TRANS-SEPARABILITY IN THE STRICT AND COMPACT-OPEN TOPOLOGIES

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Abstract. We give a characterization of trans-separability for the function spaces \((C_b(X, E), \beta), (C(X, E), k)\) and \((C_b(X, E), u)\) in the case of \(E\) any general topological vector space.

1. Introduction

The fundamental result on the characterization of separability of \((C_b(X), u)\) was obtained by M. Krein and S. Krein [12] in 1940. Later, similar results were obtained by Gulick and Schmets [5] and, independently, by Summers [15] for \((C_b(X), k)\) and \((C_b(X), \beta)\). On the other hand, Gulick and Schmets [5] also gave a characterization of seminorm-separability for \((C_b(X), u), (C_b(X), k)\) and \((C_b(X), \beta)\). Characterization of separability for vector-valued function spaces have been considered in [16, 2, 8]. In [9], the author generalised these results by giving a characterization of neighbourhood-separability for the spaces \((C_b(X, E), \beta), (C(X, E), k)\) and \((C_b(X, E), u)\) in the case of \(E\) a ‘semi-convex’ topological vector space (TVS) having non-trivial topological dual \(E'\). The purpose of this note is to extend these results further to the case of \(E\) any general TVS, using the terminology of trans-separability as in [10].

2. Preliminaries

For the convenience of the reader, we recall some terminology so that this note can be read independently of [9, 10]. Let \(X\) denote a completely regular Hausdorff space and \(E\) a non-trivial Hausdorff TVS with \(W\) a base of neighbourhoods of 0. A neighbourhood \(G\) of 0 in \(E\) is called shrinkable [11] if \(rG \subseteq \text{int} G\) for \(0 \leq r < 1\). By ([11], Theorems 4 and 5), every Hausdorff TVS has a base of shrinkable neighbourhoods of 0 and also the Minkowski functional \(\rho_G\) of any such neighbourhood \(G\) is continuous and positively homogeneous.
Definition 1. The strict topology $\beta$ [1, 7] on $C_0(X, E)$ is defined as the linear topology which has a base of neighbourhoods of 0 consisting of all sets of the form

$$U(\varphi, W) = \{f \in C_0(X, E) : (\varphi f)(X) \subseteq W\},$$

where $\varphi \in B_0(X)$, the set of all bounded scalar-valued functions on $X$ which vanish at infinity, and $W \in W$.

Let $u$ (resp. $k$) denotes the uniform (resp. compact-open topology) on $C_0(X, E)$ (resp. $C(X, E)$). Then $k \leq \beta \leq u$ on $C_0(X, E)$. For any $\varphi \in B_0(X)$, let $\| \cdot \|_\varphi$ denote the seminorm on $C^b(X)$ given by $\| f \|_\varphi = \sup \{ |\varphi(x)f(x)| : x \in X \}$, $f \in C_0(X)$. We shall denote by $C(X) \otimes E$ the vector subspace of $C(X, E)$ spanned by the set of all functions of the form $\varphi \otimes a$, where $\varphi \in C(X)$, $a \in E$, and $(\varphi \otimes a) = \varphi(x)a$, $x \in X$.

Recall that a locally convex space $L$ is called seminorm-separable [5] if, for each continuous seminorm $p$ on $L$, $(L, p)$ is separable. The following classical result is stated for reference purpose.

Theorem 1 ([5]). The following statements are equivalent:

(a) $(C_0(X), \beta)$ is seminorm-separable.
(b) $(C(X), \kappa)$ is seminorm-separable.
(c) Every compact subset of $X$ is metrizable.

Definition 2. A uniform space $L$ is called trans-separable if every uniform cover of $L$ admits a countable subcover [6]. In particular, a TVS $L$ is trans-separable if, for each neighbourhood $W$ of 0 in $L$, the open cover $\{a+W : a \in L\}$ of $L$ admits a countable subcover.

Drewnowski [3] had actually coined the word “trans-separable” and it has been further used by Robertson [13] and Ferrando-Kakol-Pellicer [4]. Khan [9] introduced a generalized notion of separability, namely, the neighbourhood-separability in the TVS setting, as follows.

Definition 3. Let $L$ be a TVS, and let $V$ be a neighbourhood of 0 in $L$. A subset $H$ of $L$ is said to be $V$-dense in $L$ if, for any $z \in L$ and $\delta > 0$, there exists an element $y \in H$ such that $y - z \in \delta V$. $L$ is called neighbourhood-separable if, for each neighbourhood $V$ of 0, there exists a countable $V$-dense subset of $L$.

Another notion of generalized separability may also be considered, as follows.

Definition 4. Let $(L, \tau)$ be a TVS whose topology is generated by a family $Q(\tau)$ of continuous $F$-seminorms [17]. Then $(L, \tau)$ is called $F$-seminorm-separable if $(L, q)$ is separable for each $q \in Q(\tau)$.

Clearly, separability implies $F$-seminorm-separability; the converse holds in metrizable spaces.

The following result establishes the equivalence of all the above notions of generalized separabilities.
Lemma 1 (cf. [10]). Let $(L, \tau)$ be a TVS. The following are equivalent:

1. $(L, \tau)$ is trans-separable.
2. $(L, \tau)$ is neighbourhood-separable.
3. $(L, \tau)$ is $F$-seminorm-separable.

Proof. (1) $\Rightarrow$ (2) Suppose $L$ is trans-separable, and let $V$ be a neighbourhood of $0$. For each $n \geq 1$, $U_n = \{x + n^{-1}V : x \in L\}$ is a uniform cover of $L$, and so it has a countable subcover $U_n^* = \{x_k^{(n)} + n^{-1}V : k \in \mathbb{N}\}$. Let $D = \bigcup_{n=1}^{\infty} \{x_k^{(n)} : k \in \mathbb{N}\}$. To show that $D$ is $V$-dense in $L$, let $y \in L$ and $\delta > 0$. Choose $N \geq 1$ such that $N^{-1} < \delta$. Since $U_N^*$ is a cover of $L$, $y \in x_K^{(N)} + N^{-1}V$ for some $K \in \mathbb{N}$. Then $y - x_K^{(N)} \in \delta V$. Hence $L$ is neighbourhood-separable.

(2) $\Rightarrow$ (3) This is trivial.

(3) $\Rightarrow$ (1) Suppose $(L, q)$ is separable for each $q \in Q(\tau)$. Let $\{x + U : x \in L\}$ be any uniform cover of $L$, where $U$ is neighbourhood of $0$ in $L$. Choose a balanced neighbourhood $V$ of $0$ in $L$ such that $\mathbb{N} \cdot V \subseteq U$. Let $\{z_n\}$ be a countable dense subset in $(L, q)$. Since $L = \bigcup_{x \in L} (x + W)$, to each $z_n \in L$, there exists some $x_n \in L$ such that $z_n - x_n \in W$. Let $y \in L$. Choose $z_k$ such that $q(y - z_k) < 1$. Then

$$y - x_k = (y - z_k) + (z_k - x_k) \in W + W \subseteq U,$$

and so $L = \bigcup_{n \geq 1} (x_n + U)$.

3. Main results

Theorem 2. Let $E$ be any non-trivial TVS. Then the following statements are equivalent:

(a) $(C_b(X) \otimes E, \beta)$ is trans-separable.
(b) $(C(X) \otimes E, k)$ is trans-separable.
(c) Every compact subset of $X$ is metrizable and $E$ is trans-separable.

Proof. (a) $\Rightarrow$ (b) This follows from the fact that $k \leq \beta$ on $C_b(X) \otimes E$ and that $C_b(X) \otimes E$ is $k$-dense in $C(X) \otimes E$.

(b) $\Rightarrow$ (c) This follows from Theorem 1 and the fact that both $(C(X), k)$ and $E$ are isomorphic to subspaces of $(C(X) \otimes E, k)$ via the maps $g \mapsto g \otimes a$ ($0 \neq a \in E$ fixed) and $a \mapsto 1_X \otimes a$, respectively.

(c) $\Rightarrow$ (a) By Theorem 1, $(C_b(X), \beta)$ is trans-separable. Fix a $\varphi \in B_0(X)$, $0 \leq \varphi \leq 1$ and a balanced $W \subseteq W$. We need to show that there is a countable set $H \subseteq C_b(X) \otimes E$ such that $C_b(X) \otimes E = H + U(\varphi, W)$.

For every pair $m, n \in \mathbb{N}$ choose a balanced $U_{m,n} \in W$ so that, denoting $V_{m,n} = U_{m,n} + mU_{m,n} + U_{m,n}$, one has

$$V_{m,n} + \cdots + V_{m,n} \ (n\text{-summands}) \subseteq W.$$

Also, choose a countable set $D_{m,n}$ in $E$ so that $E = D_{m,n} + U_{m,n}$. Let $D$ be the union of all these sets $D_{m,n}$ ($m, n \in \mathbb{N}$).
Next, for each $k \in \mathbb{N}$ denote $B_k = \{ f \in C_b(X) : ||f||_\varphi \leq 1/k \}$ and choose a countable set $G_k$ in $C_b(X)$ so that $C_b(X) = G_k + B_k$. Let $G$ be the union of all these sets $G_k$ ($k \in \mathbb{N}$).

We are going to show that the countable set $H = H_{\varphi,W}$ of all functions in $C_b(X) \otimes E$ of the form $h = \sum_{i=1}^n g_i \otimes d_i$, where $g_i \in G$ and $d_i \in D$ ($i = 1, \ldots, r$, $r \in \mathbb{N}$), is as required.

Take any $f \in C_b(X) \otimes E$. Then $f = \sum_{i=1}^n f_i \otimes a_i$ for some $f_1, \ldots, f_n \in C_b(X)$ and $a_1, \ldots, a_n \in E$. Let $m \in \mathbb{N}$ be such that $||f_i||_\varphi \leq m$ for $i = 1, \ldots, n$, and next choose $k \in \mathbb{N}$ so that $k^{-1}a_i \in U_{m,n}$ for $i = 1, \ldots, n$. By the definitions of $D_{m,n}$ and $G_k$, there are $d_1, \ldots, d_n \in D_{m,n}$ and $g_1, \ldots, g_k \in G_k$ such that

$$a_i - d_i \in U_{m,n} \quad \text{and} \quad ||f_i - g_i||_\varphi \leq 1/k \quad \text{for} \quad i = 1, \ldots, n.$$  

Now, for $i = 1, \ldots, n$ and $x \in A$,

$$\varphi(x)[f_i(x)a_i - g_i(x)d_i] = \varphi(x)[f_i(x) - g_i(x)]a_i + \varphi(x)f_i(x)(a_i - d_i)$$

$$+ \varphi(x)[g_i(x) - f_i(x)](a_i - d_i),$$

hence (using the fact that $U_{m,n}$ is balanced)

$$\varphi(x)[f_i(x)a_i - g_i(x)d_i] \in U_{m,n} + mU_{m,n} + U_{m,n} = V_{m,n}.$$  

In consequence, setting $h = \sum_{i=1}^n g_i \otimes d_i$ we have $h \in H$ and for every $x \in A$,

$$\varphi(x)[f(x) - h(x)] = \sum_{i=1}^n \varphi(x)[f_i(x)a_i - g_i(x)d_i] \in W$$

so that $f - h \in U(\varphi, W)$. \qed

Remark 1. A somewhat more transparent variant of the above proof that (c) implies (a) can be based on Lemma 1. We need to show that for any $\varphi \in B_0(X)$, $0 \leq \varphi \leq 1$, and any continuous $F$-seminorm $q$ on $E$, the space $(C_b(X) \otimes E, p_\varphi)$ is separable, where $p_\varphi(f) = \sup_{x \in X} q(\varphi(x)f(x))$. Now, let $G$ be a countable subset dense in $(C_b(X), ||\cdot||_{\varphi})$, and $D$ a countable set dense in $(E, q)$. Take any $f = \sum_{i=1}^n f_i \otimes a_i \in C_b(X) \otimes E$, and choose $m \in \mathbb{N}$ so that $||f_i||_\varphi \leq m$ for each $i$. Given $\varepsilon > 0$, let $g = \sum_{i=1}^n g_i \otimes d_i$, where $g_i \in G$ and $d_i \in D$. Assume that $||f_i - g_i||_\varphi \leq \delta$ for all $i$ and some as yet unspecified $0 < \delta < 1$. Then, making use of (3), it is easily seen that

$$p_\varphi(f - g) \leq \sum_{i=1}^n (q(||f_i - g_i||_\varphi a_i) + q(||f_i||_\varphi(a_i - d_i)) + q(||f_i - g_i||_\varphi(a_i - d_i))$$

$$\leq \sum_{i=1}^n (q(\delta a_i) + (m + 1)q((a_i - d_i)$$

and this can be made smaller than $\varepsilon$ by taking $\delta$ sufficiently small and choosing the $d_i$'s in $D$ sufficiently close to the $a_i$'s. It follows that the countable set of all $g$'s of the above form is dense in $(C_b(X) \otimes E, p_\varphi)$.  

Remark 2. If $X$ has finite covering dimension or $E$ is locally convex, or $E$ has the approximation property or $E$ is complete metrizable with a basis, then $C_b(X) \otimes E$ is $\beta$-dense in $C_b(X, E)$ and that $C(X) \otimes E$ is $k$-dense in $C(X, E)$ (see [14, 7]). Hence, under these assumptions, the above theorem holds with $C_b(X) \otimes E$, $C(X) \otimes E$ and $C_b(X) \otimes E$ replaced by $C_b(X, E)$, $C(X, E)$ and $(C_o(X, E))$, respectively. It is not known whether or not these ‘density’ results hold for $E$ a locally bounded space. However, we include the following analogue of ([8]; [9], Theorem 3.4) for the reader’s interest.

**Theorem 3.** Let $X$ be any Hausdorff space and $E$ any locally bounded space. Then $(C_b(X, E), \beta)$ is trans-separable $\iff (C(X, E), k)$ is so.

**Proof.** Suppose $(C(X, E), k)$ is trans-separable. Let $\varphi \in B_o(X)$ with $0 \leq \varphi \leq 1$, and let $W \subset W$. Let $V$ be a balanced bounded neighbourhood of $0$ in $E$, and let $S$ be a closed shrinkable neighbourhood of $0$ with $S \subset V$. The Minkowski functional $\rho = \rho_S$ of $S$ is continuous and positive homogeneous and, consequently, for each $r > 0$, the function $h_r : E \to E$ defined by

$$h_r(a) = \begin{cases} \frac{a}{\rho(a)} & \text{if } a \in rS \\ a & \text{if } a \in E \setminus rS \end{cases}$$

is continuous. Further, $h_r(E) \subseteq rS \subseteq rV$, which shows that, for each $f \in C(X, E)$, the function $h_r \circ f \in C_b(X, E)$. Choose $t \geq 1$ such that $V + V \subseteq ts$ and $V + V \subseteq tW$. For each $m = 1, 2, \ldots$, there exists a compact set $K_m \subset X$ such that $\varphi(x) < 1/tn^2$ for $x \in X \setminus K_m$. Corresponding to each $K_m$, choose $\{f_m : n \in \mathbb{N}\}$ as a $N(K_m, V)$-dense of $C(X, E)$, where

$$U(K_m, W) = \{f \in C_b(X, E) : f(K_m) \subseteq W\}.$$  

We show that $\{h_m \circ f_m : m, n = 1, 2, \ldots\}$ is $\beta$-dense in $C_b(X, E)$. Let $f \in C_b(X, E)$ and $0 \leq \delta \leq 1$. Choose integers $M \geq 1/\delta$ and $N \geq 1$ such that $f(X) \subseteq (M\delta/t)V$ and $(fMN - f)(K_M) \subseteq (\delta/t)V$. Let $y \in X$. If $y \in K_M$, then $fMN(y) \in f(y) + (\delta/t)V \subseteq (M\delta/t)V + (M\delta/t)V \subseteq MS$ and so

$$\varphi(y)[h_M \circ fMN(y) - f(y)] = \varphi(y)[fMN(y) - f(y)] \in \delta W.$$  

If $y \in X \setminus K_M$, then, since $h_M(fMN(y)) \in h_M(E) \subseteq MS$,

$$\varphi(y)[h_M \circ fMN(y) - f(y)] \in \varphi(y)[MS - \frac{M\delta}{t}V] \subseteq \frac{1}{tM}[V + \frac{\delta}{t}V] \subseteq \frac{\delta}{t}[V + V] \subseteq \delta W.$$  

Thus $h_M \circ fMN - f \in \delta U(\varphi, W)$. Consequently, $(C_b(X, E), \beta)$ is neighbourhood-separable and hence trans-separable by Lemma 1.

The converse follows from the fact that $C_b(X, E)$ is dense in $(C(X, E), k)$, using again the local boundedness of $E$. Indeed, let $f \in C(X, E)$, $K$ a compact subset of $X$ and $W \subset W$. Let $V$ and $S$ be as above with $S \subseteq V$. Choose
$r \geq 1$ with $f(K) \subseteq rS$. Then, as in the above part, we have a function $h_r \circ f \in C_b(X, E)$ such that
\[ h_r \circ f(x) - f(x) = f(x) - f(x) = 0 \in W \text{ for all } x \in K. \]

Next, we obtain:

**Theorem 4.** Let $E$ be a non-trivial TVS. Then
(a) $(C_b(X) \otimes E, u)$ is trans-separable $\iff X$ is a compact metric space and $E$ is trans-separable.
(b) Suppose $X$ is locally compact. Then $(C_o(X) \otimes E, u)$ is trans-separable $\iff X$ is a $\sigma$-compact metric space and $E$ is trans-separable.

**Proof.** (a) In this case, $(C_b(X), u)$ is trans-separable $\iff$ it is separable $\iff X$ is a compact metric space [12]. The proof now follows just as in Theorem 2.
(b) If $X$ is locally compact, then $(C_o(X), u)$ is trans-separable $\iff$ it is separable $\iff X$ is a $\sigma$-compact metric space [5, 15]. Again the proof follows just as in Theorem 2.

Again we remark that, if $C_b(X) \otimes E$ (resp. $C_o(X) \otimes E$) is $u$-dense in $C_b(X, E)$ ($C_o(X, E)$), the above theorem remains valid with $C_b(X) \otimes E$ ($C_o(X) \otimes E$) replaced by $C_b(X, E)$ ($C_o(X, E)$).

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**References**


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