A KUROSH-AMITSUR LEFT JACOBSON RADICAL FOR RIGHT NEAR-RINGS

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ABSTRACT. Let $R$ be a right near-ring. An $R$-group of type-5/2 which is a natural generalization of an irreducible (ring) module is introduced in near-rings. An $R$-group of type-5/2 is an $R$-group of type-$2$ and an $R$-group of type-$3$ is an $R$-group of type-5/2. Using it $J_{5/2}$, the Jacobson radical of type-5/2, is introduced in near-rings and it is observed that $J_2(R) \subseteq J_{5/2}(R) \subseteq J_3(R)$. It is shown that $J_{5/2}$ is an ideal-hereditary Kurosh-Amitsur radical (KA-radical) in the class of all zero-symmetric near-rings. But $J_{5/2}$ is not a KA-radical in the class of all near-rings. By introducing an $R$-group of type-(5/2)(0) it is shown that $J_{(5/2)(0)}$, the corresponding Jacobson radical of type-(5/2)(0), is a KA-radical in the class of all near-rings which extends the radical $J_{5/2}$ of zero-symmetric near-rings to the class of all near-rings.

1. Introduction

Near-rings considered are right near-rings and $R$ stands for a right near-ring. Many generalizations of the Jacobson radical of rings to near-rings were introduced and studied. Let $\nu \in \{0, 1, 2\}$. $J_\nu$, the Jacobson radical of type-$\nu$, was introduced and studied by Betsch [1] and $J_3$, the Jacobson radical of type-$3$, was introduced and studied by Holcombe [2]. In this paper an $R$-group of type-5/2 is introduced as a natural generalization of an irreducible (ring) module. The corresponding Jacobson radical $J_{5/2}$ is also introduced in near-rings. Moreover, $J_2(R) \subseteq J_{5/2}(R) \subseteq J_3(R)$. $J_{5/2}$ is an ideal-hereditary Kurosh-Amitsur radical (KA-radical) in the class of all zero-symmetric near-rings. But $J_{5/2}$ is not a KA-radical in the class of all near-rings. By introducing an $R$-group of type-(5/2)(0) it is proved that $J_{(5/2)(0)}$, the corresponding Jacobson radical of type-(5/2)(0), is a KA-radical in the class of all near-rings which extends the radical $J_{5/2}$ of zero-symmetric near-rings to the class of all near-rings.

We recall some of the definitions related to $R$-groups and Jacobson radicals of near-rings.

Let $G$ be an $R$-group and $R_0$ be the zero-symmetric part of $R$. Then $G$ is

Received April 27, 2007.
2000 Mathematics Subject Classification. 16Y30.
Key words and phrases. near-ring, $R$-groups of type-5/2 and (5/2)(0), Jacobson radicals of type-5/2 and (5/2)(0).
(i) monogenic if there is a $g \in G$ such that $Rg = G$.

(ii) strongly monogenic if $G$ is monogenic and for each $g \in G$ either $Rg = 0$ or $G$.

(iii) an $R$-group of type-$0$ if $G \neq 0$ and is a monogenic simple $R$-group.

(iv) an $R$-group of type-$1$ if $G$ is of type-$0$ and strongly monogenic.

(v) an $R$-group of type-$2$ if $G \neq 0$, monogenic and $R_0$-simple.

(vi) an $R$-group of type-$3$ if $G$ is an $R$-group of type-$2$ and $x, y \in G$ and $rx = ry$ for all $r \in R$ implies $x = y$.

If $I$ is an ideal of $R$, then it is denoted by $I \triangleleft R$.

Let $Q$ be a mapping which assigns to each near-ring $R$ an ideal $Q(R)$ of $R$. Such mappings are called ideal-mappings. We consider the following properties which $Q$ may satisfy:

(H1) $h(Q(R)) \subseteq Q(h(R))$ for all homomorphisms $h$ of $R$;

(H2) $Q(R/Q(R)) = \{0\}$ for all $R$;

- $Q$ is $r$-hereditary if $I \cap Q(R) \subseteq Q(I)$ for all ideal $I$ of $R$;
- $Q$ is $s$-hereditary if $Q(I) \subseteq I \cap Q(R)$ for all ideals $I$ of $R$;
- $Q$ is ideal-hereditary if it is both $r$-hereditary and $s$-hereditary, that is, if $Q(I) = I \cap Q(R)$ for all ideals $I$ of $R$;
- $Q$ is idempotent if $Q(Q(R)) = Q(R)$ for all $R$;
- $Q$ is complete if $Q(I) = I$ and $I$ is an ideal of $R$ implies $I \subseteq Q(R)$.

With $Q$ we associate two classes of near-rings $R_Q$ and $S_Q$ defined by $R_Q := \{R \mid Q(R) = R\}$, $S_Q := \{R \mid Q(R) = 0\}$ and are called $Q$-radical class and

- $Q$-semisimple class respectively.

- An ideal-mapping $Q$ is a Hoehnke radical (H-radical) if it satisfies conditions (H1) and (H2).

- An ideal-mapping $Q$ is a Kurosh-Amitsur radical (KA-radical) if it is a complete idempotent $H$-radical.

Let $\mathcal{M}$ be a class of near-ring. Classes of near-rings always assumed to be abstract, that is, they contains the one element near-ring and are closed under isomorphic copies. With every near-ring $R$, we associate two ideals of $R$, depending on $\mathcal{M}$. These ideals are defined by:

$\mathcal{M}(R) := \Sigma\{I \mid I$ is an ideal of $R$ and $I \in \mathcal{M}\}$ and

$(R/\mathcal{M}) := \cap\{I \mid I$ is an ideal of $R$ and $R/I \in \mathcal{M}\}$.

$\mathcal{M}$ is called regular if $0 \neq I \triangleleft R \in \mathcal{M}$ implies that $0 \neq I/K \in \mathcal{M}$ for some $K \triangleleft I$; hereditary if $I \triangleleft R \in \mathcal{M}$ implies $I \in \mathcal{M}$ and; $c$-hereditary if $I$ is a left invariant ideal of $R \in \mathcal{M}$, then $I \in \mathcal{M}$. (An ideal $I$ of $R$ is left invariant if $RI \subseteq I$.)

A class of near-rings $\mathcal{M}$ is a Kurosh-Amitsur radical class (KA-radical class) if it satisfies the following:

(R1) $\mathcal{M}$ is closed under homomorphic images;

(R2) $\mathcal{M}(R) \in \mathcal{M}$ for all near-rings $R$;

(R3) $\mathcal{M}(R/\mathcal{M}(R)) = \{0\}$ for all near-rings $R$. 
With a KA-radical class \( \mathcal{R} \) we associate its semisimple class \( \mathcal{S} = \{ R \mid \mathcal{R}(R) = \{0\} \} \).

The following properties for a KA-radical class \( \mathcal{R} \) are well known.

(i) \( \mathcal{R} \) is hereditary if and only if \( \mathcal{R}(R) \cap I \subseteq \mathcal{R}(I) \) for all \( I \triangleleft R \).

(ii) \( \mathcal{S} \) is hereditary if and only if \( \mathcal{S}(I) \subseteq \mathcal{R}(R) \cap I \) for all \( I \triangleleft R \).

(iii) \( \mathcal{R} \) is c-hereditary if and only if \( \mathcal{R}(R) \cap I \subseteq \mathcal{R}(I) \) for all left invariant ideals \( I \) of \( R \).

We say that a class \( M \) of near-rings satisfy condition \((F_1)\) if \( K \triangleleft I \) and \( I \) is a left invariant ideal of \( R \) with \( I/K \in \mathcal{M} \), then \( K \triangleleft R \).

**Theorem 1.1** (Corollary 2.3 of [5]). Let \( \mathcal{M} \) be a class of zero-symmetric near-rings and \( \mathcal{L} \) be defined by \( \mathcal{L}(R) := (R)\mathcal{M} \) and \( \mathcal{L}_o \) be the restriction of \( \mathcal{L} \) to the class of all zero-symmetric near-rings. Then the following are equivalent.

1. \( \mathcal{L} \) is a KA-radical in the class of all near-rings with \( \mathcal{L}(I) \subseteq \mathcal{L} \cap I \) for all \( I \triangleleft R \) and equality holds if \( I \) is left invariant.

2. \( \mathcal{L}_o \) is an ideal-hereditary KA-radical in the class of all zero-symmetric near-rings and \( \mathcal{M} \) satisfies condition \((*)\):

\((*)\) If \( K \triangleleft I \triangleleft R \) with \( I \) a left invariant ideal of \( R \) and \( I/K \in \mathcal{M} \), then \( R/I \subseteq K \), where \( R/I \) is the ideal of \( R \) generated by the subnear-ring \( R/I \).

**Theorem 1.2** (Theorem 4.2.3 of [5]). The class of all zero-symmetric 2-primitive near-rings satisfy condition \((F_1)\).

2. **\( R \)-groups of type-5/2**

Throughout this section \( R \) stands for a right near-ring.

**Definition 2.1.** Let \( G \) be an \( R \)-group. Then \( G \) is called an \( R \)-group of type-5/2 if \( G \) is an \( R \)-group of type-2 and \( Rg = G \) for all \( 0 \neq g \in G \).

**Remark 2.2.** From the definition we have that an \( R \)-group of type-5/2 is an \( R \)-group of type-2.

**Proposition 2.3.** An \( R \)-group of type-3 is an \( R \)-group of type-5/2.

**Proof.** Let \( G \) be an \( R \)-group of type-3. So, \( G \) is an \( R \)-group of type-2. Let \( 0 \neq g \in G \). Since \( G \) is an \( R \)-group of type-2, it is an \( R \)-group of type-1. So, either \( Rg = G \) or \( Rg = \{0\} \). Suppose that \( Rg = 0 \). Now \( R0 = Rg0 = Rg \subseteq Rg = 0 \) and hence \( R0 = \{0\} \). So, \( rg = r0 \) for all \( r \in R \). Since \( G \) is an \( R \)-group of type-3, \( g = 0 \). This is a contradiction to the fact that \( g \neq 0 \). Therefore, \( Rg = G \). \( \square \)

**Proposition 2.4.** Let \( R \) be a zero-symmetric near-ring and \( \{0\} \neq G \) be an \( R \)-group. Then \( G \) is an \( R \)-group of type-5/2 if and only if \( Rg = G \) for all \( 0 \neq g \in G \).
Proof. If $G$ is an $R$-group of type-5/2, then obviously $Rg = G$ for all $0 \neq g \in G$. Suppose that $Rg = G$ for all $0 \neq g \in G$. Let $\{0\} \neq H$ be an $R$-subgroup of $G$. Let $0 \neq h \in H$. Now $G = Rh \subseteq H$ and hence $H = G$. Therefore, $G$ is an $R$-group of type-2 and hence it is an $R$-group of type-5/2. \qed

We present an example of an $R$-group of type-5/2 which is not an $R$-group of type-3.

**Example 2.5.** Let $(R, +)$ be a group of order $\geq 3$. Let $a, b \in R$. Define $ab = a$ if $b \neq 0$ and $ab = 0$ if $b = 0$. Now $R$ is a zero-symmetric near-ring. Moreover, $Ra = R$ for all $0 \neq a \in R$. Therefore, by Proposition 2.4, $R$ is an $R$-group of type-5/2. Let $0 \neq b, 0 \neq c \in R$ and $b \neq c$. Now $ab = a = ac$ for all $a \in R$. So, $R$ is not an $R$-group of type-3.

Now we give an example of an $R$-group of type-2 which is not an $R$-group of type-5/2.

**Example 2.6.** Let $(R, +)$ be a group of order $\geq 3$. Let $S$ be a non-empty subset of $R \setminus \{0\}$ such that $R \setminus S$ contains no non-zero subgroup of $(R, +)$. Let $a, b \in R$. Define $ab = a$ if $b \in S$ and $ab = 0$ if $b \notin S$. Now $R$ is a zero-symmetric near-ring. We have that $Rb = \{0\}$ if $b \notin S$ and $Rb = R$ if $b \in S$. Now it is clear that $R$ is an $R$-group of type-2. But, by Proposition 2.4, $R$ is not an $R$-group of type-5/2.

**Definition 2.7.** A modular left ideal $L$ of $R$ is said to be a $5/2$-modular left ideal of $R$ if $R/L$ is an $R$-group of type-5/2.

**Proposition 2.8.** Let $G$ be an $R$-group of type-5/2 and $0 \neq g \in G$. Then $(0: g)$ is a $5/2$-modular left ideal of $R$ and $R/(0: g)$ and $G$ are isomorphic $R$-groups.

Proof. The mapping $h : R \to G$ defined by $h(r) = rg$ is an $R$-homomorphism of $R$ onto $G$ with $\text{Ker } h = (0: g)$ which is a modular left ideal of $R$. Now $R/(0: g)$ is isomorphic to $G$ as $R$-groups. So, $(0: g)$ is a $5/2$-modular left ideal of $R$. \qed

**Definition 2.9.** $R$ is called a $5/2$-primitive near-ring if $R$ has a faithful $R$-group of type-5/2.

**Definition 2.10.** An ideal $I$ of $R$ is called a $5/2$-primitive ideal of $R$ if $R/I$ is a $5/2$-primitive near-ring.

One can easily verify the following.

**Proposition 2.11.** Let $I$ be an ideal of $R$. Then
(1) If $G$ is an $R$-group of type $5/2$ and $I \subseteq (0 : G)$, then $G$ is also an $R/I$-group of type $5/2$, where $(r + I)g := rg, r + I \in R/I$ and $g \in G$. If in addition $I = (0 : G)$, then $G$ is a faithful $R/I$-group.

(2) If $G$ is an $R/I$ group of type $5/2$, then $G$ is also an $R$-group of type $5/2$, where $rg := (r + I)g, r \in R$ and $g \in G$. If in addition $G$ is a faithful $R/I$-group, then $I = (0 : G)_R$.

An immediate consequence of Propositions 2.8 and 2.11 is the following.

**Proposition 2.12.** Let $I$ be an ideal of $R$. Then the following are equivalent.
(i) $I$ is a $5/2$-primitive ideal of $R$.
(ii) $I = (0 : G)$ for some $R$-group $G$ of type $5/2$.
(iii) $I = (L : R)$ for some $5/2$-modular left ideal $L$ of $R$.

**Corollary 2.13.** The following are equivalent
(i) $\{0\}$ is a $5/2$-primitive ideal of $R$.
(ii) $R$ is $5/2$-primitive.
(iii) $R$ has a $5/2$-modular left ideal $L$ such that $(L : R) = \{0\}$.

We know that an ideal $P$ of $R$ is a 3-prime ideal of $R$ if $a, b \in R$ and $aRb \subseteq P$ implies $a \in P$ or $b \in P$.

**Proposition 2.14.** Let $P$ be a $5/2$-primitive ideal of $R$. Then $P$ is a 3-prime ideal of $R$.

**Proof.** Let $P$ be a $5/2$-primitive ideal of $R$. We get an $R$-group $G$ such that $P = (0 : G)$. Let $a, b \in R$ and $aRb \subseteq P = (0 : G)$. Suppose that $b \notin P$. Now $bg \neq 0$ for some $g \in G$. So $R(bg) = G$ as $G$ is an $R$-group of type $5/2$. Therefore, $aG = aR(bg) = (aRb)g = \{0\}$. So $a \in (0 : G) = P$. Hence $P$ is 3-prime. □

We know that a 3-primitive ideal of a zero-symmetric near-ring is equiprime and 3-prime. So with the introduction of $5/2$-primitive ideals, we have primitive ideals which are 3-prime but not equiprime.

3. The Jacobson radical of type $5/2$

**Definition 3.1.** The Jacobson radical of $R$ of type $5/2$, denoted by $J_{5/2}(R)$, is defined as the intersection of all 5/2-primitive ideals of $R$ and if $R$ has no such ideals, then $J_{5/2}(R)$ is defined as $R$.

**Remark 3.2.** By Proposition 2.12, $J_{5/2}(R) = \cap \{(0 : G) \mid G$ is an $R$-group of type $5/2\} = \cap \{(L : R) \mid L$ is a $5/2$-modular left ideal of $R\}$.

The following proposition is immediate.

**Proposition 3.3.** $J_{5/2}(R) = \cap \{P \mid R/P$ is a $5/2$-primitive near-ring$\}$.

**Proposition 3.4.** $J_{5/2}(R) = \cap \{L \mid L$ is a $5/2$-modular left ideal of $R\}$. 

Proof. If $R$ has no $5/2$-primitive ideals, then by Proposition 2.12, $R$ has no $5/2$-modular left ideals. So, if $J_{5/2}(R) = R$, then the result follows. Now suppose that $R$ has a $5/2$-primitive ideal. So there is an $R$-group of type-$5/2$. We have $J_{5/2}(R) = \cap \{(0 : G) \mid G$ is an $R$-group of type-$5/2\}$. Let $G$ be an $R$-group of type-$5/2$. Let $0 \neq g \in G$. Since $Rg = G$, we get that $r \rightarrow rg$ is an $R$-homomorphism of $R$ onto $G$ with Kernel $(0 : g)$. So $R/(0 : g)$ and $G$ are isomorphic $R$-groups and hence $(0 : g)$ is a $5/2$-modular left ideal of $R$. Therefore $(0 : G)$ is an intersection of $5/2$-modular left ideals of $R$. This shows that $J_{5/2}(R)$ is an intersection of $5/2$-modular left ideals of $R$. Let $T$ be a $5/2$-modular left ideal of $R$. Now $R/T$ is an $R$-group of type-$5/2$. Since $T$ is modular, by Corollary 3.24 of [3], we get that $(T : R) \subseteq T$. So $J_{5/2}(R) \subseteq (T : R) \subseteq T$. Hence $J_{5/2}(R)$ is the intersection of all $5/2$-modular left ideals of $R$.

\[\square\]

Lemma 3.5. Let $R$ be a zero-symmetric near-ring and $S$ be an invariant subnearring of $R$. If $L$ is a $5/2$-modular left ideal of $S$, then $L$ is an ideal of the $R$-group $S$ and $S/L$ is an $R$-group of type-$5/2$.

Proof. Let $L$ be a $5/2$-modular left ideal of $S$. Since an $R$-group of type-$5/2$ is an $R$-group of type-$2$, $L$ is a $2$-modular left ideal of $S$. Therefore, by Theorem 3.34 of [3], $L$ is an ideal of the $R$-group $S$ and $S/L$ is an $R$-group of type-$2$. Let $0 \neq s + L \in S/L$. Since $S/L$ is an $S$-group of type-$5/2$, $S(s + L) = S/L$. Therefore $S/L = S(s + L) \subseteq R(s + L) \subseteq S/L$. So $R(s + L) = S/L$ and hence $S/L$ is an $R$-group of type-$5/2$.

\[\square\]

Theorem 3.6. Let $S$ be an invariant subnear-ring of a zero-symmetric near-ring $R$. Then $J_{5/2}(S) \subseteq J_{5/2}(R) \cap S$.

Proof. If $S$ has no $5/2$-primitive ideals then $J_{5/2}(S) = S \subseteq J_{5/2}(R) \cap S$. So, suppose that $S$ has $5/2$-primitive ideals. Let $P$ be a $5/2$-primitive ideal of $S$. We get an $S$-group $G$ of type-$5/2$ such that $P = (0 : G)_S$. Let $0 \neq g \in G$. Now $S/(0 : g)_S$ and $G$ are isomorphic as $S$-groups and that $L := (0 : g)_S$ is a $5/2$-modular left ideal of $S$ and $P = (0 : G)_S = (0 : S/L)_S = (L : S)_S$. By Lemma 3.5, $S/L$ is an $R$-group of type-$5/2$. So $Q := (0 : S/L)_R = (L : S)_R$ is a $5/2$-primitive ideal of $R$. Therefore $P = (L : S)_S = (L : S)_R \cap S = Q \cap S$. Hence $J_{5/2}(S) \subseteq J_{5/2}(R) \cap S$.

\[\square\]

Lemma 3.7. Let $S$ be an invariant subnear-ring of a zero-symmetric near-ring $R$. Let $L$ be a $5/2$-modular left ideal of $R$ and $S \not\subseteq L$. Then $L \cap S$ is a $5/2$-modular left ideal of $S$.

Proof. We have that $L$ is a $5/2$-modular left ideal of $R$ and $S \not\subseteq L$. Now $R = S + L$. So $R/L = (S + L)/L \cong S/(S \cap L)$ and that $S/(S \cap L)$ is an $R$-group of type-$5/2$. Let $L$ be modular by $e$. Now $r - re \in L$ for all $r \in R$. Let $s \in S - (S \cap L)$. Since $0 \neq s + L \in R/L$, $R(s + L) = R/L$ and that $Rs + L = R$. 

\[\square\]
Now \( e = rs + l, r \in R, l \in L \). \( S \cap L \) is a left ideal of \( S \) modular by \( rs \). Let \( t \in S \). Now \( te - t \in L \). So \( te - t = t(rs + l) - t(= (t(rs + l) - t(rs)) + t(rs) - t) \in L \) and that \( t(rs) - t \in L \cap S \). Therefore \( t + (L \cap S) = (t(rs) + (L \cap S) \in (S + L) \cap S) \) and that \( S/(L \cap S) = (S + L) \cap S) = S/(L \cap S) \) is an \( S \)-group of type 5/2. Since \( L \cap S \) is a modular left ideal of \( S \), \( L \cap S \) is a 5/2-modular left ideal of \( S \).

**Theorem 3.8.** Let \( R \) be a zero-symmetric near-ring and \( S \) be an invariant subnearring of \( R \). Then \( J_{5/2}(S) \subseteq J_{5/2}(R) \cap S \).

**Proof.** If \( J_{5/2}(R) = R \), then \( J_{5/2}(S) \subseteq R \cap S = J_{5/2}(R) \cap S \). Suppose that \( J_{5/2}(R) \neq R \). So \( R \) has 5/2-modular left ideals. Let \( L \) be a 5/2-modular left ideal of \( R \). If \( S \subseteq L \), then \( J_{5/2}(S) \subseteq S \cap L \). Now suppose that \( S \nsubseteq L \). By Lemma 3.7, \( S \cap L \) is a 5/2-modular left ideal of \( S \). So \( J_{5/2}(S) \subseteq S \cap L \). Therefore, by Proposition 3.4, \( J_{5/2}(S) \subseteq J_{5/2}(R) \cap S \).

**Theorem 3.9.** Let \( R \) be a zero-symmetric near-ring and \( S \) be an invariant subnearring of \( R \). Then \( J_{5/2}(S) = J_{5/2}(R) \cap S \).

**Theorem 3.10.** \( J_{5/2} \) is an ideal-hereditary Kurosh-Amitsur radical in the class of all zero-symmetric near-rings.

We show now that \( J_{5/2} \) is not a KA-radical in the class of all near-rings.

Consider the dihedral group \( D_8 = \{0, a, 2a, 3a, b, a + b, 2a + b, 3a + b\} \). Let \( T \) be the near-ring given in Example 11 of [3], (p.418) whose additive group is \( D_8 \). As mentioned in [4], \( \{0\} \), \( J = \{0, a, 2a, 3a\} \) and \( T \) are the ideals of \( T \). Moreover, these are the only left ideals of \( T \). Now \( T/J \) is the constant near-ring on \( Z_2 \). Since \( T/J \) is a \( T \)-group of type 5/2, \( J \) is a 5/2-primitive ideal and is the only 5/2-primitive ideal of \( T \). So \( J_{5/2}(T) = J \).

We need the following result.

**Proposition 3.11 (Proposition 3.3 of [4]).** Let \( Q \) be an ideal-mapping which satisfies (H1) and for which \( Q(T) = J \) and \( F \in S_Q \), where \( F \) is the field of order 2. Then \( Q \) is not idempotent and hence not a KA-radical mapping.

**Theorem 3.12.** \( J_{5/2} \) is not a KA-radical in the class of all near-rings.

**Proof.** By Proposition 3.3, we have that \( J_{5/2} \) is the H-radical corresponding to the class of all 5/2-primitive near-rings. As seen above \( J_{5/2}(T) = J \). Moreover, the two element field is in \( S_{5/2} \). So, by Proposition 3.11, \( J_{5/2} \) is not a KA-radical in the class of all near-rings.

4. The Jacobson radical of type -(5/2)(0)

It is known that Jacobson radicals of type 2 and 3 are ideal-hereditary KA-radicals in the class of all zero-symmetric near-rings and the Jacobson radical of type 2 is not even a KA-radical in the class of all near-rings. S. Veldsman [5] introduced \( R \)-groups of type-(2)(0) and (3)(0) and the corresponding Jacobson
radicals of type-2(0) and 3(0) which are extensions of the Jacobson radicals of
type-2 and 3 respectively of zero-symmetric near-rings to the class of all near-
rings and has shown that these two new radicals are KA-radicals in the class
of all near-rings.

In this section we introduce R-groups of type-(5/2)(0) and the corresponding
Jacobson radical of type-(5/2)(0). We show that it is a KA-radical in the class
of all near-rings.

**Definition 4.1.** Let G be an R-group of type-5/2. G is called an R-group of
type-(5/2)(0) if R0 = {0}, where 0 is the additive identity in G.

**Proposition 4.2.** Let G be an R-group of type-5/2. Then G is an R-group
of type-(5/2)(0) if and only if R_c ⊆ (0 : G), where R_c is the constant part of
R.

**Proof.** Let G be an R-group of type-(5/2)(0). R_cg = (R0)g = R(og) = R0 =
{0} for all g ∈ R. So, R_c ⊆ (0 : G). Suppose now that R_c ⊆ (0 : G). Now R_c,0
= {0}, where 0 is the additive identity in G. So R0 = {0} and hence G is an
R-group of type-(5/2)(0).

**Corollary 4.3.** Let R is a zero-symmetric near-ring and G be an R-group.
Then G is type-(5/2)(0) if and only if it is of type-5/2.

**Definition 4.4.** A near-ring R is said to be (5/2)(0)-primitive if it has a
faithful R-group of type-(5/2)(0). An ideal I of R is called (5/2)(0)-primitive
if R/I is a (5/2)(0)-primitive near-ring.

**Proposition 4.5.** Let I be an ideal of R. Then the following are equivalent.
(i) I is (5/2)(0)-primitive ideal of R.
(ii) I = (0 : G) for some R-group G of type-(5/2)(0).

**Proof.** Suppose that I is a (5/2)(0)-primitive ideal of R. R/I is a (5/2)(0)-
primitive on some R/I-group G of type-(5/2)(0). Since G is a faithful R/I-
group of type-(5/2)(0), G is an R-group of type-5/2 and I = (0 : G). Also,
since R/I is zero-symmetric, R_c ⊆ I = (0 : G) and hence G is an R-group
of type-(5/2)(0). Conversely, suppose that I = (0 : G) for an R-group G of
type-(5/2)(0). Since G is an R-group of type-(5/2)(0) and I = (0 : G), G is
a faithful R/I-group of type-5/2. Also since R_c ⊆ (0 : G) = I, R/I is a zero-
symmetric near-ring and hence G is a faithful R/I-group of type-(5/2)(0). So
R/I is a (5/2)(0)-primitive near-ring and hence I is a (5/2)(0)-primitive ideal
of R.

**Corollary 4.6.** The following are equivalent
(i) {0} is a (5/2)(0)-primitive ideal of R.
(ii) R is (5/2)(0)-primitive.

**Corollary 4.7.** R is (5/2)(0)-primitive if and only if R is a zero-symmetric
and (5/2)-primitive.
Remark 4.8. It is clear that a $(5/2)(0)$-primitive ideal of $R$ contains $R_c$, the constat part of $R$.

Definition 4.9. Let $R$ be a near-ring. $J_{(5/2)(0)}(R)$ is defined as the intersection of all $(5/2)(0)$-primitive ideal of $R$ and $J_{(5/2)(0)}(R) = R$ if $R$ has no $(5/2)(0)$-primitive ideals. $J_{(5/2)(0)}$ is called the Jacobson radical of type $(5/2)(0)$.

Remark 4.10. If $R$ is a ring, then $J_{(5/2)(0)}(R)$ is the Jacobson radical of $R$.

We show now that $J_{(5/2)(0)}$ is a KA-radical in the class of all near-rings, its semisimple class is hereditary and radical class is $c$-hereditary.

Theorem 4.11. The class of all zero-symmetric $5/2$-primitive near-rings satisfy condition $(F_i)$.

Proof. Since a zero-symmetric $5/2$-primitive near-ring is a $2$-primitive near-ring, by Theorem 1.2, we get that the class of all zero-symmetric $5/2$-primitive near-rings also satisfy condition $(F_i)$. \qed

Theorem 4.12. Let $R$ be a near-ring. $J_{(5/2)(0)}$ is a KA-radical in the class of all near-rings, $J_{(5/2)(0)}(I) \subseteq J_{(5/2)(0)}(R) \cap I$ for all $I \lhd R$ and the equality holds if $I$ is a left invariant ideal.

Proof. Let $\mathcal{M}$ be the class of all zero-symmetric $5/2$-primitive near-rings. Now by Corollary 4.7, $J_{(5/2)(0)}(R) = (R)\mathcal{M}$ for all near-rings $R$. By Theorem 3.10, $J_{5/2}$ is an ideal-hereditary KA-radical in the class of all zero-symmetric near-rings. In view of Theorem 1.1, it is enough to show that $\mathcal{M}$ satisfies condition $(\ast)$ of Theorem 1.1. Let $K \lhd I \lhd R$ and $I$ be a left invariant ideal of $R$ with $I/K \in \mathcal{M}$. By Theorem 4.11, $\mathcal{M}$ satisfies condition $(F_i)$. So $K \lhd R$. Since $I$ is a left invariant ideal of $R$, $R_c \subseteq I$. Also since $I/K$ is a zero-symmetric near-ring, $R_c = I_c \subseteq K$. Since $R_c \subseteq K$ and $K \lhd R$, we get that $\overline{R_c} \subseteq K$. This completes the proof. \qed

References


