tilde{M}$ is a f-section if there exists a vector $X \in T_p\tilde{M}$ orthogonal to $\xi_1, \ldots, \xi_s$ such that $\{X, fX\}$ span the section. The sectional curvature of a f-section is called a f-sectional curvature. It is known that [5]
in an $S$-manifold of constant $f$-sectional curvature $c$

\begin{equation}
\tilde{R}(X,Y)Z = \sum_{\alpha, \beta} (\eta^\alpha(X)\eta^\beta(Z) f^2 Y - \eta^\alpha(Y)\eta^\beta(Z) f^2 X
- \langle fX, fZ \rangle \eta^\alpha(Y)\xi_\beta + \langle fY, fZ \rangle \eta^\alpha(X)\xi_\beta)
+ \frac{c + 3s}{4} \{ - \langle fY, fZ \rangle f^2 X + \langle fX, fZ \rangle f^2 Y
+ \frac{c - s}{4} \langle \langle fX, fZ \rangle fY - \langle fY, fZ \rangle fX + 2 \langle fX, fY \rangle fZ \} \}
\end{equation}

for all $X, Y, Z \in T\tilde{M}$, where $\tilde{R}$ is the curvature tensor of $\tilde{M}$. An $S$-manifold of constant $f$-sectional curvature $c$ is called an $S$-space form $\tilde{M}(c)$.

When $s = 1$, an $S$-space form $\tilde{M}(c)$ reduces to a Sasakian space form $\tilde{M}(c)$ [3] and (14) reduces to

\begin{equation}
\tilde{R}(X,Y)Z = \frac{c + 3}{4} \{ \langle Y, Z \rangle X - \langle X, Z \rangle Y \}
+ \frac{c - 1}{4} \{ \langle X, fZ \rangle fY - \langle Y, fZ \rangle fX + 2 \langle fX, fY \rangle fZ
+ \eta(X)\eta(Z)X - \eta(Y)\eta(Z)X
+ \langle X, Z \rangle \eta(Y)\xi - \langle Y, Z \rangle \eta(X)\xi \},
\end{equation}

where $\xi_1 \equiv \xi$ and $\eta^1 \equiv \eta$. When $s = 0$, an $S$-space form $\tilde{M}(c)$ becomes a complex space form and (14) moves to

\begin{equation}
4\tilde{R}(X,Y)Z = c \{ \langle Y, Z \rangle X - \langle X, Z \rangle Y
+ \langle X, fZ \rangle fY - \langle Y, fZ \rangle fX + 2 \langle fX, fY \rangle fZ \}.
\end{equation}

4. Ricci curvature of integral submanifolds

Let $\tilde{M}$ be an $S$-manifold equipped with an $S$-structure $(f, \xi, \eta^\alpha, \tilde{g})$. A submanifold $M$ of $\tilde{M}$ is an integral submanifold if $\eta^\alpha(X) = 0, \alpha = 1, \ldots, s$, for every tangent vector $X$. A submanifold $M$ of $\tilde{M}$ is an anti-invariant submanifold if $f(TM) \subseteq T^\perp M$. An integral submanifold is identical with an anti-invariant submanifold normal to the structure vector fields $\xi_1, \ldots, \xi_s$. In particular case of $s = 1$, an integral submanifold $M$ of a Sasakian manifold is a C-totally real submanifold [16]. It is known that [6] an $n$-dimensional integral submanifold $M$, of an $S$-manifold $\tilde{M}$ of dimension $(2n + s)$, is of constant curvature $s$ if and only if the normal connection is flat.

First, we give the following Lemma.

**Lemma 4.1.** Let $M$ be an $n$-dimensional integral submanifold of an $S$-space form $\tilde{M}(c)$. Let $\{e_1, \ldots, e_n\}$ be an orthonormal basis of the tangent space $T_pM$. 

Then

\begin{equation}
\bar{K}_{ij} = \frac{1}{4} (c + 3s),
\end{equation}

\begin{equation}
\text{Ric}_{(T_p M)}(e_i) = \frac{1}{4} (n - 1)(c + 3s),
\end{equation}

\begin{equation}
\bar{\tau}(T_p M) = \frac{1}{8} n(n - 1)(c + 3s).
\end{equation}

\textbf{Proof.} Equation (15) follows from (14). Using $\text{Ric}_{(T_p M)}(e_i) = \sum_{j \neq i} K_{ij}$ in (15), we get (16). Next, using $2\bar{\tau}(T_p M) = \sum_{i=1}^{n} \text{Ric}_{(T_p M)}(e_i)$, from (16) we get (17). \hfill \Box

Now, we have the following Theorem.

\textbf{Theorem 4.2.} If $M$ is an $n$-dimensional integral submanifold of an $S$-space form $\bar{M}(c)$, then the following statements are true.

(a) For $X \in T_p^1 M$, it follows that

\begin{equation}
\text{Ric}(X) \leq \frac{1}{4} \left\{ n^2 \|H\|^2 + (n - 1)(c + 3s) \right\}.
\end{equation}

(b) The equality case of (18) is satisfied by $X \in T_p^1 M$ if and only if (7) is true. If $H(p) = 0$, $X \in T_p^1 M$ satisfies equality in (18) if and only if $X \in N_p$.

(c) The equality case of (18) holds for all $X \in T_p^1 M$ if and only if either $p$ is a totally geodesic point or $n = 2$ and $p$ is a totally umbilical point.

\textbf{Proof.} Using (16) in (6), we find the inequality (18). Rest of the proof is straightforward. \hfill \Box

By polarization, from Theorem 4.2, we derive

\textbf{Theorem 4.3.} Let $M$ be an $n$-dimensional integral submanifold of an $S$-space form $\bar{M}(c)$. Then the Ricci tensor $S$ satisfies

\begin{equation}
S \leq \frac{1}{4} \left\{ n^2 \|H\|^2 + (n - 1)(c + 3s) \right\} g,
\end{equation}

where $g$ is the induced Riemannian metric on $M$. The equality case of (19) is true if and only if either $M$ is a totally geodesic submanifold or $M$ is a totally umbilical surface.

When $s = 0$, we have the following two results.

\textbf{Theorem 4.4.} If $M$ is an $n$-dimensional totally real submanifold (or isotropic submanifold) of a complex space form $\bar{M}(c)$, then the following statements are true.
(a) It follows that

\[ \text{Ric}(X) \leq \frac{1}{4} \left\{ n^2 \| H \|^2 + (n - 1)c \right\}, \quad X \in T_p^1 M. \]

(b) The equality case of (20) is satisfied by \( X \in T_p^1 M \) if and only if (7) is true. If \( H(p) = 0, \) \( X \in T_p^1 M \) satisfies equality in (20) if and only if \( X \in N_p. \)

(c) The equality case of (20) holds for all \( X \in T_p^1 M \) if and only if either \( p \) is a totally geodesic point or \( n = 2 \) and \( p \) is a totally umbilical point.

**Theorem 4.5.** If \( M \) is an \( n \)-dimensional totally real submanifold (or isotropic submanifold) of a complex space form \( \bar{M}(c), \) then the following statements are true.

(a) It follows that

\[ S \leq \frac{1}{4} \left\{ n^2 \| H \|^2 + (n - 1)c \right\} g. \]

(b) The equality case of (21) holds identically if and only if either \( M \) is totally geodesic submanifold or \( M \) is a totally umbilical surface.

For \( s = 1, \) we again have the following two results.

**Theorem 4.6.** If \( M \) is an \( n \)-dimensional \( C \)-totally real submanifold of a Sasakian space form \( \bar{M}(c), \) then the following statements are true.

(a) It follows that

\[ \text{Ric}(X) \leq \frac{1}{4} \left\{ n^2 \| H \|^2 + (n - 1)(c + 3) \right\}, \quad X \in T_p^1 M. \]

(b) The equality case of (22) is satisfied by \( X \in T_p^1 M \) if and only if (7) is true. If \( H(p) = 0, \) \( X \in T_p^1 M \) satisfies equality in (22) if and only if \( X \in N_p. \)

(c) The equality case of (22) holds for all \( X \in T_p^1 M \) if and only if either \( p \) is a totally geodesic point or \( n = 2 \) and \( p \) is a totally umbilical point.

(d) The equality case of (23) holds identically if and only if either \( M \) is totally geodesic submanifold or \( M \) is a totally umbilical surface.

**Theorem 4.7.** If \( M \) is an \( n \)-dimensional \( C \)-totally real submanifold of a Sasakian space form \( \bar{M}(c), \) then the following statements are true.

(a) It follows that

\[ S \leq \frac{1}{4} \left\{ n^2 \| H \|^2 + (n - 1)(c + 3) \right\} g. \]

(b) The equality case of (23) holds identically if and only if either \( M \) is totally geodesic submanifold or \( M \) is a totally umbilical surface.
It is known that (Theorem 4, [14]) if $M$ is an $n$-dimensional compact minimal $C$-totally real submanifold of a Sasakian space form $M^{2n+1}(c)$, $c > -3$, such that $M$ has positive sectional curvature, then $M$ is totally geodesic. Therefore, in view of Theorem 4.7, we have the following

**Theorem 4.8.** An $n$-dimensional compact minimal $C$-totally real submanifold of a Sasakian space form $M^{2n+1}(c)$, $c > -3$ with positive sectional curvature is an Einstein manifold and satisfies $4S = (n - 1)(c + 3)g$.

The inequality (22) is the inequality (2.1) in Theorem 2.1 of [12]. The inequality (23) is the inequality (9) in Theorem 3.1 of [10]. The inequality (21) is the inequality (2.1) in Theorem 1 of [8]. Here, we find the proofs very much simplified.

### 5. Minimality of integral submanifolds of maximum dimension

We already know the following result [6]. If $M$ is an $n$-dimensional integral submanifold of any $(2n+s)$-dimensional $S$-space form $\tilde{M}(c)$, then the following four statements are equivalent: (i) $M$ is totally geodesic. (ii) $M$ is of constant curvature $\frac{1}{4}(c + 3s)$. (iii) The Ricci tensor is $\frac{1}{4} (n - 1)(c + 3s)g$. (iv) The scalar curvature is $\frac{1}{4} n(n - 1)(c + 3s)$. In Theorem 5.2, we find a condition for minimality.

Now, we begin with the following

**Theorem 5.1.** Let $M$ be an $n$-dimensional integral submanifold of a $(2n+s)$-dimensional $S$-space form $\tilde{M}(c)$. If a unit vector of $T_pM$ satisfies the equality case of (18), then $H(p) = 0$.

**Proof.** Choose an orthonormal basis $\{e_1, \ldots, e_n\}$ of $T_pM$ such that $e_1$ satisfies the equality case of (18). Then, $\{e_{n+1}, \ldots, e_{2n}, e_{2n+1} = \xi_1, \ldots, e_{2n+s} = \xi_s\}$ is an orthonormal basis of $T_{e_1}M$ such that $e_{n+j} = f_{e_j}$, $j \in \{1, \ldots, n\}$. We then have $A_{\xi_\alpha} = 0$ for all $\alpha \in \{1, \ldots, s\}$ and $A_{f_XY} = A_{f_YX}$ for $X, Y \in TM$. Using these two facts along with (7), for any $Y = \sum_{j=1}^{n} a_j e_{n+j} + \sum_{\alpha=1}^{s} a_\alpha \xi_\alpha \in T_pM$, we have

$$
\langle \sigma(e_1, e_1), Y \rangle = a_1 \langle \sigma(e_1, e_1), fe_1 \rangle \\
+ \sum_{j=2}^{n} a_j \langle \sigma(e_j, e_1), fe_j \rangle + \sum_{\alpha=1}^{s} a_\alpha \langle \sigma(e_1, e_1), \xi_\alpha \rangle
$$

$$
= a_1 \left(\sum_{j=2}^{n} \sigma(e_j, e_1), fe_1\right) + \sum_{j=2}^{n} a_j \langle \sigma(e_1, e_1), fe_j \rangle + 0
$$

$$
= a_1 \sum_{j=2}^{n} \langle \sigma(e_1, e_j), fe_j \rangle + \sum_{j=2}^{n} a_j \langle \sigma(e_1, e_1), fe_j \rangle
$$

$$
= 0 + 0 = 0.
$$

Hence in view of (7), $H(p) = 0$. \qed
The maximum Ricci curvature function ([8]) on a Riemannian manifold $M$, denoted $\overline{\text{Ric}}$, is defined as

$$\overline{\text{Ric}}(p) = \max \{\text{Ric}(X) \mid X \in T^1_p M\}.$$ 

Now, in view of Theorem 5.1, we immediately have the following

**Theorem 5.2.** Let $M$ be an $n$-dimensional integral submanifold of a $(2n + s)$-dimensional $S$-space form $\widetilde{M}(c)$. Then

$$\overline{\text{Ric}} \leq \frac{1}{4} \left\{ n^2 \|H\|^2 + (n - 1)(c + 3s) \right\}.$$ 

If $M$ satisfies the equality case of (24) identically, then $M$ is a minimal submanifold and

$$\overline{\text{Ric}} = \frac{1}{4} (n - 1)(c + 3s).$$

When $s = 0$, from Theorem 5.2 we have the following

**Theorem 5.3.** ([8], Theorem 2) Let $M$ be a Lagrangian submanifold of a 2n-dimensional complex space form $\widetilde{M}(c)$. Then

$$\overline{\text{Ric}} \leq \frac{1}{4} \left\{ n^2 \|H\|^2 + (n - 1)c \right\}.$$ 

If $M$ satisfies the equality case of (24) identically, then $M$ is a minimal submanifold and

$$\overline{\text{Ric}} = \frac{1}{4} (n - 1)c.$$ 

When $s = 1$, from Theorem 5.2 we have the following (Theorem 4.1 of [10] or Theorem 3.1 of [11])

**Theorem 5.4.** ([10], Theorem 4.1 or Theorem 3.1 of [11]) Let $M$ be an $n$-dimensional $C$-totally real submanifold of a $(2n + 1)$-dimensional Sasakian space form $\widetilde{M}(c)$. Then

$$\overline{\text{Ric}} \leq \frac{1}{4} \left\{ n^2 \|H\|^2 + (n - 1)(c + 3) \right\}.$$ 

If $M$ satisfies the equality case of (24) identically, then $M$ is a minimal submanifold and

$$\overline{\text{Ric}} = \frac{1}{4} (n - 1)(c + 3).$$

Following the arguments as in [8], we can prove

**Theorem 5.5.** Let $M$ be an $n$-dimensional minimal integral submanifold of a $(2n + s)$-dimensional $S$-space form $\widetilde{M}(c)$. Then the following statements are true.

1. The submanifold $M$ satisfies the equality case of (24) if and only if $\dim(\mathcal{N}_p) \geq 1$. 


(2) If \( \dim(N_p) \) is a positive constant \( d \), then \( N_p \) is completely integral distribution and \( M \) is \( d \)-ruled, that is, for each \( p \in M \), \( M \) contains a \( d \)-dimensional totally geodesic submanifold \( M' \) of \( M(c) \) passing through \( p \).

(3) If the submanifold \( M \) is also ruled, then it satisfies the equality case of (24) identically if and only if, for each ruling \( M' \) in \( M \), the normal bundle \( T^\perp M \) restricted to \( M' \) is a parallel normal subbundle of the normal bundle \( T^\perp M' \) along \( M' \).

References


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