ON $\phi$-RECURRENT $(k, \mu)$-CONTACT METRIC MANIFOLDS

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Abstract. In this paper we prove that a $\phi$-recurrent $(k, \mu)$-contact metric manifold is an $\eta$-Einstein manifold with constant coefficients. Next, we prove that a three-dimensional locally $\phi$-recurrent $(k, \mu)$-contact metric manifold is the space of constant curvature. The existence of $\phi$-recurrent $(k, \mu)$-manifold is proved by a non-trivial example.

1. Introduction

The notion of local symmetry of a Riemannian manifold has been weakened by many authors in several ways to a different extent. As a weaker version of local symmetry, T. Takahashi [13] introduced the notion of locally $\phi$-symmetry on a Sasakian manifold. Generalizing the notion of $\phi$-symmetry, one of the authors, De [10] introduced the notion of $\phi$-recurrent Sasakian manifold. In the context of contact geometry the notion of $\phi$-symmetry is introduced and studied by Boeckx, Buecken, and Vanhecke [8] with several examples. In [E. Boeckx, A class of locally $\phi$-symmetric contact metric spaces, Arch. Math. 72 (1999), 466–472], he proved that every non-Sasakian $(k, \mu)$-manifold is locally $\phi$-symmetric in the strong sense.

In the present paper we introduce a type of $(k, \mu)$-contact metric manifolds called $\phi$-recurrent $(k, \mu)$-contact metric manifold which generalizes the notion of $\phi$-symmetric $(k, \mu)$-contact metric structure of Boeckx. The $(k, \mu)$-contact metric manifold is one of special interest as it contains both the class of Sasakian and non-Sasakian cases. Hence, in our opinion, this is the first time that the notion of $\phi$-recurrent manifold for the non-Sasakian case is appearing in the literature. After preliminaries in Section 3, it is proved that a $\phi$-recurrent $(k, \mu)$-contact metric manifold is an $\eta$-Einstein manifold with constant coefficients. Also it is shown that the characteristic vector field of the $(k, \mu)$-contact metric manifold is an $\eta$-Einstein manifold with constant coefficients.
manifold and the vector field associated to the 1-form of recurrence are co-directional. In Section 4, we study 3-dimensional locally \( \phi \)-recurrent \((k, \mu)\)-contact metric manifold. The last section provides the existence of the locally \( \phi \)-recurrent \((k, \mu)\)-contact metric manifold by an example which is neither locally symmetric nor locally \( \phi \)-symmetric.

2. Contact metric manifolds

A \((2n + 1)\)-dimensional manifold \( M^{2n+1} \) is said to admit an almost contact structure if it admits a tensor field \( \phi \) of type \((1,1)\), a vector field \( \xi \) and a 1-form \( \eta \) satisfying

\[
\begin{align*}
(a) \quad & \phi^2 = -I + \eta \otimes \xi, \quad (b) \quad \eta(\xi) = 1, \quad (c) \quad \phi \xi = 0, \quad (d) \quad \eta \circ \phi = 0.
\end{align*}
\]

An almost contact metric structure is said to be normal if the induced almost complex structure \( J \) on the product manifold \( M^{2n+1} \times \mathbb{R} \) defined by \( J(X, f \frac{d}{dt}) = (\phi X - f \xi, \eta(X) \frac{d}{dt}) \) is integrable, where \( X \) is tangent to \( M \), \( t \) is the coordinate of \( \mathbb{R} \) and \( f \) is a smooth function on \( M \times \mathbb{R} \). Let \( g \) be a compatible Riemannian metric with almost contact structure \((\phi, \xi, \eta)\), that is,

\[
g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y).
\]

Then \( M \) becomes an almost contact metric manifold equipped with an almost contact metric structure \((\phi, \xi, \eta, g)\). From (2.2) it can be easily seen that

\[
\begin{align*}
(a) \quad & g(X, \phi Y) = -g(Y, \phi X), \quad (b) \quad g(X, \xi) = \eta(X)
\end{align*}
\]

for all vector fields \( X \) and \( Y \). An almost contact metric structure becomes a contact metric structure if

\[
g(X, \phi Y) = d\eta(X, Y)
\]

for all vector fields \( X \) and \( Y \). The 1-form \( \eta \) is then called a contact form and \( \xi \) is the characteristic vector field. We define a \((1,1)\)-tensor field \( h \) by \( h = \frac{1}{2} L_\phi \), where \( L \) denotes the Lie differentiation. Blair [3] proved that the tensor \( h \) is a symmetric operator. Then \( h \) satisfies \( h\phi = -\phi h \). We have \( \text{Tr}(h) = \text{Tr}(\phi h) = 0 \) and \( h\xi = 0 \). Also,

\[
\nabla_X \xi = -\phi X - \phi h X
\]

holds in a contact metric manifold. A normal contact metric manifold is a Sasakian manifold. An almost contact metric manifold is Sasakian if and only if

\[
(\nabla_X \phi)Y = g(X, Y)\xi - \eta(Y)X, \quad X, Y \in TM,
\]

where \( \nabla \) is Levi-Civita connection of the Riemannian metric \( g \). A contact metric manifold \( M^{2n+1}(\phi, \xi, \eta, g) \) for which \( \xi \) is a Killing vector field is said to be a \( K \)-contact manifold. A Sasakian manifold is \( K \)-contact but not conversely. However a 3-dimensional \( K \)-contact manifold is Sasakian [11]. It is well known that the tangent sphere bundle of a flat Riemannian manifold admits a contact
metric structure satisfying $R(X, Y)\xi = 0$ [2]. On the other hand, on a Sasakian manifold the following holds:

$(2.7) \quad R(X, Y)\xi = \eta(Y)X - \eta(X)Y.$

It is well known that there exist contact metric manifolds for which the curvature tensor $R$ and the direction of the characteristic vector field $\xi$ satisfying $R(X, Y)\xi = 0$ for any vector fields $X$ and $Y$. For example, the tangent sphere bundle of a flat Riemannian manifold admits such a structure.

As a generalization of both $R(X, Y)\xi = 0$ and the Sasakian case: D. E. Blair, T. Koufogiorgos, and B. J. Papantoniou [5] considered the $(k, \mu)$-nullity condition on a contact metric manifold and gave several reasons for studying it. The $(k, \mu)$-nullity distribution $N(k, \mu)$ ([5], [12]) of a contact metric manifold $M$ is defined by

$$N(k, \mu) : p \rightarrow N_p(k, \mu) = \{ W \in T_pM \mid R(X, Y)W = (kI + \mu h)(g(Y, W)X - g(X, W)Y) \}$$

for all $X$ and $Y \in TM$, where $(k, \mu) \in \mathbb{R}^2$. A contact metric manifold $M^{2n+1}$ with $\xi \in N(k, \mu)$ is called a $(k, \mu)$-contact metric manifold. We have

$(2.8) \quad R(X, Y)\xi = k[\eta(Y)X - \eta(X)Y] + \mu[\eta(Y)hX - \eta(X)hY].$

Applying a $D$-homothetic deformation to a contact metric manifold with $R(X, Y)\xi = 0$, we obtain a contact metric manifold satisfying $(2.8)$. In [5], it is proved that the standard contact metric structure on the tangent sphere bundle $T_1(M)$ satisfies the condition that $\xi$ belongs to the $(k, \mu)$-nullity distribution if and only if the base manifold is the space of constant curvature. There exist examples in all dimensions and the condition that $\xi$ belongs to the $(k, \mu)$-nullity distribution is invariant under $D$-homothetic deformations; in dimensions greater than 5, the condition determines the curvature completely; dimension 3 include the 3-dimensional unimodular Lie groups with a left invariant metric.

On a $(k, \mu)$-contact metric manifold, $k \leq 1$. If $k = 1$, the structure is Sasakian ($h = 0$ and $\mu$ is indeterminant) and if $k < 1$, the $(k, \mu)$-nullity condition completely determines the curvature of $M^{2n+1}$ [5]. In fact, for a $(k, \mu)$-manifold, the condition of being a Sasakian manifold, a $K$-contact manifold, $k = 1$ and $h = 0$ are all equivalent.

In a $(k, \mu)$-contact metric manifold, the following relations hold ([5], [7]):

$(2.9) \quad h^2 = (k - 1)\phi^2, \quad k \leq 1,$

$(2.10) \quad (\nabla_X \phi)Y = g(X + hX, Y)\xi - \eta(Y)(X + hX),$

$(2.11) \quad R(\xi, X)Y = k[\eta(Y)X - \eta(X)Y] + \mu[\eta(hX, Y)\xi - \eta(Y)hX],$

$(2.12) \quad S(X, \xi) = 2nk\eta(X).$
\[ S(X, Y) = [2(n - 1) - n\mu]g(X, Y) + [2(n - 1) + \mu]g(hX, Y) + [2(1 - n) + n(2k + \mu)]\eta(X)\eta(Y), \quad n \geq 1, \]

\[ \tau = 2n(2n - 2 + k - n\mu), \]

\[ S(\phi X, \phi Y) = S(X, Y) - 2nk\eta(X)\eta(Y) - 2(2n - 2 + \mu)g(hX, Y), \]

where \( S \) is the Ricci tensor of type \((0,2)\) and \( \tau \) is the scalar curvature of the manifold. From (2.5), it follows that

\[ (\nabla_X \eta)Y = g(X + hX, \phi Y). \]

Also in a \((k, \mu)\)-manifold, the following holds

\[ \eta(R(X, Y)Z) = k[g(Y, Z)\eta(X) - g(X, Z)\eta(Y)] + \mu[g(hY, Z)\eta(X) - g(hX, Z)\eta(Y)]. \]

Especially for the case \( \mu = 2(1 - n) \), from (2.13) it follows that the manifold is \( \eta \)-Einstein.

The \( k \)-nullity distribution \( N(k) \) of a Riemannian manifold \( M \) \([10]\) is defined by

\[ N(k) : p \rightarrow N_p(k) = \{Z \in T_pM \mid R(X, Y)Z = k(g(Y, Z)X - g(X, Z)Y)\}, \]

\( k \) being a constant. If the characteristic vector field \( \xi \in N(k) \), then we call a contact metric manifold an \( N(k) \)-contact metric manifold \([4]\). The \( \phi \)-recurrent \( N(k) \)-contact metric manifolds have been studied by De and Gazi \([9]\).

If \( k = 1 \), then \( N(k) \)-contact metric manifold is Sasakian and if \( k = 0 \), then \( N(k) \)-contact metric manifold is locally isometric to the product \( E^{n+1} \times S^n(4) \) for \( n > 1 \) and flat for \( n = 1 \). If \( k < 1 \), the scalar curvature is \( \tau = 2n(2n - 2 + k) \).

3. \( \phi \)-recurrent \((k, \mu)\)-contact metric manifolds

**Definition 3.1** \(([13])\). A Sasakian manifold is said to be locally \( \phi \)-symmetric if the relation

\[ \phi^2((\nabla_W R)(X, Y, Z)) = 0 \]

holds for all vector fields \( X, Y, Z, W \) orthogonal to \( \xi \).

**Definition 3.2** \(([10])\). A \((k, \mu)\)-contact metric manifold is said to be \( \phi \)-recurrent if and only if there exists a non-zero 1-form \( A \) such that

\[ \phi^2((\nabla_W R)(X, Y, Z)) = A(W)R(X, Y, Z) \]

for all vector fields \( X, Y, Z, W \). Here \( X, Y, Z, W \) are arbitrary vector fields which are not necessarily orthogonal to \( \xi \).

If \( X, Y, Z, W \) are orthogonal to \( \xi \), then the manifold is called \textit{locally} \( \phi \)-recurrent. If the 1-form \( A \) vanishes identically, then the manifold is said to be a \textit{locally} \( \phi \)-symmetric manifold.
Definition 3.3 ([5]). A contact metric manifold is said to be \( \eta \)-Einstein if the Ricci tensor \( S \) of type \((0,2)\) satisfies the condition

\[
S(X,Y) = ag(X,Y) + bh(X)\eta(Y),
\]

where \( a \) and \( b \) are smooth functions on \( M^{2n+1} \).

Now we prove the main theorem of the paper.

Theorem 3.1. A \( \phi \)-recurrent \((k, \mu)\)-contact metric manifold is an \( \eta \)-Einstein manifold with constant coefficients.

Proof. By virtue of (2.1)(a) and (3.1) we have

\[
-(\nabla W)R(X,Y,Z) + \eta((\nabla W)R(X,Y,Z)\xi) = A(W)R(X,Y,Z),
\]

from which it follows that

\[
-g((\nabla W)R(X,Y,Z),U) + \eta((\nabla W)R(X,Y,Z)\eta(U)) = A(W)g(R(X,Y,Z),U).
\]

Let \( \{e_i\}, i = 1,2,3,\ldots,2n+1, \) be an orthonormal basis of the tangent space at any point of the manifold. Putting \( X = U = \{e_i\} \) in (3.4) and taking summation over \( i, 1 \leq i \leq 2n+1, \) we get

\[
-(\nabla W)S(Y,Z) + \sum_{i=1}^{2n+1} \eta((\nabla W)R(e_i,Y)Z)\eta(e_i) = A(W)S(Y,Z).
\]

The second term of (3.5) by putting \( Z = \xi \) takes the form \( g((\nabla W)R(e_i,Y)\xi, \xi)g(e_i,\xi), \) which is denoted by \( E. \) In this case \( E \) vanishes. Since the following equation is well known,

\[
g((\nabla W)R(e_i,Y)\xi,\xi) = g((\nabla W)R(e_i,Y)\xi) - g(R(\nabla W e_i,Y)\xi,\xi) - g(R(e_i,\nabla W Y)\xi,\xi) - g(R(e_i,Y)\nabla W \xi,\xi)
\]

at \( p \in M. \) Using (2.3)(b) and (2.8), we obtain

\[
g(R(e_i,\nabla W Y)\xi,\xi) = g(k[\eta(\nabla W Y)e_i - \eta(e_i)\nabla W Y] + \mu[\eta(\nabla W Y)he_i - \eta(e_i)h\nabla W Y],\xi)
\]

\[
= k[\eta(\nabla W Y)\eta(e_i) - \eta(e_i)\eta(\nabla W Y)] = 0,
\]

since \( g(hX,Y) = g(X,hY). \)

Thus, we obtain

\[
g((\nabla W)R(e_i,Y)\xi,\xi) = g((\nabla W)R(e_i,Y)\xi) - g(R(e_i,Y)\nabla W \xi,\xi).
\]

In virtue of \( g(R(e_i,Y)\xi,\xi) = 0, \) we have

\[
g((\nabla W)R(e_i,Y)\xi,\xi) + g(R(e_i,Y)\xi,\nabla W \xi) = 0,
\]

since \( (\nabla W)g = 0, \) which implies

\[
g((\nabla W)R(e_i,Y)\xi,\xi) = -g(R(e_i,Y)\xi,\nabla W \xi) - g(R(e_i,Y)\nabla W \xi,\xi) = 0.
\]
Using (2.5) and applying skew-symmetry of \( R \), we get

\[
g((\nabla_W R)(e_i, Y)\xi, \xi) = g(R(e_i, Y)\xi, \phi W + \phi h W) + g(R(e_i, Y)(\phi W + \phi h W), \xi) = g(R(\phi W + \phi h W, \xi)Y, e_i) + g(R(\xi, \phi W + \phi h W)Y, e_i).
\]

Hence, we obtain

\[
E = \sum_{i=1}^{2n+1} [g(R(\phi W + \phi h W, \xi)Y, e_i)g(\xi, e_i) + g(R(\xi, \phi W + \phi h W)Y, e_i)g(\xi, e_i)]
\]

\[
= g(R(\phi W + \phi h W, \xi)Y, \xi) + g(R(\xi, \phi W + \phi h W)Y, \xi) = 0.
\]

Replacing \( Z \) by \( \xi \) in (3.5) and using (2.12), we have

\[
(\nabla_W S)(Y, \xi) = 2nkA(W)\eta(Y).
\]

Now, we have

\[
(\nabla_W S)(Y, \xi) = \nabla_W S(Y, \xi) - S(\nabla_W Y, \xi) - S(Y, \nabla_W \xi).
\]

Using (2.5) and (2.12) in the above relation, it follows that

\[
(\nabla_W S)(Y, \xi) = 2nk(\nabla_W \eta)Y + S(Y, \phi W + \phi h W).
\]

By virtue of (2.3)(a) and (2.16), we get from (3.7)

\[
(\nabla_W S)(Y, \xi) = -2nk g(\phi W + \phi h W, Y) + S(Y, \phi W + \phi h W).
\]

By virtue of (3.6) and (3.8), we have

\[
2nkA(W)\eta(Y) = -2nk g(\phi W + \phi h W, Y) - S(Y, \phi W + \phi h W).
\]

Replacing \( Y \) by \( \phi Y \) in (3.9) and using (2.1)(d), (2.2) and (2.15), we get

\[
2nk g(\phi W + \phi h W, \phi Y) - S(\phi Y, \phi W + \phi h W) = 0,
\]

or,

\[
2nk g(W + h W, Y) - \eta(W + h W) g(Y) - S(Y, W + h W)
+ 2nk \eta(Y) g(W + h W) + 2(2n - 2 + \mu) g(W + h W, Y) = 0,
\]

or,

\[
2nk g(Y, W) + 2nk g(h W, Y) - S(Y, W) - S(Y, h W)
+ 2(2n - 2 + \mu) g(h W, Y) + 2(2n - 2 + \mu) g(h^2 W, Y) = 0,
\]

since \( g(X, h Y) = g(h X, Y) \).

Now by (2.9), the above equation takes the form

\[
2nk g(Y, W) + 2nk g(h W, Y) - S(Y, W) - S(Y, h W)
+ 2(2n - 2 + \mu) g(Y, h W) + 2(2n - 2 + \mu) (k - 1) g(Y, -W + \eta(W)\xi).
\]

Now, by using (2.13), it follows that

\[
S(Y, h W) = (2n - 2 - \mu) g(Y, h W) - (2n - 2 + \mu) (k - 1) g(Y, W)
+ (2n - 2 + \mu) (k - 1) \eta(W) \eta(Y).
\]
Hence from (3.10), we get
\[(3.12)\]
\[S(Y, W) = (2n - 2 - n\mu)g(Y, hW) - (2n - 2 + \mu)(k - 1)g(Y, W) + (2n - 2 + \mu)(k - 1)\eta(Y)\eta(W)\]
\[= 2nk g(Y, W) + [2nk + 2(2n - 2 + \mu)]g(Y, hW) - 2(2n - 2 + \mu)(k - 1)g(Y, W) + 2(2n - 2 + \mu)(k - 1)\eta(Y)\eta(W),\]
or,
\[(3.13)\]
\[S(Y, W) = [\mu(1 - k) + 2(n - 1) + 2k]g(Y, W) + [2(nk + n - 1) + \mu(n + 2)]g(Y, hW) + (2n - 2 + \mu)(k - 1)\eta(Y)\eta(W).\]
Replacing \(W\) by \(hW\) and using (2.1)(a), we get from (3.13)
\[(3.14)\]
\[S(Y, hW) = [\mu(1 - k) + 2(n - 1) + 2k]g(Y, hW) + [2(nk + n - 1) + \mu(n + 2)]g(Y, h^2 W).\]
From (3.11) and (3.14), using (2.9), it follows that
\[(3.15)\]
\[\mu(k - 1 - n) - 2k]g(Y, hW) = (k - 1)[2nk - \mu(n + 1)]g(Y, W) + (k - 1)[2nk + \mu(n + 1)]\eta(Y)\eta(W).\]
From (3.13) and (3.15), we get
\[S(Y, W) = \alpha g(Y, W) + \beta\eta(Y)\eta(W),\]
where \(\alpha = [\mu(1 - k) + 2(n - 1) + 2k] + [2(nk + n - 1) + \mu(n + 2)]\frac{[-2nk - \mu(n + 1)(k - 1)]}{\mu[(k - 1 - n) - 2k]}\]
and \(\beta = [2(n - 1) + \mu](k - 1) + [2(nk + n - 1) + \mu(n + 2)]\frac{[2nk + \mu(n + 1)](k - 1)}{\mu[(k - 1 - n) - 2k]}\).
So, the manifold is an \(\eta\)-Einstein manifold with constant coefficients. Hence the theorem is proved. \(\square\)

**Theorem 3.2.** In a \(\phi\)-recurrent \((k, \mu)\)-contact metric manifold \((M^{2n+1}, g)\) \((n > 1)\) the characteristic vector field \(\xi\) and the vector field \(\rho\) associated to the 1-form \(A\) are co-directional and the 1-form \(A\) is given by
\[A(W) = \eta(W)\eta(\rho),\]
provided that \((2n - 1)^2k^2 + \mu^2(k - 1) \neq 0.\)

**Proof.** In a \((k, \mu)\)-contact metric manifold, the relation (3.3) holds. Changing \(W, X, Y\) cyclically in (3.3) and then adding the results we obtain
\[= A(W)R(X, Y)Z + A(X)R(Y, W)Z + A(Y)R(W, X)Z,\]
which yields by virtue of Bianchi’s identity that
\[(3.16)\]
\[A(W)\eta(R(X, Y)Z) + A(X)\eta(R(Y, W)Z) + A(Y)\eta(R(W, X)Z) = 0.\]
With the help of (2.17), (3.16) reduces to

\[
A(W)[k\{g(Y, Z)\eta(X) - g(X, Z)\eta(Y)\} \\
+ \mu\{g(hY, Z)\eta(X) - g(hX, Z)\eta(Y)\}]
+ A(X)[k\{g(W, Z)\eta(Y) - g(Y, Z)\eta(W)\} \\
+ \mu\{g(hW, Z)\eta(Y) - g(hY, Z)\eta(W)\}]
+ A(Y)[k\{g(X, Z)\eta(W) - g(W, Z)\eta(X)\} \\
+ \mu\{g(hX, Z)\eta(W) - g(hW, Z)\eta(X)\}]
\]

\[
= 0.
\]

Putting \(Y = Z = e_i\) in (3.17) and taking summation over \(i, 1 \leq i \leq 2n + 1\), we get

(3.18) \((2n - 1)k[A(W)\eta(X) - A(X)\eta(W)] + \mu[A(hX)\eta(W) - A(hW)\eta(X)] = 0\).

Substituting \(X\) by \(\xi\) in (3.18), we have

(3.19) \((2n - 1)k[A(W)\eta(X) - A(\xi)\eta(W)] - \mu A(hW) = 0\).

Replacing \(W\) by \(hW\) in (3.20) and using (2.9), we get

(3.20) \((2n - 1)kA(hW) = \mu(k - 1)[-A(W) + \eta(W)A(\xi)]\).

From (3.19) and (3.20), we obtain

\[
A(W) = A(\xi)\eta(W) = \eta(\rho)\eta(W),
\]

provided that

\[
(2n - 1)^2k^2 + \mu^2(k - 1) \neq 0,
\]

where \(A(\xi) = g(\xi, \rho)\). This proves the theorem. \(\square\)

4. 3-dimensional locally \(\phi\)-recurrent \((k, \mu)\)-contact metric manifolds

On any 3-dimensional Riemannian manifold we have

(4.1) \(R(X, Y)Z = g(Y, Z)QX - g(X, Z)QY + S(Y, Z)X - S(X, Z)Y - \frac{\tau}{2}[g(Y, Z)X - g(X, Z)Y]\)

for any vector fields \(X, Y, Z\), where \(Q\) is the Ricci operator, that is, \(g(QX, Y) = S(X, Y)\) and \(\tau\) is the scalar curvature of the manifold. Moreover, using Remark 3.2 [5], we have

(4.2) \(QX = \mu(\lambda - 1)X\),

where \(\lambda = \sqrt{1 - k}, k < 1\). Therefore, it follows from (4.2) that

(4.3) \(S(X, Y) = \mu(\lambda - 1)g(X, Y)\).
Thus from (4.1), (4.2), and (4.3), we get

\begin{equation}
R(X, Y)Z = 2\mu(\lambda - 1)[g(Y, Z)X - g(X, Z)Y]
- \frac{\tau}{2}[g(Y, Z)X - g(X, Z)Y].
\end{equation}

Taking the covariant differentiation to the both sides of the equation (4.4), we get

\begin{equation}
(\nabla_W R)(X, Y)Z = -\frac{d\tau(W)}{2}[g(Y, Z)X - g(X, Z)Y].
\end{equation}

Applying \(\phi^2\) to the both sides of (4.5) and using (2.1)(a) and (2.1)(c), we get

\begin{equation}
\phi^2(\nabla_W R)(X, Y)Z = \frac{d\tau(W)}{2}[g(X, Z)\phi^2 Y - g(Y, Z)\phi^2 X].
\end{equation}

By (3.1) the equation (4.6) reduces to

\begin{equation}
A(W)R(X, Y)Z = \frac{d\tau(W)}{2}[g(Y, Z)X - g(X, Z)Y].
\end{equation}

Putting \(W = \{e_i\}\), where \(\{e_i\}, i = 1, 2, 3\), is an orthonormal basis of the tangent space at any point of the manifold and taking summation over \(i\), \(1 \leq i \leq 3\), we obtain

\begin{equation}
R(X, Y)Z = \lambda[g(Y, Z)X - g(X, Z)Y],
\end{equation}

where \(\lambda = \frac{d\tau(e_i)}{d\tau(e_j)}\) is a scalar, since \(A\) is a non-zero 1-form. Then by Schur’s theorem \(\lambda\) will be a constant on the manifold. Therefore, \(M^3\) is of constant curvature \(\lambda\). Thus we get the following theorem:

**Theorem 4.1.** A 3-dimensional connected locally \(\phi\)-recurrent \((k, \mu)\)-contact metric manifold is the space of constant curvature.

5. Existence of locally \(\phi\)-recurrent \((k, \mu)\)-contact metric manifolds

In this section, we construct an example of a locally \(\phi\)-recurrent \((k, \mu)\)-contact metric manifold to prove the existence. We consider the 3-dimensional manifold \(M = \{(x, y, z) \in \mathbb{R}^3 \mid x \neq 0\}\), where \((x, y, z)\) are the standard coordinates in \(\mathbb{R}^3\). Let \(\{e_1, e_2, e_3\}\) be linearly independent global frame on \(M\) given by

\begin{align*}
e_1 &= 2 \frac{\partial}{\partial y}, & e_1 &= 2 \frac{\partial}{\partial x} \frac{\partial}{\partial y} - \frac{4z}{x} \frac{\partial}{\partial y} + xy \frac{\partial}{\partial z}, & e_3 &= \frac{\partial}{\partial z}.
\end{align*}

Let \(g\) be the Riemannian metric defined by

\begin{align*}
g(e_1, e_3) &= g(e_2, e_3) = g(e_1, e_2) = 0, & g(e_1, e_1) &= g(e_2, e_2) = g(e_3, e_3) = 1.
\end{align*}
Let \( \eta \) be the 1-form defined by
\[
\eta(U) = g(U, e_3)
\]
for any \( U \in \chi(M) \). Let \( \phi \) be the \((1,1)\)-tensor field defined by
\[
\phi e_1 = e_2, \quad \phi e_2 = -e_1, \quad \phi e_3 = 0.
\]
Then using the linearity of \( \phi \) and \( g \) we have
\[
\eta(e_3) = 1,
\]
\[
\phi^2(U) = -U + \eta(U)e_3
\]
and
\[
g(\phi U, \phi W) = g(U, W) - \eta(U)\eta(W)
\]
for any \( U, W \in \chi(M) \). Moreover
\[
he_1 = -e_1, \quad he_2 = e_2, \quad \text{and} \quad he_3 = 0.
\]
Thus for \( e_3 = \xi \), \((\phi, \xi, \eta, g)\) defines a contact metric structure on \( M \).

Let \( \nabla \) be the Levi-Civita connection with respect to the Riemannian metric \( g \) and \( R \) be the curvature tensor of \( g \). Then we have
\[
\{ e_1, e_2 \} = 2e_3 + \frac{2}{x} e_1, \quad \{ e_1, e_3 \} = 0, \quad \{ e_2, e_3 \} = 2e_1.
\]
The Riemannian connection \( \nabla \) of the metric tensor \( g \) is given by
\[
2g(\nabla X Y, Z) = X g(Y, Z) + Y g(Z, X) - Z g(X, Y)
\]
\[
- g([X, Y], Z) - g(Y, [X, Z]) + g(Z, [X, Y]).
\]

Taking \( e_3 = \xi \) and using the above formula for the Riemannian metric \( g \), we can easily calculate that
\[
\nabla_{e_1} e_3 = 0, \quad \nabla_{e_2} e_3 = 2e_1, \quad \nabla_{e_3} e_3 = 0, \quad \nabla_{e_1} e_2 = \frac{2}{x} e_1,
\]
\[
\nabla_{e_1} e_1 = -2e_3, \quad \nabla_{e_2} e_2 = 0, \quad \nabla_{e_3} e_2 = 0, \quad \nabla_{e_2} e_1 = -\frac{2}{x} e_2.
\]

From the above it can be easily seen that \((\phi, \xi, \eta, g)\) is a \((k, \mu)\)-contact metric structure on \( M \). Consequently \( M^3(\phi, \xi, \eta, g) \) is a \((k, \mu)\)-contact metric manifold with \( k = -\frac{2}{x} \neq 0 \) and \( \mu = -\frac{2}{x} \neq 0 \).

Using the above relations, we can easily calculate the non-vanishing components of the curvature tensor as follows:
\[
R(e_2, e_3)e_2 = -\frac{4}{x} e_1, \quad R(e_2, e_3)e_1 = \frac{4}{x} e_2,
\]
and components which can be obtained from these by the symmetry properties.

We shall now show that such a \((k, \mu)\)-contact metric manifold is \( \phi \)-recurrent. Since \( \{ e_1, e_2, e_3 \} \) form a basis of \( M^3 \), any vector field \( X \in \chi(M) \) can be taken as
\[
X = a_1 e_1 + a_2 e_2 + a_3 e_3,
\]
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where $a_i$ are positive real numbers, $i = 1, 2, 3$. Thus the covariant derivatives of the curvature tensor are given by

$$(\nabla_X R)(e_2, e_3)e_1 = -\frac{8a_2}{x^2} e_2, \quad (\nabla_X R)(e_2, e_3)e_2 = \frac{8a_2}{x^2} e_1.$$  

This implies that

$$\phi^2((\nabla_X R)(e_2, e_3)e_1) = \frac{8a_2}{x^2} e_2, \quad \phi^2((\nabla_X R)(e_2, e_3)e_2) = -\frac{8a_2}{x^2} e_1.$$  

Let us consider the non-vanishing 1-form

$$A(X) = \frac{2a_2}{x}$$

at any point $p \in M^3$. Then we get

$$\phi^2((\nabla_X R)(e_2, e_3)e_1) = A(X) R(e_2, e_3)e_1,$$

and

$$\phi^2((\nabla_X R)(e_2, e_3)e_2) = A(X) R(e_2, e_3)e_2.$$  

This implies that the manifold under consideration is a locally $\varphi$-recurrent $(k, \mu)$-contact metric manifold which is neither locally symmetric nor locally $\varphi$-symmetric.

References


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