METANILPOTENT GROUPS WITH CHAIN CONDITIONS FOR NORMAL SUBGROUPS OF INFINITE ORDER OR INDEX

DAE HYUN PAEK

Abstract. We study the structure of metanilpotent groups satisfying the maximal condition on infinite normal subgroups or the minimal condition on normal subgroups of infinite index.

1. Introduction

A group $G$ is said to satisfy the weak maximal condition on normal subgroups if there are no infinite ascending chains $G_1 < G_2 < \cdots$ of normal subgroups of $G$ such that all the indices $[G_{i+1} : G_i]$ are infinite. The weak minimal condition on normal subgroups is defined similarly by substituting descending for ascending chains. Kurdachenko [2] considered groups satisfying the weak maximal or weak minimal conditions on normal subgroups.

A group $G$ is said to satisfy $\text{max-}\omega n$ (the maximal condition on infinite normal subgroups) if there are no infinite ascending chains of infinite normal subgroups of $G$. Similarly a group $G$ is said to satisfy $\text{min-}\omega n$ (the minimal condition on normal subgroups of infinite index) if there are no infinite descending chains of normal subgroups with infinite index in $G$. Clearly $\text{max-}\omega n$ (min-$\omega n$) implies the weak maximal condition (weak minimal condition, respectively) on normal subgroups. We note that the additive group of $p$-adic rationals satisfies both the weak maximal and weak minimal conditions on subgroups, but it does not satisfy $\text{max-}\omega$ (the maximal condition on infinite subgroups) or $\text{min-}\omega$ (the minimal condition on subgroups of infinite index). Thus we are dealing with stronger properties than the usual weak chain conditions.

Since the chain conditions $\text{max-}\omega n$ and $\text{min-}\omega n$ are weaker than the chain conditions $\text{max-}n$ and $\text{min-}n$ (the maximal and minimal conditions on normal subgroups, respectively), we define a group satisfies $\text{max-}\omega n^*$ if it satisfies

Received July 1, 2007.

2000 Mathematics Subject Classification. 20E15.

Key words and phrases. maximal condition, minimal condition.

This work was supported by the Korea Research Foundation Grant funded by the Korean Government(MOEHARD, Basic Research Promotion Fund)(KRF-2006-003-C00010).

©2008 The Korean Mathematical Society

105
max-\(\infty n\), but not max-\(n\) and a group satisfies min-\(\infty n^*\) if it satisfies min-\(\infty n\), but not min-\(n\).

De Giovanni et al. [1] characterized the structure of groups satisfying max-\(\infty n^*\) or min-\(\infty n^*\). In addition, the structure of nonfinitely generated solvable groups satisfying max-\(\infty n^*\) and solvable groups satisfying min-\(\infty n^*\) were investigated in detail. In this paper, we consider metanilpotent groups with these chain conditions.

2. Metanilpotent groups with max-\(\infty n^*\)

We describe our first result on the structure of metanilpotent groups satisfying max-\(\infty n^*\).

**Theorem 2.1.** Let \(G\) be a metanilpotent group with \(F = \text{Fit}(G)\) and \(Z = Z(F)\). Then \(G\) satisfies max-\(\infty n^*\) if and only if the following conditions hold:

1. \(F\) is infinite nilpotent and \(G/F\) has max-\(n\);
2. if \(L\) is an infinite normal subgroup of \(G\), then \(Z/Z \cap L\) is finite.

**Proof.** Suppose that \(G\) satisfies max-\(\infty n^*\). Let \(N\) be a normal nilpotent subgroup of \(G\) such that \(G/N\) is nilpotent.

**Case: \(N\) is finite.** \(G/N\) is Prüfer-by-finite and so is \(G\) ([3, Theorem 4.2]). Let \(P\) be a normal Prüfer subgroup of \(G\) with \(G/P\) finite. Then \(F/P\) is finite and so \(F = \langle a_1, a_2, \ldots, a_k, P \rangle\) for some \(a_1, a_2, \ldots, a_k\). Since each \(a_i\) is contained in some normal nilpotent subgroup of \(G\), it follows that \(F\) is a product of finitely many normal nilpotent subgroups of \(G\). Thus \(F\) is infinite nilpotent and \(G/F\) has max-\(n\).

**Case: \(N\) is infinite.** \(G/N\) is finitely generated nilpotent and so is \(F/N\). Hence \(F\) is a product of finitely many normal nilpotent subgroups of \(G\) by the argument of the last paragraph. Thus \(F\) is an infinite nilpotent subgroup of \(G\) and \(G/F\) has max-\(n\).

Let \(G = \langle g_1, g_2, \ldots, g_r, F \rangle\) for some \(g_1, g_2, \ldots, g_r\). Then any element \(g\) in \(G\) can be written as \(g = aw\) where \(a \in F\) and \(w \in \langle g_1, g_2, \ldots, g_r \rangle\). If \(x \in Z\), then \(x^y = x^{aw} = x^w \in \langle g_1, g_2, \ldots, g_r, x \rangle\), which implies that \(\langle x \rangle^G \leq \langle g_1, g_2, \ldots, g_r, x \rangle\). Thus \(\langle x \rangle^G\) is a finitely generated module over \(G/F\). Since \(G/F\) is finitely generated nilpotent, it is polycyclic; hence \(\langle x \rangle^G\) has max-\(G\) (the maximal condition for \(G\)-invariant subgroups) ([7, 15.3.3]). It follows that \(F/\langle x \rangle^G\) does not have max-\(G\). Therefore \(\langle x \rangle^G\) must be finite and so \(Z\) is torsion.

Let \(L\) be an infinite normal subgroup of \(G\). Then \(Z/Z \cap L \simeq ZL/L\) has max-\(G\). Hence \(G/Z \simeq (G/Z \cap L)/(Z/Z \cap L)\) does not have max-\(G\) and thus \(Z/Z \cap L\) is finite.

Conversely, let \(Z_i = Z_i(F)\). Since \(F\) is infinite, \(Z_i\) is finite and \(Z_{i+1}\) is infinite for some \(i\). Now pass to the group \(G/Z_i\), that is, assume that \(i = 0\). Hence we can assume that \(Z_1\) is infinite. Suppose that the theorem is false and let \(G_1 < G_2 < \cdots\) be an infinite ascending chain of infinite normal subgroups.
Then $Z/Z \cap G_i$ is finite for all $i$. Thus $Z \cap G_i = Z \cap G_{i+1} = \cdots$ for all large $i$. Also $ZG_i = ZG_{i+1} = \cdots$ for all large $i$. Therefore 

$$G_{i+1} = ZG_i \cap G_{i+1} = (Z \cap G_{i+1})G_i = G_i,$$

a contradiction. \hfill $\Box$

**Proposition 2.2.** Let $G$ be a metanilpotent group with max-$\infty n^*$ and let $N$ be a normal nilpotent subgroup of $G$ such that $G/N$ is nilpotent. Then $G$ is finitely generated if and only if $N'$ is infinite.

**Proof.** Suppose that $G$ is finitely generated. Assume that $N'$ is finite. Then $G/N'$ is a finitely generated abelian-by-nilpotent group, so it has max-$n$, as must $G$. By this contradiction $N'$ is infinite.

Conversely, suppose that $N'$ is infinite. Then $G/N'$ has max-$n$, so it is finitely generated. Hence let $G = XN'$ with $X$ a finitely generated subgroup. Then $G = X\gamma_i(N)$ for all $i > 0$ ([6, Lemma 2.2]). But then since $N$ is nilpotent, $G = X$ and so $G$ is finitely generated. \hfill $\Box$

**Example 2.3.** Let $F = \langle x_i, y_j \mid i, j \in \mathbb{Z} \rangle$ be a group where

$$x_{i+1}^2 = x_i, \quad y_{i+1}^2 = y_i, \quad \text{and} \quad [x_i, y_j] \in Z(F)$$

for all integers $i$ and $j$.

Note that $[x_i, x_j] = 1 = [y_i, y_j]$. Put $c_{i+j} = [x_i, y_j]$, which depends only on $i + j$. Then $M$ is a nilpotent group of class 2. Define $\sigma \in \text{Aut}(F)$ by

$$x_i^\sigma = x_{i+1} \quad \text{and} \quad y_j^\sigma = y_{j-1}.$$ 

Then $c_k^\sigma = c_k$. Put

$$G = \langle \sigma \rangle \rtimes F = \langle \sigma, x_0, y_0 \rangle$$

and finally let "bars" denote images modulo $\langle c_0 \rangle$; then $\overline{G} = G/\langle c_0 \rangle$ and $\overline{Z} = \langle \overline{c}_1, \overline{c}_2, \ldots \rangle$ is a Priifer group of type $2\infty$. Note that $\overline{G}/\overline{Z}$ is finitely generated metabelian, so it has max-$n$. Note also that $\text{Fit}(\overline{G}) = \overline{F}$ is not torsion.

Let $\overline{L}$ be an infinite normal subgroup of $\overline{G}$. We claim that $\overline{Z} \leq \overline{L}$. If $\overline{L} \cap \overline{F} = 1$, then $[\overline{L}, \overline{F}] = 1$. Since $C^{\overline{G}}(\overline{F}) = Z(\overline{F})$, it follows that $\overline{L} \leq Z(\overline{F})$. This is a contradiction. Hence $\overline{L} \cap \overline{Z} \neq 1$. Since $\overline{L} \cap \overline{F}$ is a non-trivial normal subgroup of a nilpotent group $\overline{F}$, we have

$$\overline{L} \cap Z(\overline{F}) = \overline{L} \cap \overline{F} \cap Z(\overline{F}) \neq 1,$$

which means that $\overline{L}$ contains an element involving $\overline{c}_i$, say. Therefore $\overline{Z} \leq \overline{L}$. Hence $\overline{G}$ has max-$\infty n^*$ because $\overline{Z}$ is of type $2\infty$.

3. Metanilpotent groups with min-$\infty n^*$

We now determine the structure of metanilpotent groups satisfying min-$\infty n^*$. We note that next result is the generalization of Wilson's theorem on groups with min-$n$ (the minimal condition on normal subgroups). This is used for investigating metanilpotent groups with min-$\infty n^*$. 

Lemma 3.1 ([5, Proposition 2.3]). Let $M$ be a $G$-operator group and let $H$ be a subgroup of $G$ of finite index. If $M$ has $\text{min-}G$, then it has $\text{min-}H$.

Lemma 3.2 ([5, Lemma 3.3]). A polycyclic group $G$ satisfies $\text{min-}\infty^*$ if and only if it is a finite extension of a $G$-rationally irreducible free abelian subgroup of finite rank.

Theorem 3.3. Let $G$ be a metanilpotent group with $F = \text{Fit}(G)$. Then $G$ satisfies $\text{min-}\infty^*$ if and only if $F$ is infinite normal nilpotent and:

(1) $F$ is a $G$-rationally irreducible free abelian subgroup of finite rank such that $G/F$ is finite

or else

(2) $G/F$ is infinite cyclic-by-finite, $F$ has $\text{min-}G$, and $F/[F, x]$ is finite where $x$ is any element of infinite order in $G$.

Proof. Suppose that $G$ has $\text{min-}\infty^*$. Let $N$ be a normal nilpotent subgroup of $G$ such that $G/N$ is nilpotent. If $N$ is finite, then $G$ is infinite cyclic-by-finite ([4, Lemma 3.1]). Hence $G$ is polycyclic. So assume that $N$ is infinite. We first argue that $G/N$ is finitely generated nilpotent. Assume that $G/N$ is infinite. Then $N$ has $\text{min-}G$ and so $G/N$ does not satisfy $\text{min-}n$. Hence $G/N$ is infinite cyclic-by-finite ([4, Lemma 3.1]). Therefore $G/N$ is finitely generated nilpotent. It follows that $N \leq F$ and $F/N$ is finitely generated; hence $F$ is infinite nilpotent.

Case: $G/F$ is finite. We will show that $G$ is polycyclic in this case. Let $Z_i = Z_i(F)$ and let $i$ be maximum subject to $F/Z_i$ being infinite. Then $Z_i$ has $\text{min-}G$, so $F/Z_i$ does not. Since $G/F$ is finite, $Z_i$ has $\text{min-}F$ by Lemma 3.1. Hence $Z_i$ has $\text{min-}G$ and so $G/Z_i$ does not have $\text{min-}G$. Now pass to the group $G/Z_i$, that is, assume that $i = 0$. Put $Z = Z(F)$. Then $G/Z$ is finite. Let $a \in Z$ have infinite order and put $B = \langle a \rangle^G$. Then $G/B$ is finite since otherwise $B$ has $\text{min-}G$. Therefore $G$ is polycyclic. Thus it will suffice to show that $Z$ is not torsion.

Next after factoring out $Z_i$ we can suppose that $Z$ is torsion: we will argue that this cannot occur. Let $a \in Z$: then $\langle a \rangle^G$ is finite since $G/Z$ is finite. Hence $Z$ contains minimal normal subgroups of $G$. Also, if $N$ is a minimal $G$-invariant subgroup of $Z$, then let $1 \neq a \in N$, and note that $N = \langle a \rangle^G$ is finite. Let $S$ be the $G$-socle of $Z$. Then $S$ is a direct product of finitely many minimal $G$-invariant subgroups of $Z$. Thus $S$ is finite. Now $Z$ does not have $\text{min-}G$, so there exists an infinite descending chain $N_1 > N_2 > \cdots$ of $G$-invariant subgroups of $Z$. Put $I = \bigcap_i N_i$. Then $Z/I$ is infinite, so $I$ has $\text{min-}G$, but $Z/I$ does not. Pass to the group $G/I$, i.e., assume that $I = 1$. Since $S$ is finite, $S \cap N_i = S \cap N_{i+1} = \cdots$ for some $i$; hence $S \cap N_i = 1$ since $S \cap N_j \leq N_j$ for all $j$. But this implies that $N_i = 1$, a contradiction. Therefore $Z$ is not torsion and our discussion of this case is complete.
Case: $G/F$ is infinite. Assume that $z \in Z$ has infinite order. Now $G/F$ does not have min-$n$ since $F$ has min-$G$. Thus $G/F$ is infinite cyclic-by-finite. Hence $(z)^G$ is a finitely generated $Z(G/F)$-module, so it is $Z(G/F)$-noetherian. Also $(z)^G$ is $Z(G/F)$-artinian. Thus $(z)^G$ is finite ([7, 15.4.4]), which is a contradiction. Hence $Z$ is torsion. Pass to the group $\bar{G} = G/Z(F)$. Then same argument shows that $Z(\bar{F})$ is torsion; hence so is $Z_2(F)$. Repetition of this argument shows that every term of the upper central series of $F$ is torsion. Therefore $F$ is torsion.

Now if $\bar{F} = F/[F, x]$, then $\bar{F}/\bar{F}'$ is finite. Since $\bar{F}$ is nilpotent, $\bar{F}$ is finite for all $k > 0$ where $x$ is any element of infinite order in $G$.

Conversely, if (1) holds, then $G$ is polycyclic and the result follows from Lemma 3.2. Hence we assume that (2) holds. Suppose that the theorem is false and let $G_1 > G_2 > \cdots$ be an infinite descending chain of normal subgroups of $G$ with infinite index. Assume that $G_i F/F$ is infinite for some $i$. Then $G_i F/F$ contains an element $xF$ of infinite order where $x \in G_i$ and $G/G_i A$ is finite. Since $[F, x] \leq F \cap G_i$, it follows that $F/F \cap G_i \simeq FG_i/G_i$ is finite and so is $G/G_i$, a contradiction. Thus each $G_i F/F$ is finite. Hence there is an $i$ such that $G_i F = G_{i+1} F$ and $G_i \cap F = G_{i+1} \cap F$, which implies that

$$G_{i+1} = G_{i+1} \cap G_i F = G_i (G_{i+1} \cap F) = G_i.$$

This is a contradiction.

Therefore $G$ has min-$\infty n$. Finally if $G$ has min-$n$, then it is locally finite ([6, Theorem 5.25]). Hence $G/F$ is finitely generated locally finite and so is finite, a contradiction. 

References


Department of Mathematics Education
Busan National University of Education
Busan 611-736, Korea
E-mail address: paek@bnue.ac.kr