ON PRIME AND SEMIPRIME RINGS WITH PERMUTING 3-DERIVATIONS

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ABSTRACT. Let $R$ be a 3-torsion free semiprime ring and let $I$ be a nonzero two-sided ideal of $R$. Suppose that there exists a permuting 3-derivation $\Delta : R \times R \times R \to R$ such that the trace is centralizing on $I$. Then the trace of $\Delta$ is commuting on $I$. In particular, if $R$ is a 3-torsion free prime ring and $\Delta$ is nonzero under the same condition, then $R$ is commutative.

1. Introduction and preliminaries

Throughout this paper, $R$ will represent an associative ring, and $Z$ will be its center. Let $x, y \in R$. The commutator $yx - xy$ will be denoted by $[y, x]$. We will also use the identities $[xy, z] = [x, z]y + x[y, z]$ and $[x, yz] = [x, y]z + y[x, z]$. Let $S$ be a nonempty subset of $R$. Then a map $f : R \to R$ is said to be commuting (resp. centralizing) on $S$ if $[f(x), x] = 0$ (resp. $[f(x), x] \in Z$) for all $x \in S$. Recall that $R$ is semiprime if $Rx = \{0\}$ implies $x = 0$ and $R$ is prime if $xRx = \{0\}$ implies $x = 0$ or $y = 0$.

An additive map $d : R \to R$ is called a derivation if the Leibniz rule $d(xy) = d(x)y + xd(y)$ holds for all $x, y \in R$.

By a bi-derivation we mean a bi-additive map $D : R \times R \to R$ (i.e., $D$ is additive in both arguments) which satisfies the relations

$$D(xy, z) = D(x, z)y + xD(y, z),$$

$$D(x, yz) = D(x, y)z + yD(x, z)$$

for all $x, y \in R$. Let $D$ be symmetric, that is, $D(x, y) = D(y, x)$ for all $x, y \in R$. The map $\tau : R \to R$ defined by $\tau(x) = D(x, x)$ for all $x \in R$ is called the trace of $D$. If $R$ is a noncommutative 2-torsion free prime ring and $D : R \times R \to R$ is a symmetric bi-derivation, then it follows from [1, Theorem 3.5] that $D = 0$.

A map $\Delta : R \times R \times R \to R$ will be said to be permuting if the equation $\Delta(x_1, x_2, x_3) = \Delta(x_{\pi(1)}, x_{\pi(2)}, x_{\pi(3)})$ holds for all $x_1, x_2, x_3 \in R$ and for every permutation $\{\pi(1), \pi(2), \pi(3)\}$.

A map $\delta : R \to R$ defined by $\delta(x) = \Delta(x, x, x)$

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for all \( x \in R \), where \( \Delta : R \times R \times R \to R \) is a permuting map, is called the \textit{trace} of \( \Delta \). It is obvious that, in case when \( \Delta : R \times R \times R \to R \) is a permuting map which is also 3-additive (i.e., additive in each argument), the trace \( \delta \) of \( \Delta \) satisfies the relation
\[
\delta(x + y) = \delta(x) + \delta(y) + 3\Delta(x, x, y) + 3\Delta(x, y, y)
\]
for all \( x, y \in R \).

Since we have
\[
\Delta(0, y, z) = \Delta(0 + 0, y, z) = \Delta(0, y, z) + \Delta(0, y, z)
\]
for all \( y, z \in R \), we obtain \( \Delta(0, y, z) = 0 \) for all \( y, z \in R \). Hence we get
\[
0 = \Delta(0, y, z) = \Delta(x - x, y, z) = \Delta(x, y, z) + \Delta(-x, y, z)
\]
and so we see that \( \Delta(-x, y, z) = -\Delta(x, y, z) \) for all \( x, y, z \in R \). This tells us that \( \delta \) is an odd function.

Here we introduce the following map:

A 3-additive map \( \Delta : R \times R \times R \to R \) will be called a 3-\textit{derivation} if the relations
\[
\Delta(x_1 x_2, y, z) = \Delta(x_1, y, z)x_2 + x_1 \Delta(x_2, y, z),
\]
\[
\Delta(x, y_1 y_2, z) = \Delta(x, y_1, z)y_2 + y_1 \Delta(x, y_2, z)
\]
and
\[
\Delta(x, y, z_1 z_2) = \Delta(x, y, z_1)z_2 + z_1 \Delta(x, y, z_2)
\]
are fulfilled for all \( x, y, z, x_i, y_i, z_i \in R, \) \( i = 1, 2 \). If \( \Delta \) is permuting, then the above three relations are equivalent to each other.

For example, let \( R \) be commutative. A map \( \Delta : R \times R \times R \to R \) defined by \( (x, y, z) \mapsto d(x)d(y)d(z) \) for all \( x, y, z \in R \) is a permuting 3-derivation, where \( d \) is a derivation on \( R \).

On the other hand, let
\[
R = \left\{ \begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix} \mid a, b \in \mathbb{C} \right\},
\]
where \( \mathbb{C} \) is a complex field. It is clear that \( R \) is a noncommutative ring under matrix addition and matrix multiplication. We define a map \( \Delta : R \times R \times R \to R \) by
\[
\begin{pmatrix} a_1 & b_1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} a_2 & b_2 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} a_3 & b_3 \\ 0 & 0 \end{pmatrix} \mapsto \begin{pmatrix} 0 & a_1a_2a_3 \\ 0 & 0 \end{pmatrix}.
\]
Then it is easy to see that \( \Delta \) is a permuting 3-derivation.

A study concerning the theory of centralizing (commuting) maps on prime rings was initiated by the classical result of E. C. Posner [3] which states that the existence of a nonzero centralizing derivation on a prime ring \( R \) implies that \( R \) is commutative. Since then, a great deal of work in this context has been done by a number of authors (see, e.g., [1] and references therein). For
instance, as a study concerning centralizing (commuting) maps, J. Vukman [4, 5] investigated symmetric bi-derivations on prime and semiprime rings.

In this paper, we apply the results due to E. C. Posner [3] and J. Vukman [4] to permuting 3-derivations, respectively.

2. The main results

We first need the following well-known lemma [2].

Lemma 2.1. Let \( R \) be a prime ring. Let \( d : R \to R \) be a derivation and \( a \in R \). If \( ad(x) = 0 \) holds for all \( x \in R \), then we have either \( a = 0 \) or \( d = 0 \).

We begin our investigation of permuting 3-derivations with the next result.

Lemma 2.2. Let \( R \) be a noncommutative 3-torsion free prime ring and let \( I \) be a nonzero two-sided ideal of \( R \). Suppose that there exists a permuting 3-derivation \( \Delta : R \times R \times R \to R \) such that \( \delta \) is commuting on \( I \), where \( \delta \) is the trace of \( \Delta \). Then we have \( \Delta = 0 \).

Proof. Suppose that

\[
\delta(x), x] = 0 \quad \text{for all} \quad x \in I.
\]

The substitution \( x = x + y \) to linearize (2.1) leads to

\[
0 = [\delta(x), y] + [\delta(y), x] + 3[\Delta(x, x, y), x] + 3[\Delta(x, y, y), x]
+ 3[\Delta(x, x, y), y] + 3[\Delta(x, y, y), y] \quad \text{for all} \quad x, y \in I.
\]

Putting \(-x\) instead of \( x \) in (2.2) and comparing (2.2) with the result, we arrive at

\[
[\Delta(x, y, y), x] + [\Delta(x, x, y), y] = 0 \quad \text{for all} \quad x, y \in I
\]

since \( \delta \) is odd. We set \( x = x + y \) in (2.3) and then use (2.1) and (2.3) to obtain

\[
[\delta(y), x] + 3[\Delta(x, y, y), y] = 0 \quad \text{for all} \quad x, y \in I.
\]

Let us write in (2.4) \( yx \) instead of \( x \). Then we get

\[
0 = [\delta(y), yx] + 3[\Delta(yx, y, y), y]
= y[\delta(y), x] + 3\delta(y)[x, y] + 3y[\Delta(x, y, y), y]
= y\{[\delta(y), x] + 3[\Delta(x, y, y), y]\} + 3\delta(y)[x, y]
\]

which implies that

\[
\delta(y)[x, y] = 0 \quad \text{for all} \quad x, y \in I
\]

on account of (2.4). Since \( I \) is a nonzero noncommutative prime ring, it follows from (2.5) and Lemma 2.1 that, for all \( y \in I \) with \( y \notin Z \), we have \( \delta(y) = 0 \) since for every fixed \( y \in I \), a map \( x \mapsto [x, y] \) is a derivation on \( I \).

Now, let \( x \in I \) with \( x \in Z \) and \( y \in I \) with \( y \notin Z \). Then \( x + y \notin Z \) and \(-y \notin Z \). Thus we have

\[
0 = \delta(x + y) = \delta(x) + 3\Delta(x, x, y) + 3\Delta(x, y, y)
\]
and

$$0 = \delta(x - y) = \delta(x) - 3\Delta(x, x, y) + 3\Delta(x, y, y)$$

which shows that

$$(2.6) \quad \delta(x) + 3\Delta(x, y, y) = 0.$$ 

Replacing $y \in I$ ($y \notin Z$) by $2y$ in (2.6) and using (2.6), we obtain that

$$\Delta(x, y, y) = 0$$

and so the relation (2.6) gives $\delta(x) = 0$ for all $x \in I$ with $x \notin Z$. Therefore we conclude that $\delta(x) = 0$ for all $x \in I$.

On the other hand, since the relation $\delta(x + y) = \delta(x) + \delta(y) + 3\Delta(x, x, y) + 3\Delta(x, y, y)$ is fulfilled for all $x, y \in I$, it follows that

$$(2.7) \quad \Delta(x, x, y) + \Delta(x, y, y) = 0 \quad \text{for all} \quad x, y \in I$$

and substituting $y + z$ for $y$ in (2.7) and employing (2.7), we obtain that

$$2\Delta(x, y, z) = 0 = \Delta(x, y, z) \quad \text{for all} \quad x, y, z \in I.$$

Let us substitute $rz (r \in R)$ for $x$ in the above relation $\Delta(x, y, z) = 0$ for all $x, y, z \in I$. Then we have $\Delta(r, y, z)x = 0$, that is, $\Delta(r, y, z)I = \{0\}$. Since $R$ is prime, we get $\Delta(r, y, z) = 0$ for all $y, z \in I$ and $r \in R$. Also, substituting $ys (s \in R)$ for $y$ in this relation, we have $y\Delta(r, s, z) = 0$ and so $I\Delta(r, s, z) = \{0\}$. Again, by primeness of $R$, we obtain that $\Delta(r, s, z) = 0$ for all $z \in I$ and $r, s \in R$. Furthermore, replacing $z$ by $tz (t \in R)$ in the relation $\Delta(r, s, z) = 0$, we get $\Delta(r, s, t)z = 0$, i.e., $\Delta(r, y, t)I = \{0\}$. The primness of $R$ implies that $\Delta(r, s, t) = 0$ for all $r, s, t \in R$ which completes the proof of the theorem.

We continue with the following result for permuting 3-derivations on semiprime rings.

**Theorem 2.3.** Let $R$ be a noncommutative 3-torsion free semiprime ring and let $I$ be a nonzero two-sided ideal of $R$. Suppose that there exists a permuting 3-derivation $\Delta : R \times R \times R \rightarrow R$ such that $\delta$ is centralizing on $I$, where $\delta$ is the trace of $\Delta$. Then $\delta$ is commuting on $I$.

**Proof.** Assume that

$$(2.8) \quad [\delta(x), x] \in Z \quad \text{for all} \quad x \in I.$$ 

By linearizing (2.8) and again using (2.8), we obtain

$$(2.9) \quad Z \ni [\delta(x), y] + [\delta(y), x] + 3[\Delta(x, x, y), x] + 3[\Delta(x, y, y), x]$$

$$+ 3[\Delta(x, x, y), y] + 3[\Delta(x, y, y), y] \quad \text{for all} \quad x, y \in I.$$ 

We substitute $-x$ for $x$ in (2.9) and compare (2.9) with the result to get

$$(2.10) \quad [\Delta(x, y, y), x] + [\Delta(x, x, y), y] \in Z \quad \text{for all} \quad x, y \in I$$

since $R$ is 3-torsion free.

Replacing $x$ by $x + y$ in (2.10) and using (2.10), we have

$$(2.11) \quad [\delta(y), x] + 3[\Delta(x, y, y), y] \in Z \quad \text{for all} \quad x, y \in I.$$
Taking \( x = y^2 \) in (2.11) and invoking (2.8) show that
\[
Z \ni [\delta(y), y^2] + 3[\Delta(y^2, y, y), y] = 8[\delta(y), y]y \quad \text{for all } y \in I
\]
and commuting with \( \delta(y) \) in (2.12) gives
\[
8[\delta(y), y]^2 = 0 \quad \text{for all } y \in I.
\]

On the other hand, substituting \( x \) by \( yx \) in (2.11), we obtain
\[
Z \ni [\delta(y), yx] + 3[\Delta(yx, y, y), y] \\
= y\{[\delta(y), x] + 3[\Delta(x, y, y), y]\} \\
+ 3\delta(y)[x, y] + 4[\delta(y), y]x \quad \text{for all } x, y \in I.
\]

Hence we have, for all \( x, y \in I \),
\[
[y\{[\delta(y), x] + 3[\Delta(x, y, y), y]\}, y] + [3\delta(y)[x, y] + 4[\delta(y), y]x, y] = 0
\]
and so we get
\[
3\delta(y)[x, y], y] + 7[\delta(y), y][x, y] = 0 \quad \text{for all } x, y \in I
\]
according to (2.11).

Substituting \( \delta(y)x \) for \( x \) in (2.14), it follows that
\[
0 = \delta(y)\{3\delta(y)[x, y], y] + 7[\delta(y), y][x, y]\} \\
+ 6\delta(y)[\delta(y), y][x, y] + 7[\delta(y), y]^2x \quad \text{for all } x, y \in I
\]
which, by (2.14), implies that
\[
6\delta(y)[\delta(y), y][x, y] + 7[\delta(y), y]^2x = 0 \quad \text{for all } x, y \in I.
\]
Letting \( x = [\delta(y), y] \) in (2.15), we arrive at \( 7[\delta(y), y]^3 = 0 \) and so we get
\[
7[\delta(y), y]^2R[\delta(y), y]^2 = 0.
\]
Since \( R \) is semiprime, we deduce that
\[
7[\delta(y), y]^2 = 0 \quad \text{for all } y \in I.
\]
Hence, the relations (2.13) and (2.16) yield \( [\delta(y), y]^2 = 0 \) for all \( y \in I \). Since the center of a semiprime ring contains no nonzero nilpotent elements, we conclude that \( [\delta(y), y] = 0 \) for all \( y \in I \). This completes the proof of the theorem. \( \square \)

The following result is an analogue of Posner’s theorem [3, Theorem 2].

**Theorem 2.4.** Let \( R \) be a 3!-torsion free prime ring and let \( I \) be a nonzero two-sided ideal of \( R \). Suppose that there exists a nonzero permuting 3-derivation \( \Delta : R \times R \times R \to R \) such that \( \delta \) is centralizing on \( I \), where \( \delta \) be the trace of \( \Delta \). Then \( R \) is commutative.

**Proof.** Suppose that \( R \) is noncommutative. Then it follows from Theorem 2.3 that \( \delta \) is commuting on \( I \). Hence Lemma 2.2 gives \( \Delta = 0 \) which guarantees the conclusion of the theorem. \( \square \)
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