LIMIT RELATIVE CATEGORY THEORY APPLIED TO THE CRITICAL POINT THEORY

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ABSTRACT. Let $H$ be a Hilbert space which is the direct sum of five closed subspaces $X_0, X_1, X_2, X_3$ and $X_4$ with $X_1, X_2, X_3$ of finite dimension. Let $J$ be a $C^{1,1}$ functional defined on $H$ with $J(0) = 0$. We show the existence of at least four nontrivial critical points when the sublevels of $J$ (the torus with three holes and sphere) link and the functional $J$ satisfies sup-inf variational inequality on the linking subspaces, and the functional $J$ satisfies $(P.S.)^*_c$ condition and $f|_{X_0 \oplus X_4}$ has no critical point with level $c$. For the proof of main theorem we use the nonsmooth version of the classical deformation lemma and the limit relative category theory.

1. Introduction and statement of main result

Let $H$ be a Hilbert space which is a direct sum of five closed subspaces $X_0, X_1, X_2, X_3$ and $X_4$ with $X_1, X_2, X_3$ of finite dimension. Let $J$ be a $C^{1,1}$ functional defined on $H$ with $J(0) = 0$. In this paper we investigate the number of nontrivial critical points of the $C^{1,1}$ functional $J$ under some conditions on the sublevels of $J$ and the shape of $J$. We show the existence of at least four nontrivial critical points when the sublevels of $J$ (the torus with three holes and sphere) link and the functional $J$ satisfies the sup-inf variational inequality on the linking subspaces, and the functional $J$ satisfies $(P.S.)^*_c$ condition and $J|_{X_0 \oplus X_4}$ has no critical point with level $c$. Micheletti and Saccon prove in [13] that the functional $J$ has at least two nontrivial critical points under the same conditions on $J$ except the condition that the sublevel sets are the torus with one hole and sphere. In this paper we improve this result to the case that the sublevel sets are the torus with three holes and sphere.

Now, we state the main result:

Theorem 1.1. Let $H$ be a Hilbert space and $H = X_0 \oplus X_1 \oplus X_2 \oplus X_3 \oplus X_4$, where $X_0, X_1, X_2, X_3$ and $X_4$ are five closed subspaces of $H$ with $X_1, X_2, X_3$
of finite dimension. Let $J : H \to R$ be a $C^{1,1}$ functional with $J(0) = 0$. Let $0 < \rho_1, \rho_2, \rho_3, r < R, R_1 > 0$; we define
\[
S_i(\rho_i) = \{ z \in X_i \mid \|z\| = \rho_i \}, \quad i = 1, 2, 3,
\]
\[
S_i(\rho_i) - w_i = \{ z - w_i \mid z \in S_i(\rho_i), w_i \in X_i \}, \quad i = 1, 2, 3,
\]
\[
\Delta^3_R(S_i(\rho_i) - w_1, S_2(\rho_2) - w_2, S_3(\rho_3) - w_3, X_4)
\]
\[
= \{ z = (z_1 - w_1) + (z_2 - w_2) + (z_3 - w_3) + z_4 \mid z_i \in X_i, \quad i = 1, 2, 3, 4, \quad
\rho_1 \leq \|z_1 - w_1\| \leq R, \quad \rho_2 \leq \|z_2 - w_2\| \leq R,
\rho_3 \leq \|z_3 - w_3\| \leq R, \quad \|z_4\| \leq R_1, \quad \|z\| = R \}
\]
\[
\Sigma^3_R(S_1(\rho_1) - w_1, S_2(\rho_2) - w_2, S_3(\rho_3) - w_3, X_4)
\]
\[
= \{ z = (z_1 - w_1) + (z_2 - w_2) + (z_3 - w_3) + z_4 \mid z_i \in X_i, \quad i = 1, 2, 3, 4, \quad
\|z_4\| \leq R_1, \quad \|z_1 - w_1\| = \rho_1, \quad \|z_2 - w_2\| = \rho_2, \quad \|z_3 - w_3\| = \rho_3, \quad \|z\| = R \}
\]
Assume that
\[
\alpha = \inf_{S_r(X_0 \oplus X_1 \oplus X_2 \oplus X_3)} J(z), \quad \beta = \sup_{\Delta^3_R(S_1(\rho_1) - w_1, S_2(\rho_2) - w_2, S_3(\rho_3) - w_3, X_4)} J(z).
\]
Assume that the $(P.S.)_c^c$ condition holds for $J$ for $c \in [\alpha, \beta]$. Assume that $J|_{X_0 \oplus X_1}$ has no critical points with $\alpha \leq J(z) \leq \beta$. Moreover we assume $\beta < +\infty$. Then there exist at least four nontrivial critical points $z_1, z_2, z_3$ and $z_4$ for $J$ in $X_1 \oplus X_2 \oplus X_3$ such that
\[
\inf_{z \in S_r(X_0 \oplus X_1 \oplus X_2 \oplus X_3)} J(z) \leq J(z_i) \leq \sup_{z \in \Delta^3_R(S_1(\rho_1) - w_1, S_2(\rho_2) - w_2, S_3(\rho_3) - w_3, X_4)} J(z),
\]
$i = 1, 2, 3, 4$.

In Section 2, we introduce the notion of the limit relative category and the $(P.S.)_c^c$ condition and recall the suitable version of the deformation lemma and the multiplicity theorem in [13]. In Section 3, by the nonsmooth version of the classical suit deformation lemma, the limit relative category theory and the multiplicity Theorem 2.1 prove the main theorem.
2. Recall of the critical point theory on the manifold

Now, we consider the critical point theory on the manifold with boundary induced from the limit relative category. Let \( H \) be a Hilbert space and \( M \) be the closure of an open subset of \( H \) such that \( M \) can be endowed with the structure of \( C^2 \) manifold with boundary. Let \( J : W \to R \) be a \( C^{1,1} \) functional, where \( W \) is an open set containing \( M \). For applying the usual topological methods of critical points theory we need a suitable notion of critical point for \( J \) on \( M \). We recall the following notions: lower gradient of \( J \) on \( M \), \( (P.S.)_c^* \) condition and the limit relative category (see [7]).

**Definition 2.1.** If \( u \in M \), the lower gradient of \( J \) on \( M \) at \( u \) is defined by

\[
\text{grad}_M^- J(u) = \begin{cases} 
\nabla J(u) & \text{if } u \in \text{int}(M), \\
\nabla J(u) + [\nabla J(u), \nu(u)] \nu(u) & \text{if } u \in \partial M,
\end{cases}
\]

where we denote by \( \nu(u) \) the unit normal vector to \( \partial M \) at the point \( u \), pointing outwards. We say that \( u \) is a lower critical point for \( J \) on \( M \), if \( \text{grad}_M^- J(u) = 0 \).

Since the functional \( J(u) \) is strongly indefinite, the notion of the \( (P.S.)_c^* \) condition and the limit relative category is a very useful tool for the proof of the main theorems.

Let \( (H_n)_n \) be a sequence of closed subspaces of \( H \) such that, for any \( n \), \( M_n = M \cap H_n \) is the closure of an open subset of \( H_n \) and has the structure of a \( C^2 \) manifold with boundary in \( H_n \). We assume that for any \( n \) there exists a retraction \( r_n : M \to M_n \). For given \( B \subset H \), we will write \( B_n = B \cap H_n \).

**Definition 2.2.** Let \( c \in R \). We say that \( J \) satisfies the \( (P.S.)_c^* \) condition with respect to \( (M_n)_n \), on the manifold with boundary \( M \), if for any sequence \( (k_n)_n \) in \( N \) and any sequence \( (u_n)_n \) in \( M \) such that \( k_n \to \infty \), \( u_n \in M_{k_n} \), \( \forall n \) \( J(u_n) \to c \), \( \text{grad}_{M_{k_n}}^- J(u_n) \to 0 \), there exists a subsequence of \( (u_n)_n \) which converges to a point \( u \in M \) such that \( \text{grad}_{M}^- J(u) = 0 \).

Let \( Y \) be a closed subspace of \( M \).

**Definition 2.3.** Let \( B \) be a closed subset of \( M \) with \( Y \subset B \). We define the relative category \( \text{cat}_{M,Y}(B) \) of \( B \) in \( (M,Y) \), as the least integer \( h \) such that there exist \( h + 1 \) closed subsets \( U_0, U_1, \ldots, U_h \) with the following properties:

- \( B \subset U_0 \cup U_1 \cup \cdots \cup U_h \);
- \( U_1, \ldots, U_h \) are contractible in \( M \);
- \( Y \subset U_0 \) and there exists a continuous map \( F : U_0 \times [0,1] \to M \) such that

\[
\begin{align*}
F(x,0) &= x \quad \forall x \in U_0, \\
F(x,t) &\in Y \quad \forall x \in Y, \forall t \in [0,1], \\
F(x,1) &\in Y \quad \forall x \in U_0.
\end{align*}
\]

If such an \( h \) does not exist, we say that \( \text{cat}_{M,Y}(B) = +\infty \).
Definition 2.4. Let \((X, Y)\) be a topological pair and \((X_n)_n\) be a sequence of subsets of \(X\). For any subset \(B\) of \(X\) we define the limit relative category of \(B\) in \((X, Y)\), with respect to \((X_n)_n\), by
\[
\text{cat}_{(X,Y)}^*(B) = \limsup_{n \to \infty} \text{cat}_{(X_n,Y_n)}(B_n).
\]

Let \(Y\) be a fixed subset of \(M\). We set
\[
B_i = \{B \subset M | \text{cat}_{(M,Y)}^*(B) \geq i\},
\]
\[
c_i = \inf_{B \in B_i} \sup_{x \in B} J(x).
\]

We recall the “nonsmooth” version of the classical Deformation Lemma in [5].

Lemma 2.1. (Deformation Lemma) Let \(h : H \to R \cup \{+\infty\}\) be a lower semi-continuous function and assume \(h\) to be \(\varphi\)-convex of order 2. Let \(c \in R, \delta > 0\) and \(D\) be a closed set in \(H\) such that
\[
\inf\{\|\text{grad}_x h(x)\| | c - \delta \leq h(x) \leq c + \delta, \quad \text{dist}(x, D) < \delta\} > 0.
\]
Then there exist \(\epsilon > 0\) and a continuous deformation \(\eta : h^c \cap D \times [0,1] \to h^{c+\epsilon} \cap D\) such that
(i) \(\eta(x,0) = x\) \(\forall x \in h^{c+\epsilon} \cap D\),
(ii) \(\eta(x,t) = x\) \(\forall x \in h^{c+\epsilon} \cap D, \forall t \in [0,1]\),
(iii) \(\eta(x,1) \in h^{c-\epsilon}\) \(\forall x \in h^{c+\epsilon} \cap D, \forall t \in [0,1]\).

We have the following multiplicity theorem, which was proved in [13].

Theorem 2.1. Let \(i \in \mathcal{N}\) and assume that
(1) \(c_i < +\infty\),
(2) \(\sup_{x \in Y} J(x) < c_i\),
(3) the \((P.S)_i^c\) condition with respect to \((M_n)_n\) holds.
Then there exists a lower critical point \(x\) such that \(J(x) = c_i\). If
\[
c_i = c_{i+1} = \cdots = c_{i+k-1} = c,
\]
then
\[
\text{cat}_M^*(\{x \in M | J(x) = c, \quad \text{grad}_x^M J(x) = 0\}) \geq k.
\]

3. Proof of Theorem 1.1

Now we will show that \(J\) has at least four nontrivial critical points in the subspace \(X_1 \oplus X_2 \oplus X_3\) of \(H\).
Let \(P_{X_1 \oplus X_2 \oplus X_3}\) be the orthogonal projection from \(H\) onto \(X_1 \oplus X_2 \oplus X_3\) and
\[
C = \{z \in H | \|P_{X_1 \oplus X_2 \oplus X_3} z\| \geq 1\}.
\]
Then $C$ is the smooth manifold with boundary. Let $C_n = C \cap H_n$. Let us define a functional $\Psi : H \setminus \{X_0 \oplus X_4\} \to H$ by
\begin{equation}
\Psi(z) = z - \frac{P_{X_1 \oplus X_2 \oplus X_3}z}{\|P_{X_1 \oplus X_2 \oplus X_3}z\|} = P_{X_0 \oplus X_4}z + \left(1 - \frac{1}{\|P_{X_1 \oplus X_2 \oplus X_3}z\|}\right)P_{X_1 \oplus X_2 \oplus X_3}z.
\end{equation}

We have
\begin{equation}
\nabla \Psi(z)(w) = w - \frac{1}{\|P_{X_1 \oplus X_2 \oplus X_3}z\|}
\left(P_{X_1 \oplus X_2 \oplus X_3}w - \left(\frac{P_{X_1 \oplus X_2 \oplus X_3}z}{\|P_{X_1 \oplus X_2 \oplus X_3}z\|}\right)\|P_{X_1 \oplus X_2 \oplus X_3}z\|\right).
\end{equation}

Let us define the functional $\tilde{J} : C \to R$ by
\begin{equation}
\tilde{J} = J \circ \Psi.
\end{equation}

Then $\tilde{J} \in C_{1,1}^{1,1}$. We note that if $\tilde{z}$ is the critical point of $\tilde{J}$ and lies in the interior of $C$, then $z = \Psi(\tilde{z})$ is the critical point of $J$. So it suffices to find the nontrivial critical points of $\tilde{J}$. We note that
\begin{equation}
\|\text{grad}_{\tilde{z}} \tilde{J}(\tilde{z})\| \geq \|P_{X_0 \oplus X_4} \nabla J(\Psi(\tilde{z}))\| \quad \forall \tilde{z} \in \partial C.
\end{equation}

Let us set
\begin{align*}
\tilde{S}_r &= \Psi^{-1}(S_r(X_0 \oplus X_1 \oplus X_2 \oplus X_3)), \\
\tilde{B}_r &= \Psi^{-1}(B_r(X_0 \oplus X_1 \oplus X_2 \oplus X_3)), \\
\tilde{\Sigma}_r^3 &= \Psi^{-1}(\Sigma_r^3(S_1(\rho_1) - w_1, S_2(\rho_2) - w_2, S_3(\rho_3) - w_3, X_4)), \\
\tilde{\Delta}_r^3 &= \Psi^{-1}(\Delta_r^3(S_1(\rho_1) - w_1, S_2(\rho_2) - w_2, S_3(\rho_3) - w_3, X_4)).
\end{align*}

We note that $\tilde{S}_r$, $\tilde{B}_r$, $\tilde{\Sigma}_r^3$ and $\tilde{\Delta}_r^3$ have the same topological structure as $S_r$, $B_r$, $\Sigma_r^3$ and $\Delta_r^3$, respectively. By the condition of Theorem 1.1, there exist $0 < \rho_1, \rho_2, \rho_3, r < R$ and $R_1 > 0$ such that
\begin{align*}
\sup_{\tilde{z} \in \tilde{\Sigma}_r^3} \tilde{J}(\tilde{z}) &= \sup_{z \in \Sigma_r^3(S_1(\rho_1) - w_1, S_2(\rho_2) - w_2, S_3(\rho_3) - w_3, X_4)} J(z) \\
&< \inf_{z \in S_r(X_0 \oplus X_1 \oplus X_2 \oplus X_3)} J(z) = \inf_{\tilde{z} \in \tilde{S}_r} \tilde{J}(\tilde{z}), \\
\sup_{\tilde{z} \in \tilde{\Delta}_r^3} \tilde{J}(\tilde{z}) &= \sup_{z \in \Delta_r^3(S_1(\rho_1) - w_1, S_2(\rho_2) - w_2, S_3(\rho_3) - w_3, X_4)} J(z) < \infty
\end{align*}

and
\begin{align*}
\inf_{\tilde{z} \in \tilde{B}_r} \tilde{J}(\tilde{z}) &= \inf_{z \in B_r(X_0 \oplus X_1 \oplus X_2 \oplus X_3)} J(z) > -\infty.
\end{align*}

By the condition of Theorem 1.1, $\tilde{J}$ satisfies the $(P.S.)_c$ condition with respect to $(C_n)_n$ for every real number $\tilde{c}$ such that
\begin{equation}
\inf_{\tilde{z} \in \tilde{S}_r} \tilde{J}(\tilde{z}) \leq \tilde{c} \leq \sup_{\tilde{z} \in \tilde{\Delta}_r^3} \tilde{J}(\tilde{z}).
\end{equation}
Thus we have

$$\Sigma^3_n = \Sigma^3_R(S_1(\rho_1) - w_1, S_2(\rho_2) - w_2, S_3(\rho_3) - w_3, X_4) \cap H_n,$$

$$\Delta^3_n = \Delta^3_R(S_1(\rho_1) - w_1, S_2(\rho_2) - w_2, S_3(\rho_3) - w_3, X_4) \cap H_n,$$

$$\tilde{\Sigma}^3_n = \tilde{\Sigma}^3_R \cap H_n,$$

$$\tilde{\Delta}^3_n = \tilde{\Delta}^3_R \cap H_n.$$  

We claim that

$$\text{cat}_{(C_n, \tilde{\Sigma}^3_n)}(\Delta^3_n) = 4.$$  

In fact, we consider a continuous deformation \( r : \tilde{S}_r \setminus X_0 \times [0, 1] \to \tilde{S}_r \setminus X_0 \) such that

- \( r(x, 0) = x, \quad \forall x \in \tilde{S}_r \setminus X_0, \)
- \( r(x, t) = x, \quad \forall x \in \tilde{S}_r \cap (X_1 \oplus X_2 \oplus X_3), \forall t \in [0, 1], \)
- \( r(x, 1) \in \tilde{S}_r \cap (X_1 \oplus X_2 \oplus X_3), \quad \forall x \in \tilde{S}_r \setminus X_0. \)

Now we can define, if \( x = x_0 + x_{123} + x_4 \in X_0 \oplus (X_1 \oplus X_2 \oplus X_3) \oplus X_4, \ t \in [0, 1], \)

$$r_1(x, t) = x_0 + \|x_{123} + x_4\| r\left(\frac{x_{123} + x_4}{\|x_{123} + x_4\|}, t\right).$$

Using \( r_1, \) it is easy to construct, for all \( n, \) a continuous deformation \( \eta_n : C_n \times [0, 1] \to C_n \) such that

- \( \eta_n(x, 0) = x, \quad \forall x \in C_n, \)
- \( \eta_n(x, t) = x, \quad \forall x \in \Delta^3_n, \forall t \in [0, 1], \)
- \( \eta_n(x, 1) \in \Delta^3_n, \quad \forall x \in C_n, \)
- \( \eta_n(x, t) \in C_n \setminus \tilde{S}_r, \quad \forall x \in C_n \setminus \tilde{S}_r, \quad \forall t \in [0, 1]. \)

The existence of \( \eta_n \) implies that

$$\text{cat}_{(C_n, \tilde{\Sigma}^3_n)}(\Delta^3_n) = \text{cat}_{(\Delta^3_n, \tilde{\Sigma}^3_n)}(\Delta^3_n).$$

We note that the pair \((\Delta^3_n, \tilde{\Sigma}^3_n)\) is homeomorphic to the pair \((\Delta^3_n, \Sigma^3_n)\) and the pair \((\Delta^3_n, \Sigma^3_n)\) is homeomorphic to the pair \((B^{p+1} \times \{(S^{q_1-1} - w_1) \cup (S^{q_2-1} - w_2) \cup (S^{q_3-1} - w_3)\}, S^p \times \{(S^{q_1-1} - w_1) \cup (S^{q_2-1} - w_2) \cup (S^{q_3-1} - w_3)\})\), where \( p = \dim X_0 \cap H_n, \ q_1 = \dim X_1 \cap H_n, \ q_2 = \dim X_2 \cap H_n, \ q_3 = \dim X_3 \cap H_n \) and \( B^r, S^r \) denote the \( r \)-dimensional ball, the \( r \)-dimensional sphere, respectively. Thus the pair \((\Delta^3_n, \Sigma^3_n)\) is homeomorphic to the pair \((B^{p+1} \times \{(S^{q_1-1} - w_1) \cup (S^{q_3-1} - w_2) \cup (S^{q_2-1} - w_3)\}, S^p \times \{(S^{q_1-1} - w_1) \cup (S^{q_2-1} - w_2) \cup (S^{q_3-1} - w_3)\})\).

This fact and the facts that \( q_i = 1, \ i = 1, 2, 3 \) and (b) of (3.7) in [7] imply that

$$\text{cat}_{(C_n, \tilde{\Sigma}^3_n)}(\Delta^3_n) = 4.$$  

Thus we have

$$\text{cat}_{(C, \tilde{\Sigma}^3_R)}(\Delta^3_R) = 4.$$
Let us set
\[(3.11)\]
\[A_1 = \{ A \subset C | \text{cat}^*_{(C,S^3_R)}(A) \geq 1 \}, \quad A_2 = \{ A \subset C | \text{cat}^*_{(C,S^3_R)}(A) \geq 2 \}, \quad A_3 = \{ A \subset C | \text{cat}^*_{(C,S^3_R)}(A) \geq 3 \}, \quad A_4 = \{ A \subset C | \text{cat}^*_{(C,S^3_R)}(A) \geq 4 \}.
\]
Since \(\text{cat}^*_{(C,S^3_R)}(\Delta^3_R) = 4, \Delta^3_R \in A_i, i = 1, 2, 3, 4\). Let us set
\[\bar{c}_1 = \inf_{\tilde{\pi} \in A_1} \sup_{\tilde{z} \in A} \tilde{J}(\tilde{z}), \quad \bar{c}_2 = \inf_{\tilde{\pi} \in A_2} \sup_{\tilde{z} \in A} \tilde{J}(\tilde{z}),
\]
\[\bar{c}_3 = \inf_{\tilde{\pi} \in A_3} \sup_{\tilde{z} \in A} \tilde{J}(\tilde{z}), \quad \bar{c}_4 = \inf_{\tilde{\pi} \in A_4} \sup_{\tilde{z} \in A} \tilde{J}(\tilde{z}).
\]
We claim that \(\bar{c}_i < \infty, i = 1, 2, 3, 4\). In fact, from the facts that
\[\sup_{\tilde{z} \in \Delta^3_R(S_i, S_j, s, t, r, \rho)} J(\tilde{z}) < \infty
\]
and \(\Delta^3_R \in A_i, i = 1, 2, 3, 4\), we have that
\[\bar{c}_i = \inf_{\tilde{\pi} \in A_i} \sup_{\tilde{z} \in A} \tilde{J}(\tilde{z}) \leq \sup_{\tilde{z} \in \Delta^3_R} \tilde{J}(\tilde{z}) \leq \sup_{\tilde{z} \in \Delta^3_R} J(\tilde{z}) < \infty.
\]
We also claim that \(\sup_{\tilde{z} \in \Sigma^3_R} \tilde{J}(\tilde{z}) \leq \bar{c}_i, i = 1, 2, 3, 4\). In fact, for any \(A \in A_i\) with \(\Sigma^3_R \subset A, i = 1, 2, 3, 4\),
\[\sup_{\tilde{z} \in \Sigma^3_R} \tilde{J}(\tilde{z}) \leq \sup_{\tilde{z} \in A} \tilde{J}(\tilde{z}),
\]
and hence
\[\sup_{\tilde{z} \in \Sigma^3_R} \tilde{J}(\tilde{z}) \leq \inf_{\tilde{\pi} \in A_i} \sup_{\tilde{z} \in A} \tilde{J}(\tilde{z}) = \bar{c}_i, i = 1, 2, 3, 4.
\]
By the condition of Theorem 1.1, \(\tilde{J}\) satisfies the \((P.S.)^*_R\) condition with respect to \((C_n)\) for any real number \(\tilde{c}\) with \(\inf_{\tilde{z} \in \tilde{\pi}} \tilde{J}(\tilde{z}) \leq \tilde{c} \leq \sup_{\tilde{z} \in \Delta^3_R} \tilde{J}(\tilde{z})\). Thus, by Theorem 2.1, there exist four nontrivial critical points \(\tilde{z}_1, \tilde{z}_2, \tilde{z}_3, \tilde{z}_4\) of the functional \(\tilde{J}\) such that
\[\bar{c}_1 = \tilde{J}(\tilde{z}_1), \quad \bar{c}_2 = \tilde{J}(\tilde{z}_2), \quad \bar{c}_3 = \tilde{J}(\tilde{z}_3), \quad \bar{c}_4 = \tilde{J}(\tilde{z}_4).
\]
We claim that
\[\inf_{\tilde{z} \in S_i} \tilde{J}(\tilde{z}) \leq \bar{c}_1 \leq \bar{c}_2 \leq \bar{c}_3 \leq \bar{c}_4 \leq \sup_{\tilde{z} \in \Delta^3_R} \tilde{J}(\tilde{z}).
\]
Since \(\text{cat}^*_{(C,S^3_R)}(\Delta^3_R) = 4, \Delta^3_R \in A_4\) and hence
\[\bar{c}_4 = \inf_{\tilde{\pi} \in A_4} \sup_{\tilde{z} \in A} \tilde{J}(\tilde{z}) \leq \sup_{\tilde{z} \in \Delta^3_R} \tilde{J}(\tilde{z}), \quad \forall A \in A_4.
\]
For the proof of $c_1 \geq \inf_{z \in S_r} J(\tilde{z})$, we construct a deformation $\eta'_n : C_n \setminus \tilde{S}_r \times [0, 1] \to C_n \setminus \tilde{S}_r$, for all $n$, such that

- $\eta'_n(x, 0) = x$, $\forall x \in C_n \setminus \tilde{S}_r$,
- $\eta'_n(x, t) = x$, $\forall x \in \Sigma^3_n$, $\forall t \in [0, 1]$,
- $\eta'_n(x, 1) \in \Sigma^3_n$, $\forall x \in C_n$.

Actually $\eta'_n$ can be defined by taking the retraction of $\eta_n$ on $C_n \setminus \tilde{S}_r$ followed by a retraction of $\Delta^3_n \setminus \tilde{S}_r$ to $\Sigma^3_n$. The existence of $\eta'_n$, for all $n$, implies that any $A \in A_1$ must intersect $\tilde{S}_r$. So sup $J(A) \geq \inf_{z \in S_r} J(\tilde{z}) \forall A \in A_1$. So we have $c_1 = \inf_{A \in A_1} \sup_{z \in A} J(\tilde{z}) \geq \inf_{z \in S_r} J(\tilde{z})$. Therefore there exist at least four nontrivial critical points $\tilde{z}_1, \tilde{z}_2, \tilde{z}_3, \tilde{z}_4$ for the functional $J$ such that

$$\inf_{z \in S_r} J(\tilde{z}) \leq J(\tilde{z}_1) \leq J(\tilde{z}_2) \leq J(\tilde{z}_3) \leq J(\tilde{z}_4) \leq \sup_{z \in \Delta^3_n} J(\tilde{z}).$$

Setting $z_i = \Psi(\tilde{z}_i)$, $i = 1, 2, 3, 4$, we have

$$\inf_{z \in S_r} J(z) = \inf_{z \in S_r} J(\tilde{z}) \leq J(z_1) \leq J(z_2) \leq J(z_3) \leq J(z_4) \leq \sup_{z \in \Delta^3_n} J(z).$$

(3.18)

We claim that $\tilde{z}_i \notin \partial C$, that is $z_i \notin X_0 \oplus X_4$, which implies that $z_i$ are the critical points for $J$ in $X_1 \oplus X_2 \oplus X_3$. For this we assume by contradiction that $z_i \in X_0 \oplus X_4$. From (3.5), $P_{X_0 \oplus X_4} \nabla J(z_i) = 0$, namely, $z_i$, $i = 1, 2, 3, 4$, are the critical points for $J|_{X_0 \oplus X_4}$. By the condition of Theorem 1.1, the critical points $z_i$ in $X_0 \oplus X_4$ has no critical values in

$$\left\{ \inf_{z \in S_r \setminus (X_0 \oplus X_1 \oplus X_2 \oplus X_3)} J(z), \sup_{z \in \Delta^3_n (S_1(r_1) - w_1, S_2(r_2) - w_2, S_3(r_3) - w_3, X_4)} J(z) \right\},$$

which contradicts to (3.18). Thus $z_i \notin X_0 \oplus X_4$. This proves Theorem 1.1.

References


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