ON THE CONVERGENCE BETWEEN THE MANN ITERATION AND ISHIKAWA ITERATION FOR THE GENERALIZED LIPSCHITZIAN AND Φ-STRONGLY PSEUDOCONTRACTIVE MAPPINGS

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Abstract. In this paper, we prove that the equivalence between the convergence of Mann and Ishikawa iterations for the generalized Lipschitzian and Φ-strongly pseudocontractive mappings in real uniformly smooth Banach spaces. Our results significantly generalize the recent known results of [B. E. Rhoades and S. M. Soltuz, The equivalence of Mann iteration and Ishikawa iteration for non-Lipschitz operators, Int. J. Math. Math. Sci. 42 (2003), 2645–2651].

1. Introduction

Let $E$ be a real Banach space. Let $J$ denote the normalized duality mapping from $E$ to $2^{E^*}$ defined by

\begin{equation}
Jx = \{ f \in E^* : \langle x, f \rangle = ||x|| \cdot ||f|| = ||f||^2 \},
\end{equation}

where $E^*$ denotes the dual space of $E$ and $\langle \cdot, \cdot \rangle$ denotes the generalized duality pairing. It is well known that if $E$ is a uniformly smooth Banach space, then $J$ is single-valued, $J(tx) = tJ(x)$ for all $x \in E$ and $t \geq 0$, and $J$ is uniformly continuous on any bounded subset of $E$ [1, 3]. We denote the single-valued normalized duality mapping by $j$.

A map $T$ with domain $D(T)$ and range $R(T)$ in $E$ is said to be strongly pseudocontractive if there exists a constant $k \in (0, 1)$ such that for any $x, y \in D(T)$, there exists $j(x - y) \in J(x - y)$ such that

\begin{equation}
\langle Tx - Ty, j(x - y) \rangle \leq k||x - y||^2.
\end{equation}

The map $T$ is called Φ-strongly pseudocontractive if, for all $x, y \in D(T)$, there exist $j(x - y) \in J(x - y)$ and a strictly continuous increasing function...
\( \Phi : [0, +\infty) \to [0, +\infty) \) with \( \Phi(0) = 0 \) such that
\[
(Tx - Ty, j(x - y)) \leq \|x - y\|^2 - \Phi(\|x - y\|)\|x - y\|.
\]
Set \( \Phi(t) = kt \), \( t \in [0, +\infty) \) and \( k \) is a constant of \((0, 1)\), we get the above definition of strongly pseudocontractive mapping.

The mapping \( T : D(T) \subset E \to E \) is called generalized Lipschitz map if there exists a constant \( L > 0 \) such that
\[
\|Tx - Ty\| \leq L(1 + \|x - y\|)
\]
for all \( x, y \in D(T) \). Clearly, if map \( T \) either is Lipschitz or has a bounded range, then \( T \) must be a generalized Lipschitz map. Conversely, the following example indicates that the class of generalized Lipschitz map neither is Lipschitz nor has the bounded range.

**Example.** Let \( E = (-\infty, +\infty) \) with the usual norm. Define \( T : E \to E \) by
\[
Tx = \begin{cases} 
\frac{-3x}{2} + 1, & \text{if } x \in (-\infty, -1), \\
1, & \text{if } x = -1, \\
\frac{-3x}{2} + \sqrt{-x}, & \text{if } x \in (-1, 0), \\
\frac{-3x}{2}, & \text{if } x \in [0, +\infty).
\end{cases}
\]
Then \( T \) is not continuous when \( x = -1 \); However \( T \) is both a generalized Lipschitz map with \( 0 \in \text{Fix}(T) \) and a strongly pseudocontractive map. Of course, \( T \) is also a \( \Phi \)-strongly pseudocontractive map.

Let \( D \) be a nonempty closed convex subset of \( E \) and \( T : D \to D \). Suppose that \( u_0, x_0 \in D \) are arbitrary. The Mann iteration is defined by
\[
u_{n+1} = (1 - a_n)u_n + a_nTu_n, \quad n \geq 0,
\]
where \( \{a_n\}_{n=0}^{\infty} \subseteq [0, 1] \). The Ishikawa iteration is defined by
\[
\begin{align*}
\begin{cases} 
y_n = (1 - b_n)x_n + b_nTx_n, & n \geq 0, \\
x_{n+1} = (1 - a_n)x_n + a_nTy_n, & n \geq 0,
\end{cases}
\end{align*}
\]
where \( \{a_n\}_{n=0}^{\infty} \) and \( \{b_n\}_{n=0}^{\infty} \) are two real sequences of \([0, 1]\).


**Theorem R-S1** ([6, Theorem 2.1]). Let \( X \) be a real Banach space with a uniformly convex dual and \( B \) a nonempty, closed, convex, and bounded subset of \( X \). Let \( T : B \to B \) be a continuous and strongly pseudocontractive operator. Then for \( u_1 = x_1 \in B \), the following assertions are equivalent:

(i) Mann iteration (1.1) converges to the fixed point of \( T \);  
(ii) Ishikawa iteration (1.3) converges to the fixed point of \( T \).

**Theorem R-S2** ([6, Corollary 3.1]). Let \( X \) be a real Banach space with a uniformly convex dual and \( B \) a nonempty, convex and closed subset of \( X \). Let \( S : B \to B \) be a continuous and strongly accretive operator and let \( \{x_n\} \),
given by (3.2), be bounded. Then, for \( u_1 = x_1 \in B \), the following assertions are equivalent:

(i) Mann iteration (3.3) converges to the solution of \( Sx = f \);
(ii) Ishikawa iteration (3.2) converges to the solution of \( Sx = f \).

The aim of this paper is to prove the equivalence between the convergence of above two iterations when \( T \) is generalized Lipschitz and \( \Phi \)-strongly pseudocontractive mapping in real uniformly smooth Banach spaces. Our results improve and extend the corresponding known results. For this purpose, we need the following Lemmas.

**Lemma 1.1** ([4]). Let \( E \) be a real Banach space, then for all \( x, y \in E \), there exists \( f(x + y) \in J(x + y) \) such that

\[
\|x + y\|^2 \leq \|x\|^2 + 2\langle y, f(x + y) \rangle.
\]

**Lemma 1.2** ([8]). Let \( \{\rho_n\}_{n=0}^{\infty} \) be a nonnegative sequence which satisfies the following inequality

\[
\rho_{n+1} \leq (1 - \lambda_n)\rho_n + \sigma_n, n \geq 0,
\]
where \( \lambda_n \in (0, 1), \sum_{n=0}^{\infty} \lambda_n = \infty \) and \( \sigma_n = o(\lambda_n) \). Then \( \rho_n \to 0 \) as \( n \to \infty \).

2. Main results

**Theorem 2.1.** Let \( E \) be a real uniformly smooth Banach space and \( D \) be a nonempty closed convex subset of \( E \), let \( T : D \to D \) be a generalized Lipschitz and \( \Phi \)-strongly pseudocontractive map. Let \( q \) be a fixed point of \( T \) in \( D \). Suppose that \( \{a_n\}_{n=0}^{\infty} \) and \( \{b_n\}_{n=0}^{\infty} \) are defined by (1.5) and (1.6) respectively, with the iterative parameters \( \{a_n\}_{n=0}^{\infty} \) and \( \{b_n\}_{n=0}^{\infty} \) satisfying \( a_n, b_n \to 0 \) as \( n \to \infty \); \( \sum_{n=0}^{\infty} a_n = \infty \). Then the following two assertions are equivalent:

(i) Mann iteration (1.5) converges to the fixed point of \( T \);
(ii) Ishikawa iteration (1.6) converges to the fixed point of \( T \).

**Proof.** By the definition of the strongly pseudocontractive map, we know that the fixed point of \( T \) is unique. If the Ishikawa iteration (1.6) converges to the fixed point \( q \in F(T) \), then set \( b_n = 0 \), we can get the convergence of the Mann iteration (1.5). Conversely, we only want to prove (i) \( \Rightarrow \) (ii), i.e., \( \|u_n - q\| \to 0 \) as \( n \to \infty \) \( \Rightarrow \|x_n - q\| \to 0 \) as \( n \to \infty \). Without loss of generality, we assume that \( \|u_n - q\| \leq 1 \) for all \( n \geq 0 \).

Applying Lemma 1.1, (1.5), (1.6), and (1.3), we have

\[
\|x_{n+1} - u_{n+1}\|^2 \\
\leq (1 - a_n)^2\|x_n - u_n\|^2 + 2a_n\langle Ty_n - Tu_n, j(x_{n+1} - u_{n+1}) \rangle \\
\leq (1 - a_n)^2\|x_n - u_n\|^2 + 2a_n\langle Ty_n - Tu_n, j(y_n - u_n) \rangle \\
+ 2a_n\langle Ty_n - Tu_n, j(x_{n+1} - u_{n+1}) - j(y_n - u_n) \rangle \\
\leq (1 - a_n)^2\|x_n - u_n\|^2 + 2a_n\langle y_n - u_n \|^2 - \Phi(\|y_n - u_n\|)\|y_n - u_n\| \rangle
\]
\[\begin{align*}
&+ 2a_n\|Ty_n - Tu_n\| \cdot \|j\left(\frac{x_{n+1} - u_{n+1}}{1 + \|x_n - u_n\|}\right) - j\left(\frac{y_n - u_n}{1 + \|x_n - u_n\|}\right)\| \\
&\cdot (1 + \|x_n - u_n\|) \leq (1 - a_n)^2\|x_n - u_n\|^2 + 2a_n(\|y_n - u_n\|^2 - \Phi(\|y_n - u_n\|)\|y_n - u_n\|) \\
&+ 2a_nL(1 + \|y_n - u_n\|)A_n(1 + \|x_n - u_n\|),
\end{align*}\]

where \(A_n = \|j\left(\frac{x_{n+1} - u_{n+1}}{1 + \|x_n - u_n\|}\right) - j\left(\frac{y_n - u_n}{1 + \|x_n - u_n\|}\right)\| \to 0\) as \(n \to \infty\). Indeed, since

\[\begin{align*}
|\|x_{n+1} - u_{n+1}\| - \|y_n - u_n\|&| \\
&\leq 1 - a_n + \frac{a_nL(1 + \|x_n - u_n\|)}{1 + \|x_n - u_n\|}\|x_n - u_n\| + b_n\|Ty_n - Tu_n\| \\
&\leq 1 - a_n + \frac{a_nL(1 + \|x_n - u_n\|)}{1 + \|x_n - u_n\|}\|x_n - u_n\| + b_n\|Ty_n - Tu_n\| + \|Ty_n - Tu_n\| + \|Ty_n - Tu_n\| \\
&\leq 1 - a_n + a_nL(1 + \|x_n - u_n\|) + b_n\|Ty_n - Tu_n\| + b_n\|Ty_n - Tu_n\| + \|Ty_n - Tu_n\| + \|Ty_n - Tu_n\| \\
&\leq 1 - a_n + a_nL(2 - b_n + 2b_nL + b_n(1 + L))\|u_n - q\|,
\end{align*}\]

and

\[\begin{align*}
&\|x_{n+1} - u_{n+1}\| - \|y_n - u_n\| \\
&\leq (1 - b_n)\|x_n - u_n\| + b_n\|Ty_n - Tu_n\| \\
&\leq (1 - b_n)\|x_n - u_n\| + b_n\|Ty_n - Tu_n\| + \|Ty_n - Tu_n\| + \|Ty_n - Tu_n\| \\
&\leq 1 - b_n + b_nL + b_n(L + \|x_n - u_n\|)\|u_n - q\|,
\end{align*}\]

so sequences \(\{x_{n+1} - u_{n+1}\}\) and \(\{y_n - u_n\}\) are bounded, and

\[\begin{align*}
\|x_{n+1} - u_{n+1} - (y_n - u_n)\| \\
&\leq |b_n - a_n| + 2a_nL + 2b_nL^2 + a_nb_nL\|u_n - q\| + a_nL + b_n + b_nL + b_n(1 + L)\|u_n - q\| \to 0
\end{align*}\]

as \(n \to \infty\). Using uniformly continuity of \(j\) on bounded subset of \(E\), then \(A_n \to 0\) as \(n \to \infty\).

Again using Lemma 1.1, (1.6), and (1.3), we obtain

\[\begin{align*}
&\|y_n - u_n\|^2 \\
&\leq (1 - b_n)^2\|x_n - u_n\|^2 + 2b_n\langle Tx_n - Tu_n, j(y_n - u_n) \rangle \\
&+ 2b_n\langle Tu_n - u_n, j(y_n - u_n) \rangle \\
&\leq (1 - b_n)^2\|x_n - u_n\|^2 + 2b_n\langle Tx_n - Tu_n, j(x_n - u_n) \rangle \\
&+ 2b_n\langle Tx_n - Tu_n, j(y_n - u_n) - j(x_n - u_n) \rangle + 2b_n\|Tu_n - u_n\| \cdot \|y_n - u_n\|
\end{align*}\]
\[
\leq (1 - b_n)\|x_n - u_n\|^2 + 2b_n(\|x_n - u_n\|^2 - \Phi(\|x_n - u_n\|)\|x_n - u_n\|)
+ 2b_n\|Tx_n - Tu_n\|\cdot \|j(y_n - u_n) - j(x_n - u_n)\|
+ 2b_n\|Tu_n - u_n\| \cdot \|y_n - u_n\|
\leq (1 + b_n)\|x_n - u_n\|^2 - 2b_n\Phi(\|x_n - u_n\|)\|x_n - u_n\|
+ 2b_nL(1 + \|x_n - u_n\|)\cdot \|j\left(\frac{y_n - u_n}{1 + \|x_n - u_n\|}\right) - j\left(\frac{x_n - u_n}{1 + \|x_n - u_n\|}\right)\|
\cdot (1 + \|x_n - u_n\|) + 2b_n\|Tu_n - u_n\| \cdot \|y_n - u_n\|
\leq (1 + b_n)\|x_n - u_n\|^2 - 2b_n\Phi(\|x_n - u_n\|)\|x_n - u_n\|
+ 2b_nL(1 + \|x_n - u_n\|)^2B_n + 2b_n\|Tu_n - u_n\| \cdot \|y_n - u_n\|
\leq (1 + b_n)\|x_n - u_n\|^2 - 2b_n\Phi(\|x_n - u_n\|)\|x_n - u_n\|
+ 4b_nL(1 + \|x_n - u_n\|^2)B_n + 2b_n\|Tu_n - u_n\| \cdot \|y_n - u_n\|
\leq (1 + b_n)\|x_n - u_n\|^2 - 2b_n\Phi(\|x_n - u_n\|)\|x_n - u_n\| + 4b_nB_nL
+ 2b_n\|Tu_n - u_n\| \cdot \|y_n - u_n\|
\]

where \(B_n = \|j\left(\frac{y_n - u_n}{1 + \|x_n - u_n\|}\right) - j\left(\frac{x_n - u_n}{1 + \|x_n - u_n\|}\right)\| \to 0\) as \(n \to \infty\) (By the uniformly continuous of \(j\) on a bounded set). Furthermore, we have the following estimates as a part of (2.5)

(2.6)
\[
\|y_n - u_n\|
= \|\left(1 - b_n\right)(x_n - u_n) + b_n(Tx_n - u_n)\|
= \|\left(1 - b_n\right)(x_n - u_n) + b_n(Tx_n - Tu_n) + b_n(Tu_n - u_n)\|
\leq (1 - b_n)\|x_n - u_n\| + b_nL(1 + \|x_n - u_n\|) + b_n\|Tu_n - Tq\| + b_n\|u_n - q\|
\leq (1 - b_n)\|x_n - u_n\| + b_nL(1 + \|x_n - u_n\|) + b_nL(1 + \|u_n - q\|) + b_n\|u_n - q\|
\leq 2b_nL + (1 - b_n + b_nL)\|x_n - u_n\| + (b_n + b_nL)\|u_n - q\|
\]

and

\[
= 2b_n\|Tu_n - u_n\| \cdot \|y_n - u_n\|
\leq 2b_n(\|Tu_n - Tq\| + \|u_n - q\|)
\]

(2.7)
\[
\leq 2b_nL(1 + L)\|u_n - q\|
\]

\[
\leq 2b_nL + (1 - b_n + b_nL)\|x_n - u_n\| + (b_n + b_nL)\|u_n - q\|
\]

\[
= 2b_nC_n(2b_nL + D_n\|x_n - u_n\| + E_n)
\]

\[
\leq 2b_nC_n(2b_nL + E_n) + b_nC_nD_n(1 + \|x_n - u_n\|^2),
\]

where \(C_n = L + (1 + L)\|u_n - q\|, D_n = 1 - b_n + b_nL, E_n = (b_n + b_nL)\|u_n - q\|\). Substituting (2.7) into (2.5), we obtain that

\[
\|y_n - u_n\|^2 \leq (1 + b_n^2 + 4b_nB_n + b_nC_nD_n)\|x_n - u_n\|^2
\]
\[(2.8) \quad b_n C_n (4b_n L + 2E_n + D_n) + 4b_n B_n L \leq (1 + W_n) \|x_n - u_n\|^2 + V_n,\]

where \( W_n = b_n^2 + 4b_n B_n L + b_n C_n D_n, \) \( V_n = b_n C_n (4b_n L + 2E_n + D_n) + 4b_n B_n L. \)

We now estimate \( 2a_n L (1 + \|y_n - u_n\|) A_n (1 + \|x_n - u_n\|) \) in formula \((2.1).\) Using \((2.6),\) we have

\[(2.9) \quad 2a_n L (1 + \|y_n - u_n\|) A_n (1 + \|x_n - u_n\|) \]
\[\leq 2a_n A_n L (1 + 2b_n L + (1 - b_n + b_n L)) \|x_n - u_n\| + (b_n + b_n L) \|u_n - q\| \]
\[\cdot (1 + \|x_n - u_n\|) \]
\[\leq 2a_n A_n L (1 + 2b_n L + E_n + D_n) \|x_n - u_n\| \cdot (1 + \|x_n - u_n\|) \]
\[\leq 2a_n A_n L (1 + 2b_n L + E_n + (1 + 2b_n L + E_n + D_n) \|x_n - u_n\| \]
\[+ D_n \|x_n - u_n\|^2) \]
\[\leq 2a_n A_n L (1 + 2b_n L + E_n + 2(1 + 2b_n L + E_n + D_n) \|x_n - u_n\| \]
\[+ D_n \|x_n - u_n\|^2) \]
\[\leq 2a_n A_n L (1 + 2b_n L + E_n + (1 + 2b_n L + E_n + D_n) (1 + \|x_n - u_n\|^2) \]
\[+ D_n \|x_n - u_n\|^2) \]
\[\leq 2a_n A_n L (2 + 4b_n L + 2E_n + D_n + (1 + 2b_n L + E_n + 2D_n) \|x_n - u_n\|^2) \]
\[\leq 2a_n A_n F_n L (1 + \|x_n - u_n\|^2),\]

where \( F_n = 2 + 4b_n L + 2E_n + 2D_n. \)

Taking \((2.9)\) into \((2.1),\) we get that

\[(2.10) \quad \|x_{n+1} - u_{n+1}\|^2 \leq (1 - a_n)^2 \|x_n - u_n\|^2 + 2a_n (\|y_n - u_n\|^2 - \Phi(\|y_n - u_n\|) \|y_n - u_n\|^2) \]
\[+ 2a_n A_n F_n L (1 + \|x_n - u_n\|^2).\]

Set \( \eta = \inf_{n \geq 0} \frac{\Phi(\|y_n - u_n\|)}{1 + \|y_n - u_n\|^2}. \) Then \( \eta = 0.\) If such is not the case, we assume that \( \eta > 0,\) and choose a \( \delta > 0\) such that \( 0 < \delta < \min\{1, \eta\}. \) Then we have \( \Phi(\|y_n - u_n\|) \geq \delta + \delta \|y_n - u_n\| \geq \delta \|y_n - u_n\|. \)

It follows from \((2.10)\) and \((2.8)\) that

\[(2.11) \quad \|x_{n+1} - u_{n+1}\|^2 \leq (1 - a_n)^2 \|x_n - u_n\|^2 + 2a_n (1 - \delta)(1 + W_n) \|x_n - u_n\|^2 + V_n \]
\[+ 2a_n A_n L (1 + \|x_n - u_n\|^2). \]
\[\leq (1 - a_n (2\delta - a_n - 2W_n - 2A_n F_n L)) \|x_n - u_n\|^2 + 2a_n (1 - \delta)V_n + 2a_n A_n F_n L.\]
Since $a_n + 2 W_n + 2 A_n F_n L \to 0$ as $n \to \infty$, then there exists $n_0$ such that, for all $n > n_0$, we have $a_n + 2 W_n + 2 A_n F_n L < \delta$, this leads to

\begin{equation}
\|x_{n+1} - u_{n+1}\|^2 \\
\leq (1 - a_n \delta)\|x_n - u_n\|^2 + 2 a_n (1 - \delta) V_n + 2 a_n A_n F_n L
\end{equation}

(2.12)

for all $n > n_0$. By Lemma 1.2, we get $\|x_n - u_n\| \to 0$ as $n \to \infty$. Again formula (2.6), we have $\|y_n - u_n\| \to 0$ as $n \to \infty$, contradicting condition $\eta > 0$. Hence $\eta = 0$. Since $\Phi$ is strictly increasing and continuous with $\Phi(0) = 0$, then there exists a subsequence $\{y_{n_i} - u_{n_i}\}$ of $\{y_n - u_n\}$ such that $\lim_{i \to \infty} \|y_{n_i} - u_{n_i}\| = 0$. On using (1.6), we obtain that

(2.13)

Here, without loss of generality, we assume that $1 - b_n - b_n L > 0$ for all $n \geq 0$. Thus (2.13) implies that

\begin{equation}
\|y_{n_i} - u_{n_i}\| + 2 b_n, L + (b_n + b_n L)\|u_{n_i} - q\| \\
\geq (1 - b_n - b_n L)\|x_{n_i} - u_{n_i}\| \geq 0.
\end{equation}

(2.14)

Since $\lim_{i \to \infty} \|y_{n_i} - u_{n_i}\| = 0$, $\lim_{n \to \infty} b_n = 0$, $\lim_{n \to \infty} \|u_n - q\| = 0$, then we have $\lim_{i \to \infty} \|x_{n_i} - u_{n_i}\| = 0$. Let $0 < \epsilon < 1$ be given, choose $N$ such that, for all $i \geq N$, $\|x_{n_i} - u_{n_i}\| < \epsilon$, $\|y_{n_i} - u_{n_i}\| < \epsilon$, $a_n < \frac{\epsilon}{2 + 2 L (1 + \epsilon)}$, $b_n < \frac{\epsilon}{2 a_n L + 8 L}$, and $a_n + 4 A_n F_n L < \Phi(\frac{\epsilon}{8}) (\frac{\epsilon}{8})$, $\|u_{n_i} - q\| < \epsilon$. In this case, we can prove that $\|x_{n_i + m} - u_{n_i + m}\| < \epsilon$. \forall $m = 1, 2, 3, \ldots$. First we prove that $\|x_{n_i + 1} - u_{n_i + 1}\| < \epsilon$. Suppose this is not the case. Then $\|x_{n_i + 1} - u_{n_i + 1}\| \geq \epsilon$. From (1.6), we obtain that

\begin{equation}
\|x_{n_i} - u_{n_i}\| \\
\geq \|x_{n_i + 1} - u_{n_i + 1}\| - a_n \|x_{n_i} - u_{n_i}\| - a_n \|Ty_{n_i} - Tu_{n_i}\| \\
\geq \epsilon - a_n \epsilon - a_n L (1 + \|y_{n_i} - u_{n_i}\|) \\
\geq \epsilon - a_n \epsilon - a_n L (1 + \epsilon) \\
= \epsilon - a_n (\epsilon + L (1 + \epsilon)) \\
\geq \frac{\epsilon}{2},
\end{equation}

and

\begin{equation}
\|y_{n_i} - u_{n_i}\| \\
\geq (1 - b_n)\|x_{n_i} - u_{n_i}\| - b_n \|Tx_{n_i} - u_{n_i}\| \\
\geq (1 - b_n)\|x_{n_i} - u_{n_i}\| - b_n \|Tx_{n_i} - Tu_{n_i}\| - b_n \|Tu_{n_i} - u_{n_i}\|
\end{equation}

(2.15)
In [2], Bogin showed that

\[ ||x_{n_i} - u_{n_i}|| - b_{n_i}L(1 + ||x_{n_i} - u_{n_i}||) \]

\[ - b_{n_i}(||Tu_{n_i} - Tq|| + ||u_{n_i} - q||) \]

\[ \geq (1 - b_{n_i})||x_{n_i} - u_{n_i}|| - b_{n_i}L(1 + ||x_{n_i} - u_{n_i}||) \]

\[ - b_{n_i}(L + (1 + L)||u_{n_i} - q||) \]

\[ \geq (1 - b_{n_i})||x_{n_i} - u_{n_i}|| - b_{n_i}L(1 + ||x_{n_i} - u_{n_i}||) \]

\[ - b_{n_i}(L + (1 + L)||u_{n_i} - q||) \]

\[ \geq (1 - b_{n_i})\epsilon^2 - b_{n_i}L(1 + \epsilon) - b_{n_i}(L + (1 + L)\epsilon) \]

\[ = (1 - b_{n_i})\epsilon^2 - b_{n_i}L(1 + \epsilon) - b_{n_i}(L + (1 + L)\epsilon) > \frac{\epsilon}{4} \]

for all \( i \geq N \). From (2.10), we have

\[ (2.17) \]

\[ \|x_{n_i+1} - u_{n_i+1}\|^2 \]

\[ \leq (1 - a_{n_i})^2\|x_{n_i} - u_{n_i}\|^2 + 2a_{n_i}(\|y_{n_i} - u_{n_i}\|^2 - \Phi(\|y_{n_i} - u_{n_i}\|)\|y_{n_i} - u_{n_i}\|) \]

\[ + 2a_{n_i}A_nF_nL(1 + \|x_{n_i} - u_{n_i}\|^2) \]

\[ \leq (1 - a_{n_i})^2\epsilon^2 + 2a_{n_i}\epsilon^2 - 2a_{n_i}\Phi(\frac{\epsilon}{4})\frac{\epsilon}{4} + 2a_{n_i}A_nF_nL(1 + \epsilon^2) \]

\[ \leq \epsilon^2 - a_{n_i}\Phi(\frac{\epsilon}{4})\frac{\epsilon}{4} \]

\[ < \epsilon^2, \]

which is a contradiction. Hence \( \|x_{n_i+1} - u_{n_i+1}\| < \epsilon \). Now we assume that \( \|x_{n_i+m} - u_{n_i+m}\| < \epsilon \) holds. It follows from the above argument that \( \|x_{n_i+m+1} - u_{n_i+m+1}\| < \epsilon \). Therefore, this shows that \( \lim_{n \to \infty} \|x_n - u_n\| = 0 \). Since \( \lim_{n \to \infty} \|u_n - q\| = 0 \), and the inequality

\[ 0 \leq \|x_n - q\| \leq \|x_n - u_n\| + \|u_n - q\| \to 0 \]

as \( n \to \infty \). That is, \( \lim_{n \to \infty} \|x_n - q\| = 0 \).

Remark 1. In [2], Bogin showed that \( T \) is strongly pseudocontractive if and only if \( (I - T) \) is strongly accretive, where \( I \) denotes the identity operator. By this fact, then \( T \) is \( \Phi \)-strongly pseudocontractive if and only if \( (I - T) \) is \( \Phi \)-strongly accretive. Thus we have the following results.

**Theorem 2.2.** Let \( E \) be a real uniformly smooth Banach space. Assume that \( T : E \to E \) is a generalized Lipschitz and \( \Phi \)-strongly accretive operator. For any given \( f \in E \), define \( S : E \to E \) by \( Sx = f - Tx + x \) for all \( x \in E \). Suppose that \( q \) is the solution of equation \( Tx = f \). Let \( \{a_{n}\}_{n=0}^{\infty} \) and \( \{b_{n}\}_{n=0}^{\infty} \) be two real sequences \( [0, 1] \) in satisfying the conditions: (i) \( a_n, b_n \to 0 \) as \( n \to \infty \); (ii) \( \sum_{n=0}^{\infty} a_n = \infty \). Define the Ishikawa iterative sequence \( \{x_n\}_{n=0}^{\infty} \) generated from an arbitrary \( x_0 \in E \) by

\[ \{x_{n+1} = (1 - a_n)x_n + a_nSy_n, \ n \geq 0, \]

\[ y_n = (1 - b_n)x_n + b_nSx_n, \ n \geq 0, \]

\[ \sum_{n=0}^{\infty} a_n = \infty \].
and Mann iteration sequence \( \{u_n\}_{n=0}^{\infty} \) generated from an arbitrary \( u_0 \in E \) by
\begin{equation}
 u_{n+1} = (1 - a_n)u_n + a_nSu_n, \quad n \geq 0.
\end{equation}

Then the following two assertions are equivalent:
(i) Mann iteration (2.19) converges to the solution of equation \( Tx = f \);
(ii) Ishikawa iteration (2.18) converges to the solution of equation \( Tx = f \).

Or
(i) Mann iteration (2.19) converges to the fixed point of \( S \);
(ii) Ishikawa iteration (2.18) converges to the fixed point of \( S \).

Proof. Since \( T \) is generalized Lipschitzian and \( \Phi \)-strongly accretive operator, by the definition of \( S \), then we know that \( S \) is generalized Lipschitzian \( \Phi \)-strongly pseudocontractive mapping. Applying Theorem 2.1, we obtain conclusion of Theorem 2.2.

Remark 2. Theorem 2.1 and Theorem 2.2 extend and improve Theorem 2.1 and Corollary 3.1 of [3] in the following sense:

(i). The hypotheses that both “the bounded subset \( D \) in [3, Theorem 2.1]” and “\( \{x_n\} \) is bounded in [3, Corollary 3.1]” are replaced by the more general condition generalized Lipschitz maps.

(ii). The strongly pseudocontractive in [3, Theorem 2.1] and the strongly accretive operators [3, Corollary 3.1] are replaced by \( \Phi \)-strongly pseudocontractive and \( \Phi \)-strongly accretive operators respectively.

Remark 3. A map \( T \) is said to be uniformly \( \psi \)-pseudocontractive if there exists a function \( \psi : \mathbb{R}^+ \rightarrow \mathbb{R}^+ \) which is strictly continuous increasing with \( \psi(0) = 0 \), and a \( j(x - y) \in J(x - y) \) such that
\begin{equation}
 (Tx - Ty, j(x - y)) \leq \|x - y\|^2 - \psi(\|x - y\|)
\end{equation}
for all \( x, y \in D(T) \). By the remark 1, then \( T \) is uniformly \( \psi \)-accretive if and only if \( I - T \) is uniformly \( \psi \)-pseudocontractive. Hence using the same methods, we may also obtain that Theorem 2.1 and Theorem 2.2 hold for the more general class of the generalized Lipschitzian uniformly \( \psi \)-pseudocontractive and uniformly \( \psi \)-accretive operators individually.

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References


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