ON Π-ARMENDARIZ RINGS

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Abstract. We in this note introduce a concept, so called π-Armendariz ring, that is a generalization of both Armendariz rings and 2-primal rings. We first observe the basic properties of π-Armendariz rings, constructing typical examples. We next extend the class of π-Armendariz rings, through various ring extensions.

1. Introduction

Throughout this note all rings are associative with identity unless otherwise stated. Let $R$ be a ring. The polynomial ring with an indeterminate $x$ over $R$ and the $n$ by $n$ matrix ring over $R$ are denoted by $R[x]$ and $\text{Mat}_n(R)$, respectively. The prime radical (i.e., the intersection of all prime ideals) of $R$ and the set of all nilpotent elements in $R$ are denoted by $P(R)$ and $N(R)$, respectively. $\mathbb{Z}$ denotes the ring of integers.

A ring is called reduced if it has no nonzero nilpotent elements. Over a reduced ring $R$, Armendariz [2, Lemma 1] proved that $a_ib_j = 0$ for all $i, j$ whenever $f(x)g(x) = 0$ for $f(x) = \sum_{i=0}^{m} a_ix^i, g(x) = \sum_{j=0}^{n} b_jx^j$ in $R[x]$. Due to Rege et al. [11], such rings (possibly not reduced), that satisfy Armendariz’s result, are called Armendariz. Reduced rings are Armendariz by [2, Lemma 1]. The structure of the class of non-reduced Armendariz rings was observed by many authors containing Anderson et al. [1], Hirano [5], Huh et al. [6], Kim et al. [7], Lee et al. [8], Rege et al. [11], and so on.

We call a ring $R$ a π-Armendariz provided that whenever $f(x)g(x) \in N(R[x])$ for $f(x) = \sum_{i=0}^{m} a_ix^i, g(x) = \sum_{j=0}^{n} b_jx^j$ in $R[x]$ we get $a_ib_j \in N(R)$ for all $i, j$.

Lemma 1.1. (1) [1, Proposition 1] Let $R$ be an Armendariz ring. If $f_1, \ldots, f_n \in R[x]$ are such that $f_1 \cdots f_n = 0$, then $a_1 \cdots a_n = 0$ where $a_i$ is a coefficient of $f_i$.

(2) Armendariz rings are π-Armendariz.

(3) Subrings of (π-)Armendariz rings are (π-)Armendariz.
Proof. (2) is proved by (1) and (3) is trivial. \qed

The converse of Lemma 1.1(2) need not hold by the following.

**Example 1.2.** Let $S$ be a reduced ring and

$$R = \begin{pmatrix}
  a & a_{12} & a_{13} & a_{14} \\
  0 & a & a_{23} & a_{24} \\
  0 & 0 & a & a_{34} \\
  0 & 0 & 0 & a
\end{pmatrix} \in \text{Mat}_4(S).$$

Then $R$ is $\pi$-Armendariz by Theorem 2.4 below, but $R$ is not Armendariz by [7, Example 3].

Due to Birkenmeier et al. [3], a ring $R$ is called 2-primal if $P(R) = N(R)$. It is obvious that $R$ is 2-primal if and only if $R/P(R)$ is reduced. A prime ideal $P$ of a ring $R$ is called completely prime if $R/P$ is a domain. Shin [12, Proposition 1.11] proved that a ring $R$ is 2-primal if and only if every minimal prime ideal of $R$ is completely prime, and furthermore 2-primal rings were almost completely characterized by Marks [10].

**Proposition 1.3.** 2-primal rings are $\pi$-Armendariz.

Proof. Let $R$ be a 2-primal ring and $f(x) = \sum_{i=0}^{m} a_i x^i$, $g(x) = \sum_{j=0}^{n} b_j x^j$ be in $R[x]$ such that $f(x)g(x) \in N(R[x])$. Since $R$ is 2-primal, \( \frac{R}{P(R)}[x] \cong \frac{R[x]}{P(R)[x]} \) is reduced (hence Armendariz) and so we get $a_i b_j \in N(R)$ for all $i, j$ with the help of Lemma 1.1(1). \qed

As we see in the following the converse of Proposition 1.3 need not be true by Birkenmeier et al. [4, Example 3.3] or Marks [9, Example 2.2].

**Example 1.4.** (1) Let $G$ be an abelian group which is the direct sum of a countably infinite number of infinite cyclic groups; and denote by $\{ b(0), b(1), b(-1), \ldots, b(i), b(-i), \ldots \}$ a basis of $G$. Then there exists one and only one homomorphism $u(i)$ of $G$, for $i = 1, 2, \ldots$ such that $u(i)(b(j)) = 0$ if $j \equiv 0 \pmod{2^i}$ and $u(i)(b(j)) = b(j - 1)$ if $j \not\equiv 0 \pmod{2^i}$. Denote $U$ the ring of endomorphisms of $G$ generated by the endomorphisms $u(1), u(2), \ldots$. Now let $A$ be the ring obtained from $U$ by adjoining the identity map of $G$ and let $R = A \otimes_{\mathbb{Z}} \mathbb{Q}$ and $\mathbb{Q}$ the field of rationals. Then we have $P(R) = 0$ and $0 \neq N(R) = J(R)$ by the argument in [4, Example 3.3], where $J(R)$ is the Jacobson radical of $R$. Thus $R$ is not 2-primal. Now since \( \frac{R}{J(R)}[x] \cong \frac{R[x]}{J(R)[x]} \) is reduced and $J(R)$ is nil, $R$ is $\pi$-Armendariz with the help of Lemma 1.1(1).

(2) Let $K$ be a field and let $S = K[t_i]_{i \in \mathbb{Z}}/\langle \{ t_{n_1} t_{n_2} t_{n_3} | n_3 - n_2 = n_2 - n_1 > 0 \} \rangle$, and let $R = S[x; \sigma]$ where $\sigma$ is the $K$-homomorphism of $S$ satisfying $\sigma(t_i) = t_{i+1}$ for all $i \in \mathbb{Z}$. Then we have $P(R) = 0$ and $0 \neq N(R) = N^*(R)$ by
the computation in [9, Example 2.2] where $N^*(R)$ is the sum of all nil ideals in $R$. Thus $R$ is not 2-primal. Now since $\frac{R}{N^*(R)[x]} \cong \frac{R[x]}{N^*(R)[x]}$ is reduced, $R$ is $\pi$-Armendariz with the help of Lemma 1.1(1).

2. Basic structure of $\pi$-Armendariz rings

In this section we study the properties of $\pi$-Armendariz rings and construct examples which are necessary in the process. $\prod$ denotes the direct product.

**Lemma 2.1.** (1) A finite direct product of $\pi$-Armendariz rings is $\pi$-Armendariz.

(2) A finite subdirect product of $\pi$-Armendariz rings is $\pi$-Armendariz.

**Proof.** (1) Let $R_1, R_2, \ldots, R_n$ be $\pi$-Armendariz rings and let $R = \prod_{k=1}^{n} R_k$. Consider $f(x) = \sum_{i=0}^{m} a_ix^i$, $g(x) = \sum_{j=0}^{n} b_jx^j$ in $R[x]$ such that $fg \in N(R[x])$, where $a_i = (a_{i1}, a_{i2}, \ldots, a_{in})$, $b_j = (b_{j1}, b_{j2}, \ldots, b_{jn})$ in $R$. For each $k = 1, 2, \ldots, n$, we put $f_k(x) = \sum_{i=0}^{m} a_{ik}x^i$, $g_k(x) = \sum_{j=0}^{n} b_{jk}x^j$ in $R_k[x]$. Then $f_kg_k \in N(R_k[x])$. So by $\pi$-Armendarizness of $R_k$, $a_{ik}b_{jk} \in N(R_k)$ for all $i, j$. Thus for each $i, j$, there exists positive integer $m_{ijk}$ such that $(a_{ik}b_{jk})^{m_{ijk}} = 0$. Take $m_{ij} = \max\{m_{ijk} \mid k = 1, 2, \ldots, n\}$, then $(a_{ib_j})^{m_{ij}} = ((a_{ik}b_{jk})^{m_{ijk}}) = 0$. Thus $a_{ib_j} \in N(R)$ for all $i, j$. Therefore $R$ is $\pi$-Armendariz.

(2) is obtained from (1) and Lemma 1.1 (3). $\square$

**Lemma 2.2.** For a ring $R$ suppose that $R/I$ is $\pi$-Armendariz for some ideal $I$ of $R$. If $I$ is nil then $R$ is $\pi$-Armendariz.

**Proof.** Suppose that $f(x) = \sum_{i=0}^{m} a_ix^i$, $g(x) = \sum_{j=0}^{n} b_jx^j \in R[x]$ are such that $f(x)g(x) \in N(R[x])$. Write $\bar{R} = R/I$ and $\bar{r} = r + I$. Then $\bar{f}(x)\bar{g}(x) \in N(\bar{R}[x])$. Since $\bar{R}$ is $\pi$-Armendariz, $a_ib_j \in N(\bar{R})$ for each $i, j$. But $I$ is nil, $a_{ib_j} \in N(R)$ for each $i, j$.

In Lemma 2.2 the condition “$I$ is nil” is not superfluous by the following. Let $R$ be an algebra over a commutative ring $S$. The *Dorror extension of $R$ by $S$*, written by $R \oplus_D S$, is the ring $R \oplus S$ with operations $(r_1, s_1) + (r_2, s_2) = (r_1 + r_2, s_1 + s_2)$ and $(r_1, s_1)(r_2, s_2) = (r_1r_2 + s_1r_2 + s_2r_1, s_1s_2)$, where $r_i \in R$ and $s_i \in S$.

**Example 2.3.** Let $A$ be an algebra over $\mathbb{Z}$ such that $A^2 = 0$. Then $Mat_2(A)$ is nilpotent. So by Proposition 3.4 below, $Mat_2(A) \oplus_D \mathbb{Z}$ is $\pi$-Armendariz. Next consider $R = Mat_2(\mathbb{Z} \oplus A) \oplus_D \mathbb{Z}$ and an ideal $I = Mat_2(\mathbb{Z} \times 0) \oplus_D 0$ of $R$. Then $I \cong Mat_2(\mathbb{Z})$ and $R/I \cong Mat_2(A) \oplus_D \mathbb{Z}$ is $\pi$-Armendariz. Note that $I$ is not nil and is not $\pi$-Armendariz (as a ring without identity) by the computation in Example 2.5 below. Thus $R$ is not $\pi$-Armendariz.

Let $UTM_n(R)$ (resp. $LTM_n(R)$) denotes the $n$ by $n$ upper (resp. lower) triangular matrix ring over a ring $R$. $\oplus$ denotes the direct sum. The following is one of our main results.
Theorem 2.4. Let \( R \) be a ring. Then the following conditions are equivalent:

1. \( R \) is \( \pi \)-Armendariz;
2. \( UTM_n(R) \) is \( \pi \)-Armendariz for each \( n \geq 1 \);
3. \( \begin{pmatrix} a & a_{12} & \cdots & a_{1n} \\ a & a & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a \end{pmatrix} \in Mat_n(R) \) is \( \pi \)-Armendariz;
4. \( LTM_n(R) \) is \( \pi \)-Armendariz for each \( n \geq 1 \);
5. \( \begin{pmatrix} b & 0 & \cdots & 0 \\ b_{21} & b & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ b_{n1} & b_{n2} & \cdots & b \end{pmatrix} \in Mat_n(R) \) is \( \pi \)-Armendariz.

Proof. (1) \( \Rightarrow \) (2): Let \( I = \{ A \in U \mid \text{each diagonal entry of } A \text{ is zero} \} \), where \( U = UTM_n(R) \). Then \( I \) is nilpotent ideal of \( U \) and \( U/I \cong R \oplus R \oplus \cdots \oplus R \). So \( U/I \) is \( \pi \)-Armendariz by Lemma 2.1. Thus, by Lemma 2.2, \( U \) is also \( \pi \)-Armendariz.

(2) \( \Rightarrow \) (3) \( \Rightarrow \) (1) are trivial. The proof of (1) \( \Rightarrow \) (4) \( \Rightarrow \) (5) \( \Rightarrow \) (1) is similar to (1) \( \Rightarrow \) (2) \( \Rightarrow \) (3) \( \Rightarrow \) (1). \( \square \)

From Theorem 2.4, one may suspect that if \( R \) is \( \pi \)-Armendariz then \( Mat_n(R) \) is \( \pi \)-Armendariz for \( n \geq 2 \). But the following example erases the possibility.

Example 2.5. Let \( R \) be a ring and let \( S = Mat_2(R) \). Let \( f(x) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix}x \) and \( g(x) = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}x \) be polynomials in \( S[x] \). Then \( f(x)g(x) = 0 \). But \( \begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & 0 \end{pmatrix} \) is not nilpotent. Thus \( S \) is not \( \pi \)-Armendariz.

Proposition 2.6. Let \( \{ R_\alpha \mid \alpha \in \Lambda \text{ an index set} \} \) be a family of \( \pi \)-Armendariz rings. If \( R = \prod_{\alpha \in \Lambda} R_\alpha \) is of bounded index of nilpotency, then \( R \) is \( \pi \)-Armendariz.

Proof. We put \( N(\geq 1) \) as the index of nilpotency of \( R \). Then \( R_\alpha \) is of bound index of nilpotency \( \leq N \), for each \( \alpha \in \Lambda \). Consider \( f(x) = \sum_{i=0}^m a_i x^i, g(x) = \sum_{j=0}^n b_j x^j \in R[x] \) such that \( fg \in N(R[x]) \), where \( a_i = (a_{i\alpha})_{\alpha \in \Lambda}, b_j = (b_{j\alpha})_{\alpha \in \Lambda} \in R \). For \( \alpha \in \Lambda \), we put \( f_\alpha(x) = \sum_{i=0}^m a_{i\alpha} x^i, g_\alpha(x) = \sum_{j=0}^n b_{j\alpha} x^j \in R_\alpha[x] \), then \( f_\alpha g_\alpha \in N(R_\alpha[x]) \). Since \( R_\alpha \) is a \( \pi \)-Armendariz ring of bounded index of nilpotency \( \leq N \), \((a_{i\alpha} b_{j\alpha})^N = 0 \) for all \( i, j \). Thus \( (a_i b_j)^N = ((a_{i\alpha} b_{j\alpha})_{\alpha \in \Lambda})^N = (a_{i\alpha} b_{j\alpha})^N \in (a_{i\alpha} b_{j\alpha})^N = 0 \) for each \( i, j \). Therefore \( (a_i b_j) \in N(R) \) and so \( R \) is \( \pi \)-Armendariz. \( \square \)
In Proposition 2.6, the condition "of bounded index of nilpotency" is not superfluous by the following.

**Example 2.7.** Let \( R_n = UTM_{2^n}(Z) \) \((n = 1, 2, \ldots)\). Then by Theorem 2.4, \( R_n \) is a \( \pi \)-Armendariz ring. But their direct product \( R = \prod_{n \geq 1} R_n \) is not \( \pi \)-Armendariz.

Consider two polynomials \( f(x) = A_0 + A_1 x, \ g(x) = B_0 + B_1 x \) in \( R[x] \) where

\[
A_k = \begin{pmatrix} A_{k1} & (1) & (-1)^k \\ 0 & 0 & \vdots, A_{kn} & = \begin{pmatrix} A_{k(n-1)} \\ 0 \end{pmatrix} & C_{k(n-1)} \\ A_{k(n-1)} \end{pmatrix}, \]

\[
B_k = \begin{pmatrix} B_{k1} & 0 & (-1)^{k+1} \\ 0 & 1 & \vdots, B_{kn} & = \begin{pmatrix} B_{k(n-1)} \\ 0 \end{pmatrix} & D_{k(n-1)} \\ B_{k(n-1)} \end{pmatrix}, 
\]

and \( C_{kn} = ((-1)^{k+1})_{2^n \times 2^n}, \ D_{kn} = ((-1)^{k+1})_{2^n \times 2^n} \in \text{Mat}_{2^n \times 2^n}(Z) \) for each \( k = 0, 1 \) and \( n = 1, 2, \ldots \). Now we will show that \( fg = 0 \in N(R[x]) \), but \( A_0 B_1 \not\in N(R) \), that is \( R \) is not \( \pi \)-Armendariz. To complete our result, we prove the following claim by induction on \( n \).

**Claim.** 1. \( A_{kn} B_{kn} = 0 \) for \( k = 0, 1 \) and \( n = 1, 2, \ldots \).

2. \( A_{0n} B_{1n} = -A_{1n} B_{0n} \) such that \( A_{0n} B_{1n} \in N(R_n) \setminus \{0\} \) for \( n = 1, 2, \ldots \).

If \( n = 1 \) then \( A_{k1} B_{k1} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \) for \( k = 0, 1 \). \( A_{11} B_{11} = \begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix} = -A_{11} B_{01} \not= 0 \) and \( (A_{11} B_{11})^2 = 0 \). We are done. Now suppose that it holds for \( n < l \) \((l \geq 2)\) and let \( n = l \). Note that \( A_{kn} D_{kn} = C_{kn} B_{kn} = 0 \) for \( k = 0, 1 \) and

\[
A_{0n} D_{1n} + C_{0n} B_{1n} = \begin{pmatrix} 2^n & 2^n + 2 & 2^n + 2 & \ldots & 2^{n+1} - 2 & 2^{n+1} - 2 & 2^{n+1} \\ 2^n - 2 & 2^n & 2^n & \ldots & 2^{n+1} - 4 & 2^{n+1} - 4 & 2^{n+1} - 2 \\ 2^n - 2 & 2^n & 2^n & \ldots & 2^{n+1} - 4 & 2^{n+1} - 4 & 2^{n+1} - 2 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 2 & 4 & 4 & \ldots & 2^n & 2^n & 2^n + 2 \\ 2 & 4 & 4 & \ldots & 2^n & 2^n & 2^n + 2 \\ 0 & 2 & 2 & \ldots & 2^n - 2 & 2^n - 2 & 2^n \end{pmatrix} = \begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix},
\]

that is \( A_{0n} D_{1n} + C_{0n} B_{1n} = -(A_{1n} D_{0n} + C_{1n} B_{0n}) = 2^n I_{2^n} + \text{(some matrix whose entries are all non-negative)} \), where \( I_{2^n} \) is the \( 2^n \times 2^n \) identity matrix.
By notes and inductive hypothesis,

\[
A_{kl}B_{kl} = \begin{pmatrix}
A_{k(t-1)} & C_{k(t-1)} \\
0 & A_{k(t-1)}
\end{pmatrix}
\begin{pmatrix}
B_{k(t-1)} & D_{k(t-1)} \\
0 & B_{k(t-1)}
\end{pmatrix}
= \begin{pmatrix}
A_{k(t-1)}B_{k(t-1)} & A_{k(t-1)}D_{k(t-1)} + C_{k(t-1)}B_{k(t-1)} \\
0 & A_{k(t-1)}B_{k(t-1)}
\end{pmatrix} = 0.
\]

Also

\[
A_{0l}B_{1l} = \begin{pmatrix}
A_{0(t-1)}B_{1(t-1)} & A_{0(t-1)}D_{1(t-1)} + C_{0(t-1)}B_{1(t-1)} \\
0 & A_{0(t-1)}B_{1(t-1)}
\end{pmatrix}
= \begin{pmatrix}
-A_{1(t-1)}B_{0(t-1)} & -(A_{1(t-1)}D_{0(t-1)} + C_{1(t-1)}B_{0(t-1)}) \\
0 & -A_{1(t-1)}B_{0(t-1)}
\end{pmatrix} = -A_{1l}B_{0l}.
\]

Since \(A_{0(t-1)}B_{1(t-1)}\) is in \(N(R_{i(t-1)})\), by the inductive hypothesis, there exists \(m\) in positive integers such that \((A_{0(t-1)}B_{1(t-1)})^m \neq 0\) and

\[(A_{0(t-1)}B_{1(t-1)})^{m+1} = 0.
\]

So by notes, \((A_{0l}B_{1l})^{m+1} = \begin{pmatrix} 0 & [\ast] \\ 0 & 0 \end{pmatrix}\) is not a zero matrix and \((A_{0l}B_{1l})^{2(m+1)} = 0\), that is \(A_{0l}B_{1l} \in N(R)\), \(\neq 0\). Therefore our claim is proved by the induction and so \(f(x)g(x) = 0\). Furthermore the sequence of index of \(A_{0n}B_{1n}\) is increasing. (In fact, the index of \(A_{0n}B_{1n}\) is equal to \(2^{n-1} + 1\), by using another method.) Thus \(A_0B_1\) is not a nilpotent element of \(R\).

3. More examples of \(\pi\)-Armendariz rings

In this section we extend the class of \(\pi\)-Armendariz rings through various extensions. We first consider the case of direct limits of direct systems of \(\pi\)-Armendariz rings, comparing with Lemma 2.1.

**Proposition 3.1.** The direct limit of a direct system of \(\pi\)-Armendariz rings is also \(\pi\)-Armendariz.
Proof. Let $D = \{ R_i, \alpha_{ij} \}$ be a direct system of $\pi$-Armendariz rings $R_i$ for $i \in I$ and ring homomorphisms $\alpha_{ij} : R_i \to R_j$ for each $i \leq j$ satisfying $\alpha_{ij}(1) = 1$, where $I$ is a directed partially ordered set. Set $R = \varinjlim R_i$ be the direct limit of $D$ with $\iota_i : R_i \to R$ and $\iota_j \alpha_{ij} = \iota_i$. We will prove that $R$ is a $\pi$-Armendariz ring. Take $x, y \in R$. Then $x = \iota_i(x_i), y = \iota_j(y_j)$ for some $i, j \in I$ and there is $k \in I$ such that $i \leq k, j \leq k$. Define

$$x + y = \iota_k(\alpha_{ik}(x_i) + \alpha_{jk}(y_j))$$

and

$$xy = \iota_k(\alpha_{ik}(x_i)\alpha_{jk}(y_j)),$$

where $\alpha_{ik}(x_i)$ and $\alpha_{jk}(y_j)$ are in $R_k$. Then $R$ forms a ring with $0 = \iota_i(0)$ and $1 = \iota_i(1)$.

Now suppose $f(x)g(x) \in N(R[x])$ for $f(x) = \sum_{s=0}^m a_s x^s, g(x) = \sum_{t=0}^n b_t x^t$ in $R[x]$. There are $i_s, j_t, k \in I$ such that $a_s = \iota_{i_s}(a_{i_s}), b_t = \iota_{j_t}(b_{j_t}), i_s \leq k, j_t \leq k$. So

$$a_s b_t = \iota_k(\alpha_{i_s,k}(a_{i_s})\alpha_{j_t,k}(b_{j_t})), $$

and from $f(x)g(x) \in N(R[x])$ we have

$$f(x)g(x) = \left(\sum_{s=0}^m \iota_{i_s}(\alpha_{i_s,k}(a_{i_s}))x^s\right)\left(\sum_{t=0}^n \iota_t(\alpha_{j_t,k}(b_{j_t}))x^t\right) \in N(R_k[x]).$$

But $R_k$ is $\pi$-Armendariz and so $\iota_k(\alpha_{i_s,k}(a_{i_s})\alpha_{j_t,k}(b_{j_t})) \in N(R_k)$. Thus $a_s b_t \in N(R)$ and $R$ is $\pi$-Armendariz.

Proposition 3.2. Let $R$ be a ring and $\Delta$ be a multiplicative monoid in $R$ consisting of central regular elements. Then $R$ is $\pi$-Armendariz if and only if so is $\Delta^{-1}R$.

Proof. ($\Leftarrow$) is obtained from Lemma 1.1(3). ($\Rightarrow$) Let $R$ be a $\pi$-Armendariz ring and let $S = \Delta^{-1}R$, where is a multiplicative monoid in $R$ consisting central regular elements of $R$. Note that if $f(x) = \sum_{i=0}^m a_i x^i, g(x) = \sum_{j=0}^n b_j x^j$ are in $S[x](\alpha, \beta \in S)$, then we can assume that $\alpha_i = a_i u^{-1}$ and $\beta_j = b_j v^{-1}$ for some $a_i, b_j \in R$, $u, v \in \Delta$ for all $i, j$. Now suppose that $f(x)g(x) \in N(S[x])$ then there exist a positive integer $k$ such that

$$0 = (fg)^k = \left(\sum_{i,j} a_i \beta_j x^{i+j}\right)^k$$

$$= \left(\sum_{i,j} a_i u^{-1} b_j v^{-1} x^{i+j}\right)^k = \left(\sum_{i,j} a_i b_j x^{i+j}\right)^k (uv)^k.$$

Since $(uv)^k \in \Delta$, $(\sum_{i,j} a_i b_j x^{i+j})^k = 0$ and so that $\sum_{i,j} a_i b_j x^{i+j} \in N(R[x])$.

By the hypothesis, $a_i b_j \in N(R)$ for all $i, j$. Immediately, we can show that $\alpha_i \beta_j = (a_i u^{-1} b_j v^{-1})$ is also a nilpotent element of $S$ for all $i, j$. Therefore $S$ is a $\pi$-Armendariz ring.

The ring of $Laurent$ polynomials in $x$, coefficients in a ring $R$, consists of all formal sums $\sum_{i=k}^n m_i x^i$ with obvious addition and multiplication, where $m_i \in R$ and $k, n$ are (possibly negative) integers; denotes it by $R[x; x^{-1}]$. 
Corollary 3.3. (1) A commutative ring $R$ is $\pi$-Armendariz if and only if so is the total quotient ring of $R$.

(2) Let $R$ be a ring. $R[x]$ is $\pi$-Armendariz if and only if so is $R[x; x^{-1}]$.

Proof. It suffices to show the necessity by Lemma 1.1(3).

(1) Let $\Delta$ be the set of all regular elements of $R$. Then $\Delta$ satisfies the condition of Proposition 3.2 and $\Delta^{-1}R$ is the total quotient ring of $R$. Thus the total quotient ring of $R$ is $\pi$-Armendariz.

(2) Let $\Delta = \{1, x, x^2, \ldots \} \subset R[x]$. Then $\Delta$ satisfies the condition of Proposition 3.2 and so $R[x; x^{-1}] \cong \Delta^{-1}R$ is $\pi$-Armendariz.

Proposition 3.4. Let $A$ be a nil algebra over $\mathbb{Z}$. Then $A \oplus_D \mathbb{Z}$ of $A$ by $\mathbb{Z}$ is $\pi$-Armendariz.

Proof. Since $A$ is nil, $A \oplus_D 0$ is a nil ideal of $A \oplus_D \mathbb{Z}$. Thus $A \oplus_D \mathbb{Z}$ is a ring with a nil ideal $A \oplus_D 0$ such that $\frac{A \oplus_D \mathbb{Z}}{A \oplus_D 0} \cong \mathbb{Z}$ is a $\pi$-Armendariz ring. So by Lemma 2.2, $A \oplus_D \mathbb{Z}$ is $\pi$-Armendariz.

Proposition 3.5. A ring $R$ is $\pi$-Armendariz if and only if $R[x]/(x^n)$ is $\pi$-Armendariz for any positive integer $n$, where $\langle x^n \rangle$ is the ideal of $R[x]$ generated by $x^n$.

Proof. It suffices to show the necessity by Lemma 1.1(3). Let $R$ be a $\pi$-Armendariz ring and $n$ be a positive integer. Put $S = R[x]/\langle x^n \rangle$ and $\bar{x} = x + \langle x^n \rangle$. Then $S/\bar{x} \cong R$ and so $S/\bar{x}$ is $\pi$-Armendariz. Since $S/\bar{x}$ is a nil ideal of $S$, $S$ is $\pi$-Armendariz by Lemma 2.2.

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