JORDAN θ-DERIVATIONS ON LIE TRIPLE SYSTEMS

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Abstract. In this paper we prove that every Jordan θ-derivation on a Lie triple system is a θ-derivation. Specially, we conclude that every Jordan derivation on a Lie triple system is a derivation.

1. Introduction

The concept of Lie triple system was first introduced by N. Jacobson [2, 3] (see also [4]). We recall that a Lie triple system is a vector space J over a field K with a trilinear mapping $J \times J \times J \ni (x, y, z) \mapsto [x, y, z] \in J$ satisfying the following axioms

(i) $[x, y, z] = -[y, x, z]$,  
(ii) $[x, y, z] + [y, z, x] + [z, x, y] = 0$,  
(iii) $[u, v, [x, y, z]] = [[u, v, x], y, z] + [x, [u, v, y], z] + [x, y, [u, v, z]]$

for all $u, v, x, y, z \in J$. It follows from (i) that $[x, x, y] = 0$ for all $x, y \in J$.

It is clear that every Lie algebra with product $[., .]$ is a Lie triple system with respect to $[x, y, z] := [[x, y], z]$. Conversely, any Lie triple system $J$ can be considered as a subspace of a Lie algebra (Bertram [1], Jacobson [3]).

Throughout this paper, let $\mathbb{C}$ be the complex filed and $J$ be a Lie triple system over $\mathbb{C}$.

Definition 1.1. Let $\theta : J \rightarrow J$ be a $\mathbb{C}$-linear mapping. A $\mathbb{C}$-linear mapping $D : J \rightarrow J$ is called a θ-derivation on $J$ if

$D([x, y, z]) = [D(x), \theta(y), \theta(z)] + [\theta(x), D(y), \theta(z)] + [\theta(x), \theta(y), D(z)]$

for all $x, y, z \in J$. If $\theta = 1_J$, a θ-derivation is called a derivation.

Let $u, v \in J$ and $D_{u,v} : J \rightarrow J$ be a mapping defined by

$D_{u,v}(x) := [u, v, x]$

for all $x \in J$. It is clear that $D_{u,v}$ is $\mathbb{C}$-linear and we get from (iii) that the mapping $D_{u,v}$ is a derivation on $J$. 

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**Definition 1.2.** Let $\theta : \mathcal{J} \to \mathcal{J}$ be a $\mathbb{C}$-linear mapping. A $\mathbb{C}$-linear mapping $D : \mathcal{J} \to \mathcal{J}$ is called a Jordan $\theta$-derivation on $\mathcal{J}$ if

$$D([x, y, z]) = [D(x), \theta(y), \theta(z)] + [\theta(x), D(y), \theta(z)] + [\theta(x), \theta(y), D(z)]$$

for all $x, y \in \mathcal{J}$. If $\theta = I_{\mathcal{J}}$, a Jordan $\theta$-derivation is called a Jordan derivation.

In [5], M. Sal Moslehian and Th. M. Rassias have studied the stability of derivations in normed Lie triple systems associated with a Cauchy–Jensen additive mapping.

**2. Main results**

It is clear that every $\theta$-derivation on a Lie triple system $\mathcal{J}$ is a Jordan $\theta$-derivation. In this section we prove that every Jordan $\theta$-derivation on a Lie triple system $\mathcal{J}$ is a $\theta$-derivation. So we conclude that every Jordan derivation on $\mathcal{J}$ is a derivation.

Throughout this section $D, \theta : \mathcal{J} \to \mathcal{J}$ are $\mathbb{C}$-linear mappings and $A_{D, \theta} : \mathcal{J} \times \mathcal{J} \times \mathcal{J} \to \mathcal{J}$ is a mapping defined by

$$A_{D, \theta}(x, y, z) := [D(x), \theta(y), \theta(z)] + [\theta(x), D(y), \theta(z)] + [\theta(x), \theta(y), D(z)]$$

for all $x, y, z \in \mathcal{J}$. It is clear that the mapping $A_{D, \theta}$ is trilinear and $A_{D, \theta}(x, x, y) = 0$ for all $x, y \in \mathcal{J}$.

**Theorem 2.1.** Let $D : \mathcal{J} \to \mathcal{J}$ be a Jordan $\theta$-derivation. Then $D$ is a $\theta$-derivation.

**Proof.** Since $D : \mathcal{J} \to \mathcal{J}$ is a Jordan $\theta$-derivation, $A_{D, \theta}(x, y, x) = D([x, y, x])$ for all $x, y \in \mathcal{J}$. Therefore we have

$$D([x + z, y, x + z]) = [D(x) + D(z), \theta(y), \theta(x) + \theta(z)]$$

$$+ [\theta(x) + \theta(z), D(y), \theta(x) + \theta(z)]$$

$$+ [\theta(x) + \theta(z), \theta(y), D(x) + D(z)]$$

$$= D([x, y, x]) + D([z, y, z])$$

$$+ A_{D, \theta}(x, y, z) + A_{D, \theta}(z, y, x)$$

for all $x, y, z \in \mathcal{J}$. On the other hand, we have

$$[x + z, y, x + z] = [x, y, x] + [z, y, z] + [x, y, z] + [z, y, x]$$

for all $x, y, z \in \mathcal{J}$. Therefore

$$D([x + z, y, x + z]) = D([x, y, x]) + D([z, y, z])$$

$$+ D([x, y, z]) + D([z, y, x])$$

for all $x, y, z \in \mathcal{J}$. It follows from (2.1) and (2.2) that

$$D([x, y, z]) + D([z, y, x]) = A_{D, \theta}(x, y, z) + A_{D, \theta}(z, y, x)$$

for all $x, y, z \in \mathcal{J}$. Since $[z, y, x] = [x, y, z] - [x, z, y]$, we get that

$$D([x, y, z]) + D([z, y, x]) = 2D([x, y, z]) - D([x, z, y])$$
for all $x, y, z \in \mathcal{J}$. Also

$$A_{D,\theta}(x, y, z) - A_{D,\theta}(x, z, y)$$

$$= [D(x), \theta(y), \theta(z)] + [\theta(x), D(y), \theta(z)] + [\theta(x), \theta(y), D(z)]$$

$$- [D(x), \theta(z), \theta(y)] - [\theta(x), D(z), \theta(y)] - [\theta(x), \theta(z), D(y)]$$

$$= ([D(x), \theta(y), \theta(z)] + [\theta(z), D(x), \theta(y)])$$

$$+ ([\theta(x), D(y), \theta(z)] + [\theta(z), \theta(x), D(y)])$$

$$+ ([\theta(x), \theta(y), D(z)] + [D(z), \theta(x), \theta(y)])$$

$$= [\theta(z), \theta(y), D(x)] + [\theta(z), D(y), \theta(x)] + [D(z), \theta(y), \theta(x)]$$

$$= A_{D,\theta}(z, y, x)$$

for all $x, y, z \in \mathcal{J}$. So

$$(2.5) \quad A_{D,\theta}(x, y, z) + A_{D,\theta}(z, y, x) = 2A_{D,\theta}(x, y, z)$$

for all $x, y, z \in \mathcal{J}$. We get from (2.3), (2.4) and (2.5) that

$$(2.6) \quad 2D([x, y, z]) - D([x, z, y]) = 2A_{D,\theta}(x, y, z)$$

for all $x, y, z \in \mathcal{J}$. Letting $y = z$ in (2.6), we get $D([x, y, y]) = A_{D,\theta}(x, y, y)$ for all $x, y \in \mathcal{J}$. Since $D([x, y + z, y + z]) = A_{D,\theta}(x, y + z, y + z)$ and $[., ., .], A_{D,\theta}$ are trilinear, we have

$$(2.7) \quad D([x, y, z]) + D([x, z, y]) = A_{D,\theta}(x, y, z) + A_{D,\theta}(x, z, y)$$

for all $x, y, z \in \mathcal{J}$. Adding (2.6) to (2.7), we infer that $D([x, y, z]) = A_{D,\theta}(x, y, z)$ for all $x, y, z \in \mathcal{J}$. So the proof is completed.

**Corollary 2.2.** Every Jordan derivation on a Lie triple system is a derivation.

**References**


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