ON EINSTEIN HERMITIAN MANIFOLDS II

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Abstract. We show that on a Hermitian surface $M$, if $M$ is weakly $*$-Einstein and has $J$-invariant Ricci tensor then $M$ is Einstein, and vice versa. As a consequence, we obtain that a compact $*$-Einstein Hermitian surface with $J$-invariant Ricci tensor is Kähler. In contrast with the 4-dimensional case, we show that there exists a compact Einstein Hermitian $(4n+2)$-dimensional manifold which is not weakly $*$-Einstein.

1. Introduction

The Riemannian version of the Goldberg-Sachs Theorem [1] says that the self-dual Weyl tensor $W^+$ of an Einstein Hermitian surface $M$ is degenerate, i.e., at least two of its three eigenvalues coincide. In fact, this implies that $M$ is weakly $*$-Einstein. Another weak form of the Einstein condition is the $J$-invariant condition for the Ricci tensor. In this note, we show that, on a Hermitian surface, both weak versions of the Einstein condition together are equivalent to the Einstein condition. In contrast with the 4-dimensional case, we show that there exists a compact Einstein Hermitian $(4n+2)$-dimensional manifold which is not weakly $*$-Einstein. More precisely we obtain the followings:

Theorem 1. Let $M = (M, J, g)$ be a Hermitian surface. If $M$ is weakly $*$-Einstein and has $J$-invariant Ricci tensor, then $M$ is Einstein, and vice versa.

Corollary 1. A compact $*$-Einstein Hermitian surface with $J$-invariant Ricci tensor is Kähler.

Theorem 2. If $M = (S^{2n+1} \times S^{2n+1}, J_C, g_{prod})$, then $M$ is a compact Einstein Hermitian manifold which is not weakly $*$-Einstein, where $J_C$ is the complex structure of Calabi-Eckmann and $g_{prod}$ the standard product metric on the product of the spheres.

Received May 13, 2008.

2000 Mathematics Subject Classification. 53A30, 53B35, 53C25, 53C55, 53C56.

Key words and phrases. Hermitian surface, weakly $*$-Einstein, $J$-invariant Ricci tensor, Einstein, vice versa, $*$-Einstein, Kähler, compact Einstein Hermitian $(4n+2)$-dimensional manifold.

This study is supported by Kangwon National University.

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We recall that a Hermitian manifold \((M, J, g)\) is called Einstein Hermitian if the compatible Riemannian metric \(g\) with complex structure \(J\) is Einstein by its Levi-Civita connection.

2. Preliminaries

Let \(M = (M, J, g)\) be a Hermitian manifold with complex structure \(J\) and compatible Riemannian metric \(g\), i.e., \(g(JX, JY) = g(X, Y)\) for vector fields \(X, Y\). Denote by \(\Omega(X, Y)\) the Kähler form of \(M\) defined by \(\Omega(X, Y) = g(JX, Y)\). We shall always consider \(M\) with the orientation determined by the complex structure \(J\). The Riemannian curvature \(R\), the Ricci tensor \(\text{Ric}\) and the scalar curvature \(s\) of \(M\) are defined by

\[
R(X, Y)Z = [\nabla_X, \nabla_Y]Z - \nabla_{[X,Y]}Z, \\
\text{Ric}(X, Y) = \text{Trace}(Z \to R(Z, X)Y), s = \text{Trace}_g \text{Ric}
\]

for vector fields \(X, Y, Z\).

Furthermore, we define the \(\ast\)-Ricci tensor and \(\ast\)-scalar curvature of \((J, g)\) by

\[
\text{Ric}^\ast(X, Y) = \text{Trace}(Z \to -JR(Z, X)Y), s^\ast = \text{Trace}_g \text{Ric}^\ast.
\]

The \(\ast\)-Ricci tensor is in general neither symmetric nor skew-symmetric and satisfies the equation: \(\text{Ric}^\ast(JX, JY) = \text{Ric}^\ast(Y, X)\). Note that on a Kähler manifold, the \(\ast\)-Ricci tensor and the Ricci tensor coincide; this is a consequence of the Kähler identity \(R(X, Y)(JZ) = J(R(X, Y)Z)\), which itself follows from the fact that \(\nabla J = 0\). We shall say that \(M\) is weakly \(\ast\)-Einstein. Hence the Weyl tensor satisfies the equation: \(\text{Ric}^\ast(X, Y, Z, W) = \text{Ric}^\ast(Y, X, Z, W)\). Considering the Riemannian curvature tensor \(R\) as a \((0,4)\)-tensor as follows: \(R(X, Y, Z, W) = -g(R(X, Y)Z, W)\). Considering the Riemannian curvature tensor \(R\) as a \((0,4)\)-tensor, we have the following well known \(SO(2n)\)-decomposition:

\[
R = \frac{s}{4n(2n-1)} g \wedge g + \frac{1}{2n-2} \text{Ric}_0 \wedge g + W,
\]

where \(\text{Ric}_0 = \text{Ric} - \frac{s}{2n} g\) is the traceless Ricci tensor and \(W\) is the Weyl tensor. Here the symbol \(\wedge\) is the Nomizu-Kulkarni product of symmetric \((0,2)\)-tensors generating a curvature type tensor. Note that \(\text{Ric}_0 = 0\) if and only if \(M\) is Einstein. Let \(\{e_i\}_{i=1,...,2n}\) be a local orthonormal frame and \(R_{ijkl}, r_{ij}, R_{ij}, W_{ijkl}, r^*_{ij}, R^*_{ij}\) be components of \(R, \text{Ric}, \text{Ric}_0, W, \text{Ric}^\ast, \text{Ric}^\ast_0\) with respect to \(\{e_i\}\) respectively, i.e.,

\[
R_{ijkl} = R(e_i, e_j, e_k, e_l), r_{ij} = \text{Ric}(e_i, e_j), R_{ij} = \text{Ric}_0(e_i, e_j), \\
W_{ijkl} = W(e_i, e_j, e_k, e_l), r^*_{ij} = \text{Ric}^\ast(e_i, e_j), R^*_{ij} = \text{Ric}^\ast_0(e_i, e_j).
\]

Here \(\text{Ric}^\ast_0\) is the traceless \(\ast\)-Ricci tensor. Hence the Weyl tensor \(W = (W_{ijkl})\) of \(M\) can be expressed as

\[
W_{ijkl} = R_{ijkl} - \frac{1}{2n-2} \left( R_{ik} \delta_{jl} + R_{lj} \delta_{ik} - R_{il} \delta_{jk} - R_{jk} \delta_{il} \right) - \frac{1}{2n-2} \text{Ric}_0(e_k, e_l) \delta_{ij}.
\]
\[ \frac{s}{2n(2n-1)} (\delta_{ik} \delta_{jl} - \delta_{il} \delta_{jk}). \]

Here \( \delta_{ij} \) is the Kronecker delta. We denote \( J_{ij} \) by \( J_{ij} = g(Je_i, e_j) \).

From now on the components of tensors shall be considered under orthonormal frame. Indices with an overbar are the ones with respect to \( \{ Je_i \} \), for example, \( \bar{R}_{ijkl} = R(Je_i, e_j, e_k, e_l) \).

Using this notation, we have
\[
\begin{align*}
    r_{ij} &= \sum_a R_{ia} a_j, \\
    r^*_{ij} &= \sum_a R_{ia} \bar{a}_j, \\
    s &= \sum_{a,b} R_{ab} a_b, \\
    s^* &= \sum_{a,b} R_{ab} \bar{a}_b
\end{align*}
\]

and
\[
r^*_{ij} = r^*_{ji}.
\]

The Riemannian metric \( g \) induces a metric on the bundle \( \bigwedge^2 M \) of 2-vectors on \( M \) by \( \langle X_1 \wedge X_2, X_3 \wedge X_4 \rangle = \det(g(X_i, X_j)). \) Then we also consider the curvature tensor \( R \) as an endomorphism of the bundle \( \bigwedge^2 M \) as follows: \( (R(X \wedge Y), Z \wedge W) = -g(R(X,Y)Z,W). \) In dimension 4, the Hodge star operator defines an endomorphism \( * \) of \( \bigwedge^2 M \) with \( *^2 = Id. \) Hence \( \bigwedge^2 M = \bigwedge^+ M \bigoplus \bigwedge^- M \), where \( \bigwedge^+ M \) (resp. \( \bigwedge^- M \)) is the subbundle of \( \bigwedge^2 M \) corresponding to the eigenvalue +1 (resp. -1) of \( * \). As an endomorphism of \( \bigwedge^2 M \) the Weyl tensor \( W \) commutes with the Hodge star operator \( * \) and so \( W \) preserves the decomposition \( \bigwedge^2 M = \bigwedge^+ M \bigoplus \bigwedge^- M \). Note that as an endomorphism of \( \bigwedge^2 M \) the curvature tensor \( R \) commutes with the Hodge star operator \( * \), i.e., \( *R = R* \) if and only if \( M \) is Einstein [5]. We denote the restriction of \( W \) to \( \bigwedge^+ M \) (resp. \( \bigwedge^- M \)) by \( W^+ \) (resp. \( W^- \)) called the self-dual Weyl tensor (resp. anti-self-dual Weyl tensor). Hence, in dimension 4, the Riemannian curvature (0,4)-tensor \( R \) can be obtained as follows:
\[
R = \frac{s}{24} g \wedge g + \frac{1}{2} \text{Ric}_0 \wedge g + W^+ + W^-.
\]

Let \( M = (M, J, g) \) be a Hermitian surface (i.e., a Hermitian manifold of real dimension 4). From now on we identify 2-vectors with 2-forms. Then usual type decomposition
\[
\bigwedge^2 M \otimes C = \bigwedge^0 M \bigoplus \bigwedge^1 M \bigoplus \bigwedge^2 M
\]

of complexified 2-forms induces the decomposition
\[
\bigwedge^2 M = R \Omega \bigoplus (\bigwedge^1 M)_R \bigoplus (\bigwedge^0 M \bigoplus \bigwedge^2 M)_R,
\]
where $R\Omega$ is the line bundle generated by the Kähler form $\Omega$ and $\bigwedge^{1,1}_0 M$ is the orthogonal complement of $R\Omega$ in $\bigwedge^{1,1} M$. Note that

$$R\Omega \oplus (\bigwedge^2 M \oplus \bigwedge^0 M) = \bigwedge^+ M$$

and

$$\bigwedge^0 M = \bigwedge^- M.$$

On the other hand we can extend $J$ to act on 2-forms as follows: $J(A)(X, Y) = A(JX, JY)$ for a 2-form $A$.

3. Proof of Theorem 1 and Corollary 1

Let $M = (M, J, g)$ be a Hermitian surface. From now on we assume that all tensors are continued by complex linearity. For an orthonormal frame $\{e_1, Je_1, e_2, Je_2\}$ we set $Z_k = \frac{1}{\sqrt{2}}(e_k - iJe_k), Z_\bar{k} = \frac{1}{\sqrt{2}}(e_k + iJe_k), k = 1, 2$. Let $(\cdot, \cdot)$ be the Hermitian continuation of $g$ on $\bigwedge^2 M \otimes \mathbb{C}$ and $\alpha = (Z_1 \wedge Z_2), \beta = (Z_1 \wedge Z_1 + Z_2 \wedge Z_2), \gamma = (Z_1 \wedge Z_2), \delta = \frac{1}{\sqrt{2}}(Z_1 \wedge Z_1 - Z_2 \wedge Z_2)$, $\bar{\alpha} = (Z_1 \wedge Z_2)$. Then $\{\alpha, \beta, \bar{\alpha}\}$ and $\{\gamma, \delta, \bar{\gamma}\}$ are orthonormal frames of $\bigwedge^+ M \otimes \mathbb{C}$ and $\bigwedge^- M \otimes \mathbb{C}$ respectively. Consider $W^+$ as an endomorphism of $\bigwedge^+ M \otimes \mathbb{C}$. Then the matrix of $W^+$ with respect to the frame $\{\alpha, \beta, \bar{\alpha}\}$ has the following form [2]:

$$W^+ = \begin{pmatrix}
W^+_1 & W^+_2 & W^+_3 \\
W^+_2 & -2W^+_1 & -W^+_2 \\
W^+_3 & -W^+_2 & W^+_1
\end{pmatrix}.$$ 

Here $W^+_1 = (R\alpha, \alpha) - \frac{s}{2}, W^+_2 = (R\alpha, \beta), W^+_3 = (R\alpha, \bar{\alpha})$. Since $\nabla_X Y \in T^{1,0} M$ for all $X, Y \in T^{1,0} M$, we have $W^+_3 = 0$. Hence, on a Hermitian surface $M$, $W^+$ is degenerate if and only if $W^+_2 = 0$. In order to show that Theorem 1 and Corollary 1 hold, we need the following;

**Lemma 1.** Let $M = (M, J, g)$ be a Hermitian surface. Then we have

(2) $\text{Ric}(X, Y) + \text{Ric}(JX, JY) - \text{Ric}(X, JY) - \text{Ric}(JX, Y) = \frac{s - s^*}{2}g(X, Y)$.

**Proof.** Let $\{e_1\}_{i=1, \ldots, 4}$ be an orthonormal frame and $\{e^i\}_{i=1, \ldots, 4}$ its dual frame. Then we can write the Kähler form $\Omega$ as $\Omega = \frac{1}{2} \sum J_{ij}e^i \wedge e^j$ and we have

$$W(\Omega)_{kl} = \frac{1}{2} \sum_{i,j} W_{ijkl} J_{ij} = \frac{1}{2} \sum_i W_{ikl} = \sum_i W_{ilk}.$$
From (1), we have

\[ W(\Omega)_{kl} = -r^*_k - \frac{1}{2}(R_{kl} - R_{kl}) - \frac{s}{12} J_{kl} \]

\[ = -r^*_k - r^*_{[kl]} - \frac{1}{2}(R_{kl} + R_{kl}) - \frac{s}{12} J_{kl} \]

\[ = R^*_k + \frac{s}{4} J_k - r^*_{[kl]} - \frac{1}{2}(R_{kl} + R_{kl}) - \frac{s}{12} J_{kl}, \]

where \( A_{(ij)} \) and \( A_{[ij]} \) are the symmetric and skew-symmetric part of a tensor \( A_{ij} \), respectively. It is easy to see that only \( B_{kl} = R^*_k - \frac{1}{2}(R_{kl} + R_{kl}) \) is the component of a section of \( \Lambda^{1,1}_0 M \). Since the Weyl tensor of \( M \) preserves the self-duality and \( \Lambda^{1,1}_0 M \) is identified with \( \Lambda^- \), we get \( B_{kl} = 0 \). Therefore the identity (2) holds. This completes the proof of Lemma 1. \( \square \)

Suppose that a Hermitian surface \( M \) is weakly \( * \)-Einstein. Then the above identity (2) implies \( \text{Ric}(X, Y) + \text{Ric}(JX, JY) = \frac{1}{2} g(X, Y) \). By the other assumption, i.e., \( M \) has \( J \)-invariant Ricci tensor, we get \( \text{Ric}(X, Y) = \frac{1}{2} g(X, Y) \) which means that \( M \) is Einstein. Conversely, by definition \( M \) is weakly \( * \)-Einstein if and only if \( \text{Ric}^*(X, Y) = \lambda g(X, Y) \). This is equivalent to \( R(\Omega) = \lambda \Omega \). Using the frame \( \{ \alpha, \beta, \alpha, \gamma, \delta, \tilde{\gamma} \} \), we see that \( R(\Omega) = \lambda \Omega \) if and only if \( \langle R(\beta), \alpha \rangle = \langle R(\beta), \tilde{\alpha} \rangle = \langle R(\beta), \gamma \rangle = \langle R(\beta), \delta \rangle = \langle R(\beta), \tilde{\gamma} \rangle = 0 \). Our assumption that \( M \) is an Einstein Hermitian surface implies \( W^+ \) is degenerate. Hence we have \( \langle R(\beta), \alpha \rangle = \langle R(\beta), \tilde{\alpha} \rangle = 0 \). Furthermore in dimension 4 the Einstein condition is equivalent to \( \ast R = R \ast \) which implies that \( \langle R(\beta), \gamma \rangle = \langle R(\beta), \delta \rangle = \langle R(\beta), \tilde{\gamma} \rangle = 0 \). Hence \( M \) is weakly \( * \)-Einstein and obviously its Ricci tensor is \( J \)-invariant. This completes the proof of Theorem 1.

The given condition of Corollary 1 implies that \( M \) is also a compact Einstein Hermitian surface with constant \( s \) by Theorem 1. Hence by the well-known result in [4], we can conclude that \( M \) is Kähler.

\textbf{4. A compact Einstein Hermitian and non-weakly \( * \)-Einstein manifold of dimension \((4n + 2)\)}

The product of odd dimensional spheres \( S^{2n+1} \times S^{2m+1} \) can be provided with a complex structure [3], defined as follows: let \( N_1 \) and \( N_2 \) be the outward normals to the spheres \( S^{2n+1} \) and \( S^{2m+1} \) sitting inside \( \mathbb{C}^{n+1} \) and \( \mathbb{C}^{m+1} \), respectively, and let \( J_1 \) and \( J_2 \) be the standard complex structures on these spaces. Since \( J_1 N_1 \) and \( J_2 N_2 \) are globally defined vector fields on the respective spheres, we can decompose any vector field \( X \) on \( S^{2n+1} \times S^{2m+1} \) as

\[ X = X_1 + X_2 + d_1(X)J_1 N_1 + d_2(X)J_2 N_2, \]

where \( X_1 \) is tangent to \( S^{2n+1} \) and perpendicular to \( J_1 N_1 \), while \( X_2 \) is tangent to \( S^{2m+1} \) and perpendicular to \( J_2 N_2 \). The notion of perpendicularity is defined using the standard metrics on \( R^{2n+2} = \mathbb{C}^{n+1} \) and \( R^{2m+2} = \mathbb{C}^{m+1} \), respectively.
Using this decomposition, we may now define the (1,1)-tensor $J_C$ by

$$J_CX = J_1X_1 + J_2X_2 - d_2(X)J_1N_1 + d_1(X)J_2N_2.$$  

This $J_C$ on $S^{2n+1} \times S^{2m+1}$ is in fact a complex structure [3]. For each sphere factor in $S^{2n+1} \times S^{2m+1}$ we have the Hopf fibration onto a complex projective space and so their product produces a Riemannian submersion $S^{2n+1} \times S^{2m+1} \to CP^n \times CP^m$. We obtain a decomposition of the tangent space at each point into a horizontal and vertical component. For our purpose we consider the product of odd dimension sphere $S^{2n+1}$ with itself. Obviously the product metric $g_{\text{prod}}$ of the standard metric on each sphere factor in $S^{2n+1} \times S^{2n+1}$ is Einstein.

Furthermore, the product metric $g_{\text{prod}}$ on $S^{2n+1} \times S^{2n+1}$ is compatible with $J_C$. Under the complex structure $J_C$, the $\ast$-Ricci tensor of $g_{\text{prod}}$ is not a functional multiple of the metric $g_{\text{prod}}$. In fact we have $\text{Ric}^\ast (V, V) = 0$ and $\text{Ric}^\ast (Y, Y) \neq 0$, where $V$ is a vertical vector and $Y$ is a non-trivial horizontal vector. Summing up the above argument, we can conclude that $M = (S^{2n+1} \times S^{2n+1}, J_C, g_{\text{prod}})$ is a compact Einstein Hermitian manifold which is not weakly $\ast$-Einstein. This completes the proof of Theorem 2.

References


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