HELICOIDAL SURFACES AND THEIR GAUSS MAP IN
MINKOWSKI 3-SPACE II

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Abstract. We classify and characterize the rational helicoidal surfaces
in a three-dimensional Minkowski space satisfying pointwise 1-type like
problem on the Gauss map.

1. Introduction

Nash’s imbedding theorem enables us to view every Riemannian manifold
as a submanifold of a Euclidean space. In that sense, one way to study a
Riemannian manifold is to apply the theory of submanifolds in a Euclidean
space. Since B.-Y. Chen ([3]) introduced the notion of finite type immersion of
submanifolds in a Euclidean space late 1970’s, many works have been carried
out in this area. Further, the notion of finite type can be extended to any
smooth functions on a submanifold of a Euclidean space or a pseudo-Euclidean
space. In dealing with submanifolds of a Euclidean or a pseudo-Euclidean
space, the Gauss map is a useful tool to examine the character of submanifolds
in a Euclidean space. For the last few years, two of the present authors and
D. W. Yoon introduced and studied the notion of pointwise 1-type Gauss
map in a Euclidean or a pseudo-Euclidean space ([4], [5], [7], [8]), namely the Gauss
map $G$ on a submanifold $M$ of a Euclidean space or a pseudo-Euclidean space
is said to be of pointwise 1-type if
\begin{equation}
\Delta G = F(G + C)
\end{equation}
for a non-zero smooth function $F$ on $M$ and a constant vector $C$, where $\Delta$
denotes the Laplace operator defined on $M$.

On the other hand, a helicoidal surface is well known as a kind of general-
ization of some ruled surfaces and surfaces of revolution in a Euclidean space
or a Minkowski space ([1], [2], [6]). Recently, two of the authors, H. Liu and
D. W. Yoon have classified the helicoidal surfaces with pointwise 1-type Gauss

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map in a Minkowski 3-space $\mathbb{L}^3$ ([5]). Then, we may have a natural question as follows:

What helicoidal surfaces have the harmonic Gauss map, that is, $\Delta G = 0$? Or, what helicoidal surfaces satisfy equation (1.1) whether the function $F$ is non-zero or zero?

In this paper, we mainly focus on the study of the helicoidal surfaces with harmonic Gauss map in a Minkowski 3-space and find the all solution spaces of the so-called rational helicoidal surfaces satisfying (1.1). As a consequence, we have the following characterizations:

**Theorem A.** Let $M$ be a helicoidal surface with space-like or time-like axis in a Minkowski 3-space $\mathbb{L}^3$. Then, a plane is the only rational helicoidal surface with harmonic Gauss map.

**Theorem B.** There exists no rational helicoidal surface with harmonic Gauss map which has null axis in Minkowski 3-space $\mathbb{L}^3$.

**Theorem C.** Let $M$ be a rational helicoidal surface with time-like axis in a Minkowski 3-space $\mathbb{L}^3$. Then, the Gauss map $G$ of $M$ satisfies the condition $\Delta G = F(G + C)$ for some smooth function $F$ and constant vector $C$ if and only if $M$ is an open part of a plane, a circular cylinder, a right cone, a right helicoid of type II or a helicoidal surface of elliptic type in $\mathbb{L}^3$.

2. Preliminaries

Let $\mathbb{L}^3$ be a Minkowski 3-space with the Lorentz metric

$$\langle \cdot, \cdot \rangle = -dx_0^2 + dx_1^2 + dx_2^2,$$

where $(x_0, x_1, x_2)$ is a system of the canonical coordinates in $\mathbb{R}^3$. Let $M$ be a connected 2-dimensional surface in $\mathbb{L}^3$ and $x : M \to \mathbb{L}^3$ a smooth non-degenerate isometric immersion. A surface $M$ is said to be space-like (resp. time-like) if the induced metric on $M$ is positive definite (resp. indefinite). Assuming that $M$ is orientable, we can always choose a unit normal vector field $G$ globally defined on $M$. In such a case, the unit normal vector field $G$ can be regarded as a map $G : M \to \mathbb{H}_\pm^2$ if $M$ is space-like and as a map $G : M \to \mathbb{S}_\pm^2$ if $M$ is time-like, where $\mathbb{H}_\pm^2 = \{ x \in \mathbb{L}^3 \mid \langle x, x \rangle = -1, x_2 > 0 \}$ is the hyperbolic space and $\mathbb{S}_\pm^2 = \{ x \in \mathbb{L}^3 \mid \langle x, x \rangle = 1 \}$ is the de Sitter space. The map $G$ is also called the Gauss map of the surface $M$. For the matrix $\tilde{g} = (\tilde{g}_{ij})$ consisting of the components of the induced metric on $M$, we denote by $\tilde{g}^{-1} = (\tilde{g}^{ij})$ (resp. $G$) the inverse matrix (resp. the determinant) of the matrix $(\tilde{g}_{ij})$. The Laplacian $\Delta$ on $M$ is, in turn, given by

$$\Delta = -\frac{1}{\sqrt{|G|}} \sum_{i,j} \frac{\partial}{\partial x^i} \left( \sqrt{|G|} \tilde{g}^{ij} \frac{\partial}{\partial x^j} \right).$$

Let $e$ be a non-zero vector in $\mathbb{L}^3$ and $S(e)$ the set of screw motions fixing $e$ in $\mathbb{L}^3$. In particular, if $e$ is non-null, the screw motions fixing $e$ belong to
O(e), the set of orthogonal transformations with positive determinant. Then, a helicoidal motion around the axis in the e-direction is defined by

\[ g_t(x) = A(t)x^T + (ht)e, \quad x = (x_0, x_1, x_2) \in \mathbb{L}^3, \quad t \in \mathbb{R}, \quad A \in S(e), \]

where \( h \) is a constant and \( x^T \) is the transpose of the vector \( x \).

Let \( \gamma : I = (a, b) \subset \mathbb{R} \to \Pi \) be a plane curve in \( \mathbb{L}^3 \) and \( l \) a straight line in \( \Pi \) which does not intersect the curve \( \gamma \). A helicoidal surface \( M \) with the axis \( l \) and pitch \( h \) in \( \mathbb{L}^3 \) is a non-degenerate surface which is invariant under the action of the helicoidal motion \( g_t \). Depending on the axis being space-like, time-like or null, there are three types of screw motions. If the axis \( l \) is space-like (resp. time-like), then \( l \) is transformed to the \( x_1 \)-axis or \( x_2 \)-axis (resp. \( x_0 \)-axis) by the Lorentz transformation. Therefore, we may consider \( x_2 \)-axis (resp. \( x_0 \)-axis) as the axis if \( l \) is space-like (resp. time-like). If the axis \( l \) is null, then we may assume that the axis is the line spanned by the vector \((1, 1, 0)\).

We now consider the helicoidal surfaces in \( \mathbb{L}^3 \) with space-like, time-like or null axis respectively.

**Case 1.** The axis \( l \) is space-like.

Without loss of generality we may assume that the profile curve \( \gamma \) lies in the \( x_1 x_2 \)-plane or \( x_0 x_2 \)-plane. Hence, the curve \( \gamma \) can be represented by

\[ \gamma(u) = (0, f(u), g(u)) \text{ or } \gamma(u) = (f(u), 0, g(u)) \]

for smooth functions \( f \) and \( g \) on an open interval \( I = (a, b) \). Therefore, the surface \( M \) may be parameterized by

\[ x(u, v) = (f(u) \sinh v, f(u) \cosh v, g(u) + hv), \quad f(u) > 0, \quad h \in \mathbb{R} \]

or

\[ x(u, v) = (f(u) \cosh v, f(u) \sinh v, g(u) + hv), \quad f(u) > 0, \quad h \in \mathbb{R}. \]

**Case 2.** The axis \( l \) is time-like.

In this case, we may assume that the profile curve \( \gamma \) lies in the \( x_0 x_1 \)-plane. So the curve \( \gamma \) is given by \( \gamma(u) = (g(u), f(u), 0) \) for a positive function \( f = f(u) \) on an open interval \( I = (a, b) \). Hence, the surface \( M \) can be expressed by

\[ x(u, v) = (g(u) + hv, f(u) \cos v, f(u) \sin v), \quad f(u) > 0, \quad h \in \mathbb{R}. \]

**Case 3.** The axis \( l \) is null.

In this case, we may assume that the profile curve \( \gamma \) lies in the \( x_0 x_1 \)-plane of the form \( \gamma(u) = (f(u), g(u), 0) \), where \( f = f(u) \) is a positive function and \( g = g(u) \) is a function satisfying \( p(u) = f(u) - g(u) \neq 0 \) for all \( u \in I \). Under the cubic screw motion, its parametrization has the form

\[ x(u, v) = \left( f(u) + \frac{v^2}{2} p(u) + hv, \quad g(u) + \frac{v^2}{2} p(u) + hv, \quad p(u)v \right), \quad h \in \mathbb{R}. \]
3. Helicoidal surfaces with time-like axis in Minkowski 3-space

In this section, we study the helicoidal surfaces with harmonic Gauss map which has time-like axis in Minkowski 3-space \( \mathbb{L}^3 \).

Suppose that \( M \) is a helicoidal surface in \( \mathbb{L}^3 \) with time-like axis parameterized by (2.3) for some smooth functions \( f \) and \( g \).

First, if \( f \) is constant, the parametrization of \( M \) can be written as
\[
x(u, v) = (g(u) + hv, a \cos v, a \sin v), \quad h \in \mathbb{R}
\]
for a non-zero constant \( a \). By a straightforward computation, we see that the Laplacian \( \Delta G \) of the Gauss map \( G \) satisfies \( \Delta G = \frac{1}{a} G \). Hence, \( M \) does not have the harmonic Gauss map. In fact, it has non-proper pointwise 1-type Gauss map of the first kind ([5]). Therefore, we may assume that \( f \) is not constant. Then, we may put \( f(u) = u \) and thus \( M \) is parameterized by
\[
x(u, v) = (g(u) + hv, \cos v, \sin v), \quad u > 0, \quad h \in \mathbb{R}.
\]

If \( M \) is space-like, that is, \( u^2 - u^2 g^2 - h^2 > 0 \), then the Gauss map \( G \) and its Laplacian \( \Delta G \) are obtained as follows:
\[
G = \frac{1}{\sqrt{u^2 - u^2 g^2 - h^2}} (-u, -ug' \cos v + h \sin v, -ug' \sin v - h \cos v)
\]
and
\[
\Delta G = \frac{1}{(u^2 - u^2 g^2 - h^2)^2} (D(u), A(u) \sin v + B(u) \cos v, -A(u) \cos v + B(u) \sin v),
\]
where we have put
\[
A(u) = h(2h^4 - 4h^4 g^2 + (-7h^4 g')'u + (-2h^2 + 2h^4 g^2 - h^4 g''')u^2
+ (8h^2 g'' + h^2 g^2 g')u^3 + (3h^2 g^2 g'' - h^2 g^3 g' + 2h^2 g'' + 2h^2 g'''')u^4
+ (-g' g'' + g^3 g')u^5 + (-g'' - 3g' g'' - g' g''' + g^3 g''')u^6)
\]
\[
B(u) = -3h^6 g'' + (6h^4 g' + 8h^4 g'' - h^4 g''')u + (7h^4 g' + 7h^4 g'' - h^4 g''')u^2
+ (7h^2 g' - 12h^2 g'' + 5h^2 g''' + 4h^4 g'''')u^3
+ (-5h^2 g'' - 6h^2 g'' - h^2 g''')u^4 + (-g' (1 - g^2)^3 - 8h^4 g' g^2 - 3h^2 g'' + 2h^2 g'''')u^5
+ (g'' - g^2 g')u^6 + (-g'' - 4g' g'' + g'''')u^7
\]
and
\[
D(u) = u\{-2h^4 + 4h^4 g^2 + (7h^4 g')'u + (2h^2 - 2h^4 g^2 + h^4 g''')u^2
+ (-8h^2 g'' - h^2 g^2 g')u^3 + (-3h^2 g^2 g'' + h^2 g^3 g' - 2h^2 g''')u^4
- 2h^2 g'''')u^5 + (g' g'' - g^3 g'')u^5 + (g'' + 3g^2 g'' + g' g'' - g^3 g''')u^6\}.
\]

Suppose that \( M \) has harmonic Gauss map, that is, its Gauss map \( G \) satisfies \( \Delta G = 0 \). Then, we obtain that the functions \( A(u) \), \( B(u) \) and \( D(u) \) are all vanishing.
First, we consider the case that $M$ is a helicoidal surface of polynomial kind with harmonic Gauss map, that is, $g$ is a polynomial in $u$. Then we may put

$$g(u) = a_n u^n + a_{n-1} u^{n-1} + \cdots + a_1 u + a_0,$$

where $n$ is nonnegative integer and $a_n$ is non-zero constant.

Considering the constant terms of $B(u)$, it is easy to see that $h = 0$, that is, $M$ is a surface of revolution. Therefore, $A(u) = 0$. Also, $B(u)$ and $D(u)$ are reduced to respectively:

$$B(u) = -g'(1-g^2)^3 u^5 + (g'' - g' g'') u^6 + (-g'' g'' + 4g' g'^2 + g''') u^7,$$

$$D(u) = (g' g'' - g''^2) u^5 + (g''^2 + 3g'^2 g'' + g''' - g''^3) u^6.$$

Assume that $\deg g(u) \geq 2$, where $\deg g(u)$ means the degree of the polynomial $g(u)$. Then, the term $-g'(1-g^2)^3 u^5$ in $B(u)$ includes the highest degree in $u$ and its leading coefficient must be zero, that is, $n^7 a_n^7 = 0$. Thus, $a_n = 0$, a contradiction.

Assuming $\deg g(u) = 1$, $B(u) = -a_1 (1-a_1^2)^3 u^5$. Hence, $a_1^2 = 1$, which is a contradiction since $M$ is non-degenerate.

If $g$ is constant, then $B(u) = 0$ and $D(u) = 0$. Hence, the Gauss map is harmonic. In this case, the parametrization of $M$ in (3.1) is reduced to

$$x(u,v) = (a, u \cos v, u \sin v), \quad u > 0$$

for some constant $a$. This means that $M$ is part of a plane.

Conversely, it is obvious that the Gauss map of a plane is harmonic. By a similar process as above, the same conclusion can be made in case of time-like surface. Consequently, we have:

**Theorem 3.1.** Let $M$ be a helicoidal surface of polynomial kind with time-like axis in a Minkowski 3-space $\mathbb{L}^3$. Then, $M$ has the harmonic Gauss map if and only if $M$ is part of a plane.

Next, consider $M$ is of rational kind, that is, $g(u)$ is a rational function. Suppose that $M$ is a genuine helicoidal surface of rational kind with harmonic Gauss map, i.e., $h \neq 0$. Then we may put

$$g(u) = p(u) + \frac{r(u)}{q(u)},$$

where $p(u)$ is a polynomial in $u$ and the polynomials $r(u)$ and $q(u)$ are relatively prime with $\deg r(u) < \deg q(u)$ and $\deg q(u) \geq 1$. Let $\deg p(u) = l$, $\deg r(u) = n$ and $\deg q(u) = m$ with $n < m$ and $m \geq 1$ where $l$, $m$, and $n$ are some nonnegative integers. Then, we may put

$$p(u) = a_l u^l + a_{l-1} u^{l-1} + \cdots + a_1 u + a_0,$$

$$q(u) = b_m u^m + b_{m-1} u^{m-1} + \cdots + b_1 u + b_0,$$

$$r(u) = c_n u^n + c_{n-1} u^{n-1} + \cdots + c_1 u + c_0.$$
Putting (3.2) in the equation $B(u)$ and multiplying $q^{14}(u)$ with thus obtained equation, we get a polynomial $q^{14}(u)B(u)$ in $u$.

Assume that $\deg p(u) \geq 2$. By an algebraic computation, we see that the degree of the polynomial is $7l + 14m - 2$ and so its coefficient $l^7a_l^7b_m^{14}$ must be zero. But, this is a contradiction.

Assuming $\deg p(u) = 1$, the leading coefficient of the polynomial is $-a_1(1 - a_1^2)b_1^{14}$. It must be zero and so $a_1^2 = 1$. In this case, we can consider two cases according to the value of $m - n$.

If $m - n = 1$, then the polynomial includes the term of the degree $14m + 1$ with the coefficient $2h^8a_1b_m^{14}$. Hence it must be zero, a contradiction.

Suppose $m - n = 1$. Since the Gauss map of $M$ is harmonic, the polynomials $q^{10}(u)A(u)$ and $q^{14}(u)B(u)$ are vanishing. With the help of (3.2) and (3.3), we have $b_0 = 0$. So we may put

$$q(u) = b_m u^m + \cdots + b_2 u^2 + b_1 u, \quad b_m \neq 0.$$  

Then, an algebraic computation shows that the polynomial $q^{10}(u)A(u)$ has the lowest degree 4 with the coefficient $4h^2b_1^6c_0^4$. Similarly, the polynomial $q^{14}(u)B(u)$ has the lowest degree 5 with the coefficient $-b_1^7c_0^7$. Therefore, $b_1c_0 = 0$.

If we assume $c_0 \neq 0$, then $b_1 = 0$ and we have

$$q(u) = b_m u^m + \cdots + b_2 u^2, \quad b_m \neq 0.$$  

By considering the coefficients of the terms with the lowest degree in $q^{10}(u)A(u)$ and $q^{14}(u)B(u)$, we get $b_2c_0 = 0$. Hence, $b_2 = 0$. Inductively, $b_3, \ldots, b_{m-1}$ are zero. So we put

$$g(u) = b_m u^m, \quad b_m \neq 0.$$  

Then, the polynomial $q^{14}(u)B(u)$ has the lowest degree $7m - 2$ with the coefficient $(-mb_m c_0)^7$. It must be zero, a contradiction. Thus, we conclude that $c_0 = 0$. Hence, $g(u)$ can be written as

$$g(u) = \pm u + a_0 + \frac{r(u)}{q(u)},$$  

where $r(u) = c_n u^{n-1} + \cdots + c_1$ and $q(u) = b_m u^{m-1} + \cdots + b_1$ with $c_n \neq 0$ and $b_m \neq 0$. By a similar process as above, we obtain $b_1, \ldots, b_{m-1} = 0$ and $c_1, \ldots, c_{n-1} = 0$. Consequently, we get

$$g(u) = \pm u + a_0 + \frac{c}{u}, \quad c \neq 0.$$  

Hence, $q^{14}(u)B(u)$ has the coefficient $-c^7$ of the lowest degree which is 5 and it must be zero. Thus, $c = 0$, that is, $g$ is a polynomial in $u$.

Finally, if $p$ is constant, then the degree of $q^{14}(u)B(u)$ is $13m + n + 4$ and its leading coefficient is $-(m - n)^4b_m^{14}c_n$. This must be zero, a contradiction.
By a similar argument as above, we lead to a contradiction in case of surfaces of revolution. In case of time-like surface, we have the same result. Consequently, we have:

**Theorem 3.2.** Let $M$ be a helicoidal surface with time-like axis in a Minkowski 3-space $\mathbb{L}^3$. Then, there exists no helicoidal surface of rational kind with harmonic Gauss map except polynomial kind.

Combining the above theorems we have the following:

**Theorem 3.3** (Characterization). Let $M$ be a rational helicoidal surface with time-like axis in a Minkowski 3-space $\mathbb{L}^3$. Then, $M$ has the harmonic Gauss map if and only if it is part of a plane.

Combining the results above and [5], we have the following characterization.

**Theorem 3.4** (Characterization). Let $M$ be a rational helicoidal surface with time-like axis in a Minkowski 3-space $\mathbb{L}^3$. Then, the Gauss map $G$ of $M$ satisfies the condition $\Delta G = F(G + C)$ for some smooth function $F$ and constant vector $C$ if and only if $M$ is an open part of a plane, a circular cylinder, a right cone, a right helicoid of type II or a helicoidal surface of elliptic type in $\mathbb{L}^3$.

4. Helicoidal surfaces with null axis in Minkowski 3-space

In this section, we investigate the helicoidal surfaces with harmonic Gauss map which has null axis in $\mathbb{L}^3$.

Suppose that $M$ is a helicoidal surface with null axis parameterized by

$$x(u,v) = \left( f(u) + \frac{v^2}{2} p(u) + hv, \ g(u) + \frac{v^2}{2} p(u) + hv, \ p(u) \right), \ h \in \mathbb{R},$$

where $p(u) = f(u) - g(u) \neq 0$. Since the induced metric on $M$ is non-degenerate, $(f(u) - g(u))^2(f''(u) - g''(u)) + h^2(f'(u) - g'(u))^2$ never vanishes and so $f'(u) - g'(u) \neq 0$ everywhere. Thus, we may change the variable in such a way that $p(u) = f(u) - g(u) = -2u$.

Let $k(u) = f(u) + u$. Then, the functions $f$ and $g$ in the profile curve $\gamma$ look like

$$f(u) = k(u) - u \text{ and } g(u) = k(u) + u.$$  

Thus, the parametrization of $M$ becomes

$$x(u,v) = (k(u) - u - uv^2 + hv, \ k(u) + u - uv^2 + hv, \ -2uv).$$

We now suppose that $M$ is space-like, that is, $4u^2k''(u) - h^2 > 0$. By a direct computation, the Gauss map $G$ and its Laplacian $\Delta G$ are obtained as follows:

$$G = \frac{1}{\sqrt{4u^2k''(u) - h^2}} (uk'(u) + u + uv^2 + hv, \ uk'(u) - u + uv^2 - vh, \ 2uv - h)$$

and

$$\Delta G = -\frac{1}{(4u^2k''(u) - h^2)^{\frac{3}{2}}} (2uX + Y, \ -2uX + Y, \ 2(2uv - h)X),$$
where we have put

\[(4.1)\]

\[X = X(u) = h^4 + 4h^2k'u^2 + 9h^2k''u^3 + h^3k'''u^4 - 4k'k''u^5 + 8k''^2u^6 - 4k''u^6\]

and

\[(4.2)\]

\[Y = Y(u, v) = 10h^4k'u + 7h^4k''u^2 - 32h^2k'^2u^3 + h^4k'''u^4 - 14h^2k'k''u^4 + 32k'^3u^5 + 6h^2k''u^5 - 6h^2k'k'''u^5 + 8k'^2k''u^6 - 8k'k''u^7 + 8k'^2k''u^7 - 2h^5v - 8h^3k'u^2v - 18h^3k''u^3v + 8hk'k''u^5v - 16hk'^2u^6v + 8hk'k'''u^6v + 2h^4uv^2 + 8h^2k'u^3v^2 + 18h^2k''u^4v^2 + 2h^2k''u^5v^2 - 8k'k'u^6v^2 + 16k'^2u^7v^2 - 8k''u^7v^2.\]

Suppose that \(M\) has harmonic Gauss map, that is, its Gauss map \(G\) satisfies \(\Delta G = 0\). Then the above equations \(X(u)\) and \(Y(u, v)\) are vanishing. Hence, the equation \(Y(u, v)\) in (4.2) can be rewritten as

\[Y(u, v) = Y_1(u) + Y_2(u)v + Y_3(u)v^2,\]

where we put

\[Y_1(u) = 10h^4k'u + 7h^4k''u^2 - 32h^2k'^2u^3 + h^4k'''u^4 - 14h^2k'k''u^4 + 32k'^3u^5 + 6h^2k''u^5 - 6h^2k'k'''u^5 + 8k'^2k''u^6 - 8k'k''u^7 + 8k'^2k''u^7 - 2h^5v - 8h^3k'u^2v - 18h^3k''u^3v + 8hk'k''u^5v - 16hk'^2u^6v + 8hk'k'''u^6v + 2h^4uv^2 + 8h^2k'u^3v^2 + 18h^2k''u^4v^2 + 2h^2k''u^5v^2 - 8k'k'u^6v^2 + 16k'^2u^7v^2 - 8k''u^7v^2,\]

\[Y_2(u) = -2h(h^4 + 4h^2k'u^2 + 9h^2k''u^3 + h^3k'''u^4 - 4k'k''u^5 + 8k'^2u^6 - 4k''u^6),\]

\[Y_3(u) = 2u(h^4 + 4h^2k'u^2 + 9h^2k''u^3 + h^3k'''u^4 - 4k'k''u^5 + 8k'^2u^6 - 4k''u^6).\]

Since \(X(u)\) and \(Y(u, v)\) are vanishing, we have \(Y_1(u) = 0\). Moreover, \(Y_1(u)\) can be written as \(Y_1(u) = -2k'uX(u) + uZ(u)\) and we also get \(Z(u) = 0\), where

\[(4.3)\]

\[Z(u) = 12h^4k' + 7h^4k''u - 24h^2k'^2u^2 + h^4k'''u^3 + 4h^2k'k''u^3 + 32k'^3u^4 + 6h^2k''u^4 - 4h^2k'k'''u^4 + 8k''u^6.\]

Let \(M\) be a helicoidal surface of polynomial kind with harmonic Gauss map, that is, \(k\) is a polynomial in \(u\). Then we may put

\[k(u) = a_nu^n + a_{n-1}u^{n-1} + \cdots + a_1u + a_0,\]

where \(n\) is nonnegative integer and \(a_n\) is non-zero constant.

Considering the constant terms in \(X(u)\), it is easy to see that \(h = 0\). Therefore, the equations \(X(u)\) and \(Z(u)\) can be written as

\[X(u) = -4k'k''u^5 + 8k''u^6 - 4k''u^6\]

and \(Z(u) = 32k'^3u^4 + 8k''u^6\).

Assume that \(\deg k(u) \geq 2\). Considering the equation \(X(u)\), we can easily lead to a contradiction.
If deg \( k(u) = 1 \), then \( X(u) = 0 \) and \( Z(u) = 32u^3 u^4 \). Hence, \( Z(u) \) cannot be zero and so we have a contradiction.

If \( k \) is constant, then \( X(u) = 0 \) and \( Z(u) = 0 \). But, in this case, it contradicts that \( M \) is non-degenerate, i.e., \( 4u^2 k'(u) \neq 0 \). Hence, \( M \) does not have harmonic Gauss map.

By a similar argument as above, we have the same results in case of time-like helicoidal surface of polynomial kind with null axis. Thus, we have:

**Theorem 4.1.** Suppose that \( M \) is a helicoidal surface of polynomial kind with null axis in a Minkowski 3-space \( \mathbb{L}^3 \). Then \( M \) does not have harmonic Gauss map.

We now consider a helicoidal surface of rational kind with harmonic Gauss map, that is, \( k \) is a rational function in \( u \). Then we may put

\[
k(u) = p(u) + \frac{r(u)}{q(u)},
\]

where \( p(u) \) is a polynomial in \( u \), \( r(u) \) and \( q(u) \) are relatively prime polynomials with \( \deg r(u) < \deg q(u) \) and \( \deg q(u) \geq 1 \).

Suppose that \( M \) is a genuine helicoidal surface of rational kind, that is, \( h \neq 0 \). With the help of (4.1) and (4.3), we get

\[
u^2 Z(u) - h^2 X(u) = (4u^2 k' - h^2)(h^4 - 4h^2 k' u^2 + 2h^2 k'' u^3 + 8k'^2 u^4 + 2k''^2 u^6).
\]

Since \( X(u) \) and \( Z(u) \) vanishes identically,

\[
(4u^2 k' - h^2)(h^4 - 4h^2 k' u^2 + 2h^2 k'' u^3 + 8k'^2 u^4 + 2k''^2 u^6) = 0.
\]

Because \( M \) is a nondegenerate surface, i.e., \( 4u^2 k' - h^2 \neq 0 \),

(4.4) \hspace{1cm} h^4 - 4h^2 k' u^2 + 2h^2 k'' u^3 + 8k'^2 u^4 + 2k''^2 u^6 = 0.

From the equation (4.4), we get

\[
(2k'' u^3 + h^2)^2 + (4u^2 k' - h^2)^2 = 0.
\]

It is easily seen that this is a contradiction because of \( 4u^2 k' - h^2 \neq 0 \). Thus, \( h = 0 \).

If \( h = 0 \), the equation \( Z(u) \) in (4.3) can be reduced as

\[
Z(u) = 8u^2 k'(k''^2 u^4 + 4u^2 k'^2).
\]

Since \( M \) is nondegenerate, \( k''^2 u^4 + 4u^2 k'^2 = 0 \), which implies \( k \) is constant, a contradiction.

Similarly, we prove that a time-like helicoidal surface of rational kind does not have harmonic Gauss map. Consequently, we have:

**Theorem 4.2.** Let \( M \) be a helicoidal surface with null axis in a Minkowski 3-space \( \mathbb{L}^3 \). Then, there exists no rational helicoidal surface with harmonic Gauss map.
Combining the results we obtained above and those in [5], we have the following:

**Theorem 4.3 (Characterization).** Let $M$ be a helicoidal surface of rational kind with null axis in a Minkowski 3-space $\mathbb{L}^3$. Then, the Gauss map $G$ of $M$ satisfies $\Delta G = F(G + C)$ for some smooth function $F$ and constant vector $C$ if and only if it is part of an Enneper’s surface of second kind, a de Sitter space, a hyperbolic space, a helicoidal surface of Enneper type, a helicoidal surface of hyperbolic type or a helicoidal surface of de Sitter type in $\mathbb{L}^3$.

**References**


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