BOUNDDED SOLUTIONS FOR THE SCHRÖDINGER OPERATOR ON RIEMANNIAN MANIFOLDS

SEOK WOO KIM AND YONG HAH LEE

ABSTRACT. Let $M$ be a complete Riemannian manifold and $\mathcal{L}$ be a Schrödinger operator on $M$. We prove that if $M$ has finitely many $\mathcal{L}$-nonparabolic ends, then the space of bounded $\mathcal{L}$-harmonic functions on $M$ has the same dimension as the sum of dimensions of the spaces of bounded $\mathcal{L}$-harmonic functions on each $\mathcal{L}$-nonparabolic end, which vanish at the boundary of the end.

1. Introduction

Let $M$ be a complete Riemannian manifold and $\mathcal{L} = \Delta - V$ be a Schrödinger operator on $M$, where $\Delta$ is the Laplacian on $M$ and the potential $V$ is a nonnegative function on $M$. A function $u$ on an open subset $\Omega$ of $M$ is called an $\mathcal{L}$-solution (-supersolution, -subsolution, respectively,) on $\Omega$ if $\mathcal{L}u = 0$ ($\leq 0$, $\geq 0$, respectively,) on $\Omega$. This equation is understood in the sense of distribution. We say that a function $u$ is $\mathcal{L}$-harmonic on $\Omega$ if $u$ is a continuous $\mathcal{L}$-solution on $\Omega$. In the case when the potential term $V$ of the Schrödinger operator $\mathcal{L}$ is continuous, one can achieve the continuity of $\mathcal{L}$-solutions. More generally, such a result can be extended to potentials in the local Kato class. (See [4].)

This paper is motivated by the previous works of present authors [5] and [6]. By the result of [5], the dimension of the space of bounded $\mathcal{L}$-harmonic functions on a complete Riemannian manifold is equal to the number of $\mathcal{L}$-nonparabolic ends in the case when each $\mathcal{L}$-nonparabolic end is regular. On the other hand, the present authors in [6] proved that the dimension of the space of bounded energy finite $\mathcal{L}$-harmonic functions on a complete Riemannian manifold is equal to the maximal number of $\mathcal{L}$-massive subsets of the manifold. In this paper, we propose the space of bounded $\mathcal{L}$-harmonic functions on ends of a complete Riemannian manifold and give the relation between the space of bounded $\mathcal{L}$-harmonic functions on the whole manifold and those of its ends. In particular, we prove that the dimension of the space of bounded $\mathcal{L}$-harmonic functions on

Received January 12, 2007.
2000 Mathematics Subject Classification. 58J05, 35J10.
Key words and phrases. Schrödinger operator, $\mathcal{L}$-harmonic function, $\mathcal{L}$-massive set, end.
The first author was supported by grant No. R01-2006-000-10047-0(2007) from the Basic Research Program of the Korea Science & Engineering Foundation.

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507
the whole manifold is equal to the sum of dimension of the spaces of bounded $\mathcal{L}$-harmonic functions on its ends as follows:

**Theorem 1.1.** Let $M$ be a complete Riemannian manifold and $\mathcal{L} = \Delta - V$, where $\Delta$ denotes the Laplacian on $M$ and $V$ is a nonnegative continuous function on $M$. Let $E_1, E_2, \ldots, E_l$, $l \geq 1$, be $\mathcal{L}$-nonparabolic ends of $M$. Then $\mathcal{H}B_{\mathcal{L}}(M)$ has the same dimension as the dimension of $\prod_{i=1}^{l} \mathcal{H}B_{\mathcal{L}}(E_i, \partial E_i)$, where $\mathcal{H}B_{\mathcal{L}}(X)$ and $\mathcal{H}B_{\mathcal{L}}(X, \partial X)$ denote the space of bounded $\mathcal{L}$-harmonic functions on $X$ and the subspace of elements of $\mathcal{H}B_{\mathcal{L}}(X)$ vanishing at $\partial X$, respectively.

In particular, in the case when $\mathcal{H}B_{\mathcal{L}}(M)$ is finite dimensional, there exists an isomorphism

$$
\Phi : \mathcal{H}B_{\mathcal{L}}(M) \to \prod_{i=1}^{l} \mathcal{H}B_{\mathcal{L}}(E_i, \partial E_i).
$$

In the case when the potential term of the Schrödinger operator is identically zero, $\mathcal{L}$-harmonic functions become harmonic functions. Therefore, this result partially generalizes those of Yau [10], of Grigor’yan [1], [2], [3], of Li-Tam [7], [8] and of Sung-Tam-Wang [9].

2. $\mathcal{L}$-massivity and bounded $\mathcal{L}$-harmonic functions on manifolds

Let $M$ be a complete Riemannian manifold and $o$ be a fixed point in $M$. Throughout this paper, $\Delta$ always denotes the Laplacian on $M$ and $V$ is a nonnegative continuous function on $M$. Also $\mathcal{L} = \Delta - V$ denotes a Schrödinger operator on $M$.

An open proper subset $\Omega \subset M$ is said to be $\mathcal{L}$-massive if there exists a continuous function $w$ on $M$ such that $0 \leq w \leq 1$ on $M$,

$$
\left\{ \begin{array}{ll}
\mathcal{L} w = 0 & \text{on } \Omega; \\
w = 0 & \text{on } M \setminus \Omega; \\
\sup_w w = 1.
\end{array} \right.
$$

Such a function $w$ is called an inner potential of $\Omega$.

Arguing similarly as in [3], we get the following useful properties of $\mathcal{L}$-massive sets:

**Proposition 2.1.** Suppose $\Omega' \subset \Omega$ are open proper subsets of a complete Riemannian manifold and $\mathcal{L} = \Delta - V$. Then

(i) if $\Omega'$ is $\mathcal{L}$-massive, then $\Omega$ is also $\mathcal{L}$-massive;

(ii) if $\Omega$ is $\mathcal{L}$-massive and $\partial \Omega \setminus \partial \Omega'$ is compact, then $\Omega'$ is also $\mathcal{L}$-massive.

We denote by $\mathcal{B}(M)$ the space of all bounded continuous functions on $M$. Let $\mathcal{H}B_{\mathcal{L}}(M)$ denote the subspace of all $\mathcal{L}$-harmonic functions in $\mathcal{B}(M)$. Then we can prove that the dimension of $\mathcal{H}B_{\mathcal{L}}(M)$ is equal to the supremum of the number of mutually disjoint $\mathcal{L}$-massive subsets of $M$ as follows:
Theorem 2.2. Let $M$ be a complete Riemannian manifold and $\mathcal{L} = \Delta - V$. Then for each $m \in \mathbb{N}$, $\dim \mathcal{H}_\mathcal{L}(M) \geq m$ if and only if there exist mutually disjoint $\mathcal{L}$-massive subsets $\Omega_1, \Omega_2, \ldots, \Omega_m$ of $M$.

Proof. Let $\Omega_1, \Omega_2, \ldots, \Omega_m$ be the mutually disjoint $\mathcal{L}$-massive subsets of $M$ and $w_i$ be an inner potential of $\Omega_i$ for each $i = 1, 2, \ldots, m$. Then for each $i = 1, 2, \ldots, m$ and $r > 0$, define a continuous function $f_{i,r}$ on $B_r(o)$ such that

\[
\begin{align*}
\mathcal{L} f_{i,r} &= 0 \quad \text{on } B_r(o) ; \\
f_{i,r} &= w_i \quad \text{on } \partial B_r(o) \cap \Omega_i ; \\
f_{i,r} &= 0 \quad \text{on } \partial B_r(o) \setminus \Omega_i ,
\end{align*}
\]

where $B_r(o)$ denotes the metric $r$-ball centered at $o$. By the comparison principle, $w_i \leq f_{i,r} \leq 1$ on $B_r(o)$. Since $f_{i,r'} \geq w_i = f_{i,r}$ on $\partial B_r(o)$ for $r' > r$, we have $f_{i,r'} \geq f_{i,r}$ on $B_r(o)$. Thus $\{f_{i,r}\}$ is increasing in $r$, hence has a limit function $f_i$. In particular, $f_i$ is an $\mathcal{L}$-harmonic function on $M$ satisfying $0 \leq w_i \leq f_i \leq 1$. Since $\sup_{\Omega_i} w_i = 1$, we have $\sup_{\Omega_i} f_i = 1$.

On the other hand, since $\Omega_1, \Omega_2, \ldots, \Omega_m$ are mutually disjoint, $\sum_{i=1}^m w_i = \max_{i=1,2,\ldots,m} w_i$, hence $\sup_M \sum_{i=1}^m w_i = 1$ and $\sup_M \sum_{i=1}^m f_i = 1$. Since $\sup_{\Omega_i} w_i = 1$, there is a sequence $\{x_{i,n}\}_{n \in \mathbb{N}}$ in $\Omega_i$ such that $\lim_{n \to \infty} w_i(x_{i,n}) = 1$ for each $i = 1, 2, \ldots, m$. From the fact that $0 \leq w_i \leq f_i \leq 1$ and $\sum_{i=1}^m f_i \leq 1$, the sequence $\{x_{i,n}\}$ satisfies

\[
\lim_{n \to \infty} f_j(x_{i,n}) = \delta_{ij}
\]

for each $i = 1, 2, \ldots, m$, where $\delta_{ij}$ is Kronecker's delta.

Suppose that

\[
a_1 f_1 + a_2 f_2 + \cdots + a_m f_m = 0
\]

for some $a_1, a_2, \ldots, a_m \in \mathbb{R}$. Then (2.1) implies that $a_i = 0$ for each $i = 1, 2, \ldots, m$, hence $f_1, f_2, \ldots, f_m$ are linearly independent. Consequently,

\[
\dim \mathcal{H}_\mathcal{L}(M) \geq m.
\]

Conversely, suppose that $\dim \mathcal{H}_\mathcal{L}(M) \geq m$. Then there exist linearly independent $\mathcal{L}$-harmonic functions $u_1, u_2, \ldots, u_m$ in $\mathcal{H}_\mathcal{L}(M)$. Let $\bar{M}$ be the Stone-Cech compactification of $M$ and $\partial \bar{M} = \bar{M} \setminus M$. Then every function $u \in \mathcal{B}(\bar{M})$ can be extended to a continuous function $\bar{u}$ on $\bar{M}$.

We can extend $u_i$ to $\bar{u}_i$ on $\bar{M}$ in such a way that $\bar{u}_i|_{\partial \bar{M}}$, denoted by $f_i$, is continuous on $\partial \bar{M}$. By using the linear independence of $u_1, u_2, \ldots, u_m$ and the comparison principle, $f_1, f_2, \ldots, f_m$ are also linearly independent. Then there exist continuous functions $F_1, F_2, \ldots, F_m$, each of which is a linear combination of $f_1, f_2, \ldots, f_m$ and is not identically zero, such that $\{x \in \partial \bar{M} : F_i(x) = \max_{\partial \bar{M}} F_i\}$'s are mutually disjoint. (See [3].) Since each $F_i$ is a linear combination of $f_1, f_2, \ldots, f_m$, there exists a linear combination $v_i$ of $u_1, u_2, \ldots, u_m$ such that $v_i = F_i$ on $\partial \bar{M}$. We may assume that $\max_{\partial \bar{M}} F_i > 0$ for each $i = 1, 2, \ldots, m$. For given $\epsilon > 0$, put $\Omega_i^\epsilon = \{x \in \bar{M} : u_i(x) > \max_{\partial \bar{M}} F_i - \epsilon\}$. Then $\Omega_i^\epsilon$ is an $\mathcal{L}$-massive subset of $M$.
We claim that $\Omega^*_i$'s are mutually disjoint for sufficiently small $\epsilon > 0$. If this is not the case, then for some $i \neq j$, there exists a sequence $\{\epsilon_n\}_{n \in \mathbb{N}}$ such that $\lim_{n \to \infty} \epsilon_n = 0$ and $\Omega^*_i \cap \Omega^*_j \neq \emptyset$ for all $n \in \mathbb{N}$. Let $x_n \in \Omega^*_i \cap \Omega^*_j$ for each $n \in \mathbb{N}$. Since $\hat{M}$ is compact, there exists a convergent subsequence $\{x_{n_k}\}_{k \in \mathbb{N}}$ with a limit point, say $x_0 \in \hat{M}$, as $k \to \infty$. Clearly, we have $v_i(x_0) = \max_{\partial \hat{M}} F_i = \sup v_i$ and $v_j(x_0) = \max_{\partial \hat{M}} F_j = \sup v_j$. If $x_0 \in M$, then by the maximum principle, we have a contradiction. If $x_0 \in \partial M$, then $v_i(x_0) = \max_{\partial M} F_i$ and $v_j(x_0) = \max_{\partial M} F_j$, i.e., $x_0$ is a common maximum point of $F_i$ and $F_j$, which is a contradiction. This proves the claim.

By constructing a basis from inner potentials of $L$-massive subsets of a complete Riemannian manifold, we can explicitly describe the space of bounded $L$-harmonic functions on the manifold as follows:

**Theorem 2.3.** Let $M$ be a complete Riemannian manifold whose maximal number of mutually disjoint $L$-massive subsets is $m \in \mathbb{N}$, where $L = \Delta - V$. Suppose $\Omega_1, \Omega_2, \ldots, \Omega_m$ are mutually disjoint $L$-massive subsets of $M$. Let $w_i$ be an inner potential of $\Omega_i$ for each $i = 1, 2, \ldots, m$. Then we can construct a basis $\{f_1, f_2, \ldots, f_m\}$ for $\mathcal{H}_L(M)$ such that

(i) $0 \leq w_i \leq f_i \leq 1$ on $\Omega_i$ for each $i = 1, 2, \ldots, m$;

(ii) $\sup_M \sum_{i=1}^{m} f_i = 1$.

In particular, for given real numbers $a_1, a_2, \ldots, a_m \in \mathbb{R}$, there exists an $L$-harmonic function $h \in \mathcal{H}_L(M)$ such that for each $i = 1, 2, \ldots, m$,

$$\lim_{n \to \infty} h(x_{i,n}) = a_i,$$

where $\{x_{i,n}\}_{n \in \mathbb{N}}$ is a sequence in $\Omega_i$ satisfying (2.1).

Conversely, each $L$-harmonic function $h \in \mathcal{H}_L(M)$ is uniquely determined by the values in (2.2).

**Proof.** Since the maximal number of mutually disjoint $L$-massive subsets contained in $M$ is $m$, by Theorem 2.2, $\dim \mathcal{H}_L(M) = m$. Let $\Omega_1, \Omega_2, \ldots, \Omega_m$ be the mutually disjoint $L$-massive subsets of $M$ and $w_i$ be an inner potential of $\Omega_i$ for each $i = 1, 2, \ldots, m$. Then one can check that the bounded $L$-harmonic functions $f_1, f_2, \ldots, f_m$ constructed in the proof of Theorem 2.2 form a basis for $\mathcal{H}_L(M)$ satisfying

(i) $0 \leq w_i \leq f_i \leq 1$ on $\Omega_i$ for each $i = 1, 2, \ldots, m$;

(ii) $\sup_M \sum_{i=1}^{m} f_i = 1$.

For given real numbers $a_1, a_2, \ldots, a_m \in \mathbb{R}$, define $h = \sum_{j=1}^{m} a_j f_j$. Then since the sequence $\{x_{i,n}\}$ satisfies (1), we have

$$\lim_{n \to \infty} h(x_{i,n}) = \sum_{j=1}^{m} a_j \lim_{n \to \infty} f_j(x_{i,n}) = \sum_{j=1}^{m} a_j \delta_{ij} = a_i$$

for each $i = 1, 2, \ldots, m$. 


Conversely, let $h$ be a function in $\mathcal{H}\mathcal{B}\mathcal{C}(M)$ satisfying (2.2). Clearly, a bounded $\mathcal{L}$-harmonic function $\sum_{j=1}^{m} a_j f_j$ also satisfies (2.2). Putting $g = h - \sum_{j=1}^{m} a_j f_j$, there exist $c_1, c_2, \ldots, c_m \in \mathbb{R}$ such that $g = \sum_{j=1}^{m} c_j f_j$. Then from the definition of $\{x_{i,n}\}$, we have

$$c_i = \lim_{n \to \infty} g(x_{i,n}) = \lim_{n \to \infty} h(x_{i,n}) - \sum_{j=1}^{m} a_j \lim_{n \to \infty} f_j(x_{i,n}) = a_i - \sum_{j=1}^{m} a_j \delta_{ij} = 0$$

for each $i = 1, 2, \ldots, m$. This implies that $g \equiv 0$ on $M$, i.e., $h \equiv \sum_{j=1}^{m} a_j f_j$ on $M$. \hfill \Box

3. $\mathcal{L}$-massivity and bounded $\mathcal{L}$-harmonic functions on ends

Let $M$ be a complete Riemannian manifold and $o$ be a fixed point in $M$. We denote by $\sharp(r)$ the number of unbounded components of $M \setminus B_r(o)$. It is easy to prove that $\sharp(r)$ is nondecreasing in $r > 0$. Let $\lim_{r \to \infty} \sharp(r) = l$, where $l$ may be infinity, then we say that the number of ends of $M$ is $l$. If $l$ is finite, then we can choose $r_0 > 0$ in such a way that $\sharp(r) = l$ for all $r \geq r_0$. In this case, there exist mutually disjoint unbounded components $E_1, E_2, \ldots, E_l$ of $M \setminus \overline{B}_{r_0}(o)$ and we call each $E_i$ an end of $M$ for $i = 1, 2, \ldots, l$. We say that an end $E$ of $M$ is $\mathcal{L}$-nonparabolic if there exists a continuous function $u_E$, called an $\mathcal{L}$-harmonic measure, on $E \setminus B_{r_1}(o)$ for some $r_1 \geq r_0$ such that

$$\begin{aligned}
\mathcal{L} & u_E = 0 & \text{on } E \setminus \overline{B}_{r_1}(o); \\
u_E & = 0 & \text{on } \partial B_{r_1}(o) \cap E; \\
\sup_{E \setminus \overline{B}_{r_1}(o)} u_E & = 1.
\end{aligned}$$

Otherwise, $E$ is called an $\mathcal{L}$-parabolic end.

For an end $E$ of $M$, $\mathcal{H}\mathcal{B}\mathcal{C}(E, \partial E)$ denotes the space of all $\mathcal{L}$-harmonic functions on $E$ vanishing at $\partial E$. Let $\Omega_1, \Omega_2, \ldots, \Omega_s$ be the mutually disjoint $\mathcal{L}$-massive subsets of $E$ and $w_i$ be an inner potential of $\Omega_i$ for each $i = 1, 2, \ldots, s$. For each $i = 1, 2, \ldots, s$ and sufficiently large $r > r_1$, define a continuous function $g_{i,r}$ on $B_r(o) \cap E$ such that

$$\begin{aligned}
\mathcal{L} & g_{i,r} = 0 & \text{on } B_r(o) \cap E; \\
g_{i,r} & = w_i & \text{on } (\partial B_r(o) \cap E) \cap \Omega_i; \\
g_{i,r} & = 0 & \text{on } \partial E; \\
g_{i,r} & = 0 & \text{on } (\partial B_r(o) \cap E) \setminus \Omega_i.
\end{aligned}$$

By the comparison principle, $\{g_{i,r}\}$ is increasing in $r$, hence has a limit function $g_i$. In particular, $g_1, g_2, \ldots, g_s$ are linearly independent bounded $\mathcal{L}$-harmonic functions on $E$, each of which satisfies

(i) $0 \leq w_i \leq g_i \leq 1$;
(ii) $\sup_{\Omega_i} g_i = 1$;
(iii) $\sup_{E} \sum_{i=1}^{s} g_i = 1$. 

These together with the assumption that $\Omega_1, \Omega_2, \ldots, \Omega_s$ are the mutually disjoint $L$-massive sets imply that for each $i = 1, 2, \ldots, s$, there exists a sequence $\{x_{i,n}\}_{n \in \mathbb{N}}$ in $\Omega_i$ such that
\begin{equation}
\lim_{n \to \infty} g_j(x_{i,n}) = \delta_{ij}.
\end{equation}

Arguing similarly as in the proof of Theorem 2.2, we have the following theorem:

**Theorem 3.1.** Let $E$ be an end of a complete Riemannian manifold and $L = \Delta - V$. Then for each $s \in \mathbb{N}$, \(\dim H^s(E, \partial E) \geq s\) if and only if there exist mutually disjoint $L$-massive subsets $\Omega_1, \Omega_2, \ldots, \Omega_s$ of $E$.

Suppose that the maximal number of mutually disjoint $L$-massive subsets contained in $E$ is $s \in \mathbb{N}$. Then, by Theorem 3.1, \(\dim H^s(E, \partial E) = s\). Arguing similarly as in the proof of Theorem 2.3, we have the following theorem:

**Theorem 3.2.** Let $E$ be an end of a complete Riemannian manifold, whose maximal number of mutually disjoint $L$-massive subsets in $E$ is $s \in \mathbb{N}$, where $L = \Delta - V$. Suppose $\Omega_1, \Omega_2, \ldots, \Omega_s$ are mutually disjoint $L$-massive subsets of $E$. Let $w_i$ be an inner potential of $\Omega_i$ for each $i = 1, 2, \ldots, s$. Then we can construct a basis $\{g_1, g_2, \ldots, g_s\}$ for $H^s(E, \partial E)$ such that
\begin{enumerate}
\item $0 \leq w_i \leq g_i \leq 1$ on $\Omega_i$ for each $i = 1, 2, \ldots, s$;
\item $\sup_E \sum_{i=1}^{s} g_i = 1$.
\end{enumerate}

In particular, for given real numbers $a_1, a_2, \ldots, a_s \in \mathbb{R}$, there exists an $L$-harmonic function $h \in H^s(E, \partial E)$ such that for each $i = 1, 2, \ldots, s$,
\begin{equation}
\lim_{n \to \infty} h(x_{i,n}) = a_i,
\end{equation}
where $\{x_{i,n}\}_{n \in \mathbb{N}}$ is a sequence in $\Omega_i$ satisfying (3.1).

Conversely, each $L$-harmonic function $h \in H^s(E, \partial E)$ is uniquely determined by the values in (3.2).

4. Proof of main results

In this section, we give the relation between the dimension of various spaces of $L$-harmonic functions on the whole manifold and those on its ends. To begin with, we give a characterization of $L$-parabolicity of ends in terms of $L$-massivity as follows:

**Lemma 4.1.** Suppose that the maximal number of mutually disjoint $L$-massive subsets contained in $M$ is $m$. Then we can choose mutually disjoint $L$-massive subsets $\Omega_1, \Omega_2, \ldots, \Omega_m$ in such a way that for each $\Omega_i$, there exists an $L$-nonparabolic end $E$ such that $\Omega_i \subset E$.

**Proof.** Let $\Omega_1, \Omega_2, \ldots, \Omega_m$ be mutually disjoint $L$-massive subsets of $M$. We claim that for each $i = 1, 2, \ldots, m$, there exist an $L$-massive subset $\Omega'_i \subset \Omega_i$ and an $L$-nonparabolic end $E$ such that $\Omega'_i \subset E$. 

By Proposition 2.1, \( \Omega_i \setminus \overline{B_{r_0}(o)} \), \( i = 1, 2, \ldots, m \), is also \( \mathcal{L} \)-massive. Let \( w_1 \) be an inner potential of \( \Omega_1 \setminus \overline{B_{r_0}(o)} \). If an end \( E \) of \( M \) satisfies
\[
\Omega_1 \cap E \neq \emptyset \quad \text{and} \quad \sup_{x \in \Omega_1 \cap E} w_1(x) > 0,
\]
then \( \Omega_1 \cap E \) is an \( \mathcal{L} \)-massive subset of \( \Omega_1 \). In this case, other ends cannot satisfy the property (4.1). Otherwise, there is a contradiction to the maximality of the number of mutually disjoint \( \mathcal{L} \)-massive subsets of \( M \). This implies that even if there is another end \( \tilde{E} \) of \( M \) with \( \Omega_1 \cap \tilde{E} \neq \emptyset \), \( w_1 \) must be identically zero on \( \Omega_1 \cap \tilde{E} \). Therefore,
\[
\Omega'_1 = \{ x \in \Omega_1 \setminus B_{r_0}(o) : w_1(x) > 0 \}
\]
is an \( \mathcal{L} \)-massive subset and \( E \) becomes an \( \mathcal{L} \)-nonparabolic end, hence \( \Omega'_1 \) and \( E \) are the desired ones.

Applying the above argument to other \( \mathcal{L} \)-massive subsets \( \Omega_i \), \( i = 2, 3, \ldots, m \), we have the claim. \( \square \)

We are now ready to prove our main result.

**Proof of Theorem 1.1.** In the case that \( \mathcal{H}B_\mathcal{L}(M) \) is infinite dimensional, by Theorem 3.1, \( M \) can have infinitely many mutually disjoint \( \mathcal{L} \)-massive subsets. Then by Lemma 4.1, at least one end \( E \) of \( M \) must contain infinitely many mutually disjoint \( \mathcal{L} \)-massive subsets, since the number of ends of \( M \) is finite. Thus for any \( m \in \mathbb{N} \), there are mutually disjoint \( \mathcal{L} \)-massive subsets \( \Omega_1, \Omega_2, \ldots, \Omega_m \) of the end \( E \). Then by Theorem 3.1, the dimension of the space of bounded \( \mathcal{L} \)-harmonic functions on the end \( E \), which vanish at its boundary \( \partial E \), is greater than or equal to \( m \). Since \( m \in \mathbb{N} \) is arbitrarily chosen, the function space \( \mathcal{H}B_\mathcal{L}(E, \partial E) \) is infinite dimensional.

Conversely, in the case that the function space \( \mathcal{H}B_\mathcal{L}(E, \partial E) \) on an end \( E \) is infinite dimensional, by Theorem 3.1, the end \( E \) has infinitely many mutually disjoint \( \mathcal{L} \)-massive subsets, hence so does \( M \). By Theorem 2.2, this implies that \( \mathcal{H}B_\mathcal{L}(M) \) is infinite dimensional.

Suppose that the dimension of \( \mathcal{H}B_\mathcal{L}(M) \) is \( m \in \mathbb{N} \). Then by Theorem 3.1 and Lemma 4.1, we can choose mutually disjoint \( \mathcal{L} \)-massive subsets
\[
\Omega^1_1, \Omega^2_1, \ldots, \Omega^1_{s(1)}, \Omega^2_1, \Omega^2_2, \ldots, \Omega^2_{s(2)}, \ldots, \Omega^1_1, \Omega^2_2, \ldots, \Omega^i_{s(i)},
\]
where \( \Omega^1_i, \Omega^2_i, \ldots, \Omega^i_{s(i)} \) denote the mutually disjoint \( \mathcal{L} \)-massive subsets contained in \( E_i \) for each \( i = 1, 2, \ldots, l \) and \( s(1) + s(2) + \cdots + s(l) = m \). This implies that the maximal number of mutually disjoint \( \mathcal{L} \)-massive subsets contained in \( E_i \) is \( s(i) \) for each \( i = 1, 2, \ldots, l \). Now let \( w^j_i \) be an inner potential of \( \Omega^j_i \) for each \( j = 1, 2, \ldots, s(i) \) and \( i = 1, 2, \ldots, l \). By Theorem 2.3, we can find a basis
\[
\{ f^1_1, f^2_1, \ldots, f^s_{1(1)}, f^1_2, f^2_2, \ldots, f^s_{1(2)}, \ldots, f^1_1, f^2_2, \ldots, f^s_{1(s(i))} \}
\]
for \( \mathcal{H}B_\mathcal{L}(M) \) such that for \( j = 1, 2, \ldots, s(i) \) and \( i = 1, 2, \ldots, l \),
\begin{enumerate}
\item[(i)] \( 0 \leq w^j_i \leq f^j_i \leq 1 \); 
\end{enumerate}
(ii) $\sup_M \sum_{i=1}^l \sum_{j=1}^{s(i)} f_j^i = 1$.

Since $\sup_{\Omega_j^i} w_j^i = 1$, there exists a sequence $\{x_{j,n}^i\}_{n \in \mathbb{N}}$ in $\Omega_j^i$ such that for each $j = 1, 2, \ldots, s(i)$ and $i = 1, 2, \ldots, l$, $\lim_{n \to \infty} w_j^i(x_{j,n}^i) = 1$, hence

$$\lim_{n \to \infty} f_r^i(x_{j,n}^i) = \delta_{ik} \delta_{rj}.$$  

By Theorem 3.2, we can find a basis $\{g_1^i, g_2^i, \ldots, g_{s(i)}^i\}$ for $\mathcal{H}B_L(E_i, \partial E_i)$ such that for $j = 1, 2, \ldots, s(i)$ and $i = 1, 2, \ldots, l$,

(i) $0 \leq w_j^i \leq g_j^i \leq 1$;
(ii) $\sup_{E_i} \sum_{j=1}^{s(i)} g_j^i = 1$.

Since $\sup_{\Omega_j^i} w_j^i = 1$,

$$\lim_{n \to \infty} g_j^i(x_{j,n}^i) = \delta_{rj}$$

for each $j = 1, 2, \ldots, s(i)$ and $i = 1, 2, \ldots, l$.

Let $h$ be a function in $\mathcal{H}B_L(M)$. Combining Theorem 2.3, Lemma 4.1 and Theorem 3.2, we can construct a unique function $h_i$ in $\mathcal{H}B_L(E_i, \partial E_i)$ in such a way that

$$\lim_{n \to \infty} h_i(x_{j,n}^i) = \lim_{n \to \infty} h(x_{j,n}^i)$$

for each $j = 1, 2, \ldots, s(i)$. In fact, if $h = \sum_{i=1}^l \sum_{j=1}^{s(i)} a_j^i f_j^i$, then $h_i = \sum_{j=1}^{s(i)} a_j^i g_j^i$.

Let us define $\Phi : \mathcal{H}B_L(M) \to \prod_{i=1}^l \mathcal{H}B_L(E_i, \partial E_i)$ by

$$\Phi(h) = (h_1, h_2, \ldots, h_l).$$

Then by the uniqueness of the $\mathcal{L}$-harmonic functions $h_1, h_2, \ldots, h_l$, the map $\Phi$ is well defined.

Clearly, the map $\Phi$ is linear.

If $h = \sum_{i=1}^l \sum_{j=1}^{s(i)} a_j^i f_j^i \in \ker \Phi$, i.e., $\Phi(h) = (h_1, h_2, \ldots, h_l) = (0, 0, \ldots, 0)$, then

$$a_j^i = \lim_{n \to \infty} h(x_{j,n}^i) = \lim_{n \to \infty} h_i(x_{j,n}^i) = 0$$

for each $j = 1, 2, \ldots, s(i)$ and $i = 1, 2, \ldots, l$. Hence $h \equiv 0$ on $M$. Therefore, the map $\Phi$ is injective.

Let $(h_1, h_2, \ldots, h_l) \in \prod_{i=1}^l \mathcal{H}B_L(E_i, \partial E_i)$. Then we may write

$$(h_1, h_2, \ldots, h_l) = \left( \sum_{j=1}^{s(1)} a_j^1 g_j^1, \sum_{j=1}^{s(2)} a_j^2 g_j^2, \ldots, \sum_{j=1}^{s(l)} a_j^l g_j^l \right)$$

Let $h = \sum_{i=1}^l \sum_{j=1}^{s(i)} a_j^i f_j^i$. Then $h \in \mathcal{H}B_L(M)$ and $\Phi(h) = (h_1, h_2, \ldots, h_l)$, i.e., the map $\Phi$ is surjective.

Arguing similarly as in the proof of Theorem 1.1, we get the same result in the case of bounded energy finite $\mathcal{L}$-harmonic functions as follows:

$$\square$$
Corollary 4.2. Let $M$ be a complete Riemannian manifold and $\mathcal{L} = \Delta - V$. Let $E_1, E_2, \ldots, E_l$, $l \geq 1$, be $\mathcal{L}$-nonparabolic ends of $M$. Then $\mathcal{HBD}_\mathcal{L}(M)$ has the same dimension as the dimension of $\prod_{i=1}^{l} \mathcal{HBD}_\mathcal{L}(E_i, \partial E_i)$, where $\mathcal{HBD}_\mathcal{L}(X)$ and $\mathcal{HBD}_\mathcal{L}(X, \partial X)$ denote the space of bounded energy finite $\mathcal{L}$-harmonic functions on $X$ and the subspace of elements of $\mathcal{HBD}_\mathcal{L}(X)$ vanishing at $\partial X$, respectively.

In particular, in the case when $\mathcal{HBD}_\mathcal{L}(M)$ is finite dimensional, there exists an isomorphism

$$\Phi : \mathcal{HBD}_\mathcal{L}(M) \rightarrow \prod_{i=1}^{l} \mathcal{HBD}_\mathcal{L}(E_i, \partial E_i).$$

Applying our argument to the case of harmonic functions, we have the following isomorphism between the space of bounded harmonic functions (with finite Dirichlet integral, respectively) on a complete Riemannian manifold and the Cartesian product of those on its ends:

Corollary 4.3. Let $M$ be a complete Riemannian manifold with nonparabolic ends $E_1, E_2, \ldots, E_l$, $l \geq 1$. Then $\mathcal{H}(M)$ has the same dimension as the dimension of $\prod_{i=1}^{l} \mathcal{H}(E_i, \partial E_i)$, where $\mathcal{H}(X)$ and $\mathcal{H}(X, \partial X)$ denote the space of bounded harmonic functions on $X$ and the subspace of elements of $\mathcal{H}(X)$ vanishing at $\partial X$, respectively.

In particular, in the case when $\mathcal{H}(M)$ is finite dimensional, there exists an isomorphism

$$\Phi : \mathcal{H}(M) \rightarrow \prod_{i=1}^{l} \mathcal{H}(E_i, \partial E_i).$$

Also, $\mathcal{HBD}(M)$ has the same dimension as that of $\prod_{i=1}^{l} \mathcal{HBD}(E_i, \partial E_i)$, where $\mathcal{HBD}(X)$ and $\mathcal{HBD}(X, \partial X)$ denote the space of bounded harmonic functions with finite Dirichlet integral on $X$ and the subspace of elements of $\mathcal{HBD}(X)$ vanishing at $\partial X$, respectively.

In particular, in the case when $\mathcal{HBD}(M)$ is finite dimensional, there exists an isomorphism

$$\Phi : \mathcal{HBD}(M) \rightarrow \prod_{i=1}^{l} \mathcal{HBD}(E_i, \partial E_i).$$

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