ON THE HYERS-ULAM-RASSIAS STABILITY
OF JENSEN’S EQUATION

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ABSTRACT. J. Wang [21] proposed a problem: whether the Hyers-Ulam-Rassias stability of Jensen’s equation for the case \( p, q, r, s \in (\beta, \frac{1}{\beta}) \setminus \{1\} \) holds or not under the assumption that \( G \) and \( E \) are \( \beta \)-homogeneous \( F \)-space \((0 < \beta \leq 1)\). The main purpose of this paper is to give an answer to Wang’s problem. Furthermore, we proved that the stability property of Jensen’s equation is not true as long as \( p \) or \( q \) is equal to \( \beta, \frac{1}{\beta} \), or \( \frac{\beta}{\beta_1} \) \((0 < \beta_1, \beta_2 \leq 1)\).

1. Introduction

Let \( G \) denote a linear space and \( E \) denote a real or complex Hausdorff topological vector space. \( f : G \to E \) is a mapping. We call the following equation

\[
2f\left(\frac{x + y}{2}\right) - f(x) - f(y) = \theta,
\]

as Jensen’s equation.

More than half a century ago, S. M. Ulam [20] posed the following problem:

Give a group \( G \), and a metric group \( E \) with the metric \( d(\cdot, \cdot) \) and a positive number \( \varepsilon > 0 \), does there exists a \( \delta > 0 \) such that if a function \( f : G \to E \) satisfies \( d(f(xy), f(x)f(y)) < \delta \) for all \( x, y \in G \), then there exists a homomorphism \( h : G \to E \) with \( d(h(x), f(x)) < \varepsilon \) for all \( x \in G \)?

In 1941, the case of approximately additive mapping was solved by D. H. Hyers [6] for \( G \) and \( E \) being Banach spaces. Next, Th. M. Rassias [13] generalized the conclusion of Hyers’ by introducing the unbounded Cauchy difference as follows:
Theorem. Let \( f: G \to E \) be a mapping between Banach spaces subject to the inequality
\[
\|f(x + y) - f(x) - f(y)\| \leq \varepsilon (\|x\|^p + \|y\|^p) \quad (\forall x, y \in G),
\]
where \( \varepsilon, p \) are constants with \( \varepsilon > 0 \) and \( 0 \leq p < 1 \). Then there exists a unique additive mapping \( T: G \to E \) such that
\[
\|f(x) - T(x)\| \leq \frac{2\varepsilon}{2 - 2p}\|x\|^p \quad (\forall x \in G).
\]
If, in addition \( f(tx) \) is continuous in \( t \) for each fixed \( x \in G \), then \( T \) is linear.

The proof given in [13] also works when \( p < 0 \). In 1991, Z. Gajda [3] following the spirit of the proof of Th. M. Rassias’s Theorem for the unbounded Cauchy difference by replacing \( n \) by \( -n \) solved Th. M. Rassias’s question by proving the stability theorem for all real values of \( p \) that are strictly greater than one. And in this paper, Z. Gajda found firstly that the stability problem does not hold when \( p = 1 \).

The remarkable generalization of Th. M. Rassias for D. H. Hyer’s Theorem promoted greatly the development of the stability problems of functional equations. It stimulated a number of mathematicians to study the stability problems of various functional equations. For more detailed information of such a field one can refer to [4], [14], [15], and [16].

In this paper, we deal with the stability of the Jensen’s functional Eq.(1).

The first result on the stability of Jensen’s equation was carried out by Z. Kominek [9]. New generalizations of Jensen’s functional equation were given by Th. M. Rassias [12]. In 1998, S.-M. Jung [8] gave an important generalization of the Z. Kominek’s result. In fact, he proved the following theorem:

**Theorem.** Let \( E_1 \) be a real normed space and let \( E_2 \) be a real Banach space. Assume that \( \delta, \theta \geq 0 \) are fixed, and let \( p > 0 \) be given with \( p \neq 1 \). Suppose a mapping \( f: E_1 \to E_2 \) satisfied the functional inequality
\[
\left\| 2f \left( \frac{x + y}{2} \right) - f(x) - f(y) \right\| \leq \delta + \theta (\|x\|^p + \|y\|^p)
\]
for all \( x, y \in E_1 \). Furthermore, assume \( f(0) = 0 \) and \( \delta = 0 \) in above inequality for the case of \( p > 1 \). Then there exists a unique additive function \( T: E_1 \to E_2 \) such that
\[
\|f(x) - T(x)\| \leq \delta + \|f(0)\| + \frac{\theta}{2^{1-p} - 1}\|x\|^p \quad (\text{for } p < 1)
\]
or
\[
\|f(x) - T(x)\| \leq \frac{2^{p-1}\theta}{2p-1}\|x\|^p \quad (\text{for } p > 1)
\]
for all \( x \in E_1 \).

Later, many results concerning the stability of Jensen’s equation were obtained by numerous authors, such as [11], [19], and [10]. J. Wang [22], [24]
attempted to weaken the condition of the space. She proved a generalized conclusion of S.-M. Jung. In the following, we introduce Wang’s result [24]:

**Corollary I.** Let $G$ be an $F^*$-space and $E$ be an $F$-space with the property that there exists $0 < \beta \leq 1$ such that $\|x\| \leq \frac{\|x\|}{\beta}$ for all $x \in G$, and $E$ be an $F$-space. Assume that $T(x) = f(\theta)$. If $\phi(x, y) = \delta + \varepsilon_1\|x\|p + \varepsilon_2\|y\|q(\delta, \varepsilon_1, \varepsilon_2 \geq 0, p, q < \beta)$, then there exists a unique additive mapping $T: G \to E$ such that

$$\|T(x) - f(x)\| \leq \frac{2\delta}{3^\beta - 1} + \frac{2\varepsilon_1}{3^\beta - 3} \|x\|^p + \frac{(1 + 3^\beta)\varepsilon_2}{3^\beta - 3} \|x\|^q$$

for any $x \in G$. If there exists at least one of $p, q$ such that it is strictly less than 0, then the domain of $T$ is $G \setminus \{\theta\}$ instead of $G$.

**Corollary II.** Let $G$ be an $F^*$-space with the property that there exists $0 < \beta \leq 1$ such that $\|x\| \leq \frac{\|x\|}{\beta}$ for all $x \in G$, and $E$ be an $F$-space. Assume that $T(x) = f(\theta)$. If $\phi(x, y) = \delta + \varepsilon_1\|x\|p + \varepsilon_2\|y\|q(\varepsilon_1, \varepsilon_2 \geq 0, p, q > \frac{1}{\beta})$, then there exists a unique additive mapping $T: G \to E$ such that

$$\|T(x) - f(x)\| \leq \frac{2\varepsilon_1}{3p^\beta - 3} \|x\|^p + \frac{(1 + 3^\beta)\varepsilon_2}{3p^\beta - 3} \|x\|^q$$

for any $x \in G$.

In above corollaries, $\phi(x, y) = \frac{1}{2}f(\frac{x + y}{2}) - f(x) - f(y)$.

J. Wang noticed that these results hold for $p, q < \beta$ or $p, q > \frac{1}{\beta}$. She raised the following question: What does it hold if $p, q$ satisfy $\beta < p, q < \frac{1}{\beta}$ under the assumption that $G$ and $E$ are $\beta$-homogeneous $F$-spaces $(0 < \beta \leq 1)$? In Section 2 of the present paper, by still using the ideas from the papers of Hyers [6], Rassias [13], Rassias and Šemrl [16], we provide the stability of Eq.(1) for $\beta_2 < p, q < \frac{1}{\beta_1} (p, q \neq \frac{\beta_2}{\beta_1})$ in $\beta$-homogeneous $F$-space. In Section 3, we show that the stability of Jensen’s equation is not satisfied as long as $p$ or $q$ equals $\beta_2$, $\frac{1}{\beta_1}$ or $\frac{\beta_2}{\beta_1}(0 < \beta_1, \beta_2 \leq 1)$.

2. **Stability of Eq.(1) for $\beta_2 < p, q < \frac{1}{\beta_1}$ ($p, q \neq \frac{\beta_2}{\beta_1}$)**

From now on, we let $\mathbb{N}$ denote the set of positive integers set and $\mathbb{R}$ denotes the set of real numbers set, respectively. Meanwhile, we assume $p, q$ to be different real numbers.

Firstly, we introduce the definition of $F$-space and $\beta$-homogeneous (see [18]).

Let $X$ be a linear space. A non-negative valued function $\| \cdot \|$ defined on $X$ is called an $F$-norm if it obeys the following rules:

1. $\|x\| = 0$ if and only if $x = 0$;
2. $\|ax\| = |a| \|x\|$ for all $a, |a| = 1$;
3. $\|x + y\| \leq \|x\| + \|y\|$;
Let \( F \) be an \( F^* \)-space. An \( F \)-pseudonorm \( \| x \| \) does not necessarily imply that \( x = 0 \) in (n1)) is called \( \beta \)-homogeneous \( (\beta > 0) \) if \( \| tx \| = |t|^\beta \| x \| \) for all \( x \in X \) and all \( t \in \mathbb{R} \). A complete \( F^* \)-space is said to be an \( F \)-space.

**Theorem 2.1.** Let \( G \) and \( E \) be a \( \beta_1 \)-homogeneous \( F^* \)-space and a \( \beta_2 \)-homogeneous \( F \)-space, respectively. Suppose that \( f: G \to E \) is a mapping with the property that

\[
\begin{align*}
\| 2f \left( \frac{x + y}{2} \right) - f(x) - f(y) \| &\leq \varepsilon_1 \| x \| ^p + \varepsilon_2 \| y \| ^q, \\
\end{align*}
\]

where \( \beta_1, \beta_2 \in (0, 1], \varepsilon_1, \varepsilon_2 \in (0, \infty) \) and \( p, q \in (\beta_2, \frac{1}{\beta_1}) \setminus \{ \frac{\beta_2}{\beta_1} \} \). Then there exists a unique additive mapping \( T: G \to E \) such that

\[
\| T(x) - f(x) \| \leq \frac{2\varepsilon_1}{3^{\beta_2} - 3^{\beta_1}p} \| x \| ^p + \frac{1 + 3^{\beta_1}q}{3^{\beta_2} - 3^{\beta_1}q} \| x \| ^q,
\]

in the case \( \beta_2 < p, q < \frac{\beta_2}{\beta_1} \), while in the case \( \frac{\beta_2}{\beta_1} < p, q < \frac{1}{\beta_1} \)

\[
\| T(x) - f(x) \| \leq \frac{2\varepsilon_1}{3^{\beta_q} - 3^{\beta_p}q} \| x \| ^p + \frac{1 + 3^{\beta_1}q}{3^{\beta_q} - 3^{\beta_p}q} \| x \| ^q.
\]

Moreover, if for each fixed \( x \in G \), there exists a real numbers \( \delta_x > 0 \), such that \( f(tx) \) is continuous on \( [0, \delta_x] \), then \( T(x) \) is linear.

**Proof.** Let \( g(x) = f(x) - f(\theta) \). Then \( g \) also satisfies (2). From this, we can assume that \( f(\theta) = \theta \) without loss of generality.

When \( \beta_2 < p, q < \frac{\beta_2}{\beta_1} \), we claim that

\[
\| 3^{-n} f(3^n) - f(x) \|
\]

\[
\leq \sum_{k=1}^{n} 3^{k(\beta_1 p - \beta_2)} \cdot 2 \cdot 3^{-\beta_1 p} \varepsilon_1 \| x \| ^p
\]

\[
+ \sum_{k=1}^{n} (3^{k(\beta_q - \beta_2)} \cdot 3^{-\beta_1 q} + 3^{k(\beta_1 q - \beta_2)}) \varepsilon_2 \| x \| ^q
\]

holds for any integer \( n \). The verification of (5) follows by induction on \( n \). Indeed, for \( n = 1 \), we set \( y = -x \), then

\[
\| f(x) - f(-x) \| \leq \varepsilon_1 \| x \| ^p + \varepsilon_2 \| x \| ^q.
\]

Replacing \( x \) by \( -x \) and \( y \) by \( 3x \), (5) implies

\[
\| 2f(x) - f(-x) - f(3x) \| \leq \varepsilon_1 \| x \| ^p + \varepsilon_2 \cdot 3^{\beta_1 q} \| x \| ^q.
\]
Taking the two inequality into account, then
$$\|3^{-1} f(3x) - f(x)\| = \|3^{-1} [f(3x) + f(-x) - 2f(x) - f(-x) - f(x)]\|$$
$$\leq 3^{-\beta_2} \|3^{-1} [f(3x) + f(-x) - 2f(x)]\| + \|f(-x) - f(x)\|$$
$$\leq 2 \cdot 3^{-\beta_2} \cdot \varepsilon_1 \|x\|^p + (1 + 3^{\beta_q}) \cdot 3^{-\beta_2} \varepsilon_2 \|x\|^q.$$  

Assume that the formula is true for \( n = m \), we want to examine the case when \( n = m + 1 \). We have
$$\|3^{-1} f(3^{m+1}) x - f(x)\| = \|3^{-1} [3^{-m} f(3^m (3x)) - f(3x)] + 3^{-1} f(3x) - f(x)\|$$
$$\leq 3^{-\beta_2} \sum_{k=1}^{m} 3^{k(\beta_p - \beta_q)} \cdot 2 \cdot 3^{-\beta_p} \sum_{k=1}^{m} 3^{k(\beta_q - \beta_2)} \cdot 3^{-\beta_2} \cdot 3^{k(\beta_2 - \beta_2)} \varepsilon_1 \|3x\|^p + \sum_{k=1}^{m} 3^{k(\beta_q - \beta_2)} \cdot 3^{-\beta_2} \cdot 3^{k(\beta_2 - \beta_2)} \varepsilon_2 \|3x\|^q$$
$$\leq 2 \cdot 3^{-\beta_2} \cdot \varepsilon_1 \|x\|^p + (1 + 3^{\beta_q}) \cdot 3^{-\beta_2} \varepsilon_2 \|x\|^q$$
$$= \sum_{k=1}^{m+1} 3^{k(\beta_p - \beta_q)} \cdot 2 \cdot 3^{-\beta_p} \varepsilon_1 \|x\|^p + \sum_{k=1}^{m+1} 3^{k(\beta_q - \beta_2)} \cdot 3^{-\beta_2} \cdot 3^{k(\beta_2 - \beta_2)} \varepsilon_2 \|x\|^q.$$  

Therefore (5) is proved.

Let
$$T(x) = \lim_{n \to \infty} \frac{f(3^n x)}{3^n}.$$  

It is easy to see that \( T \) exists. In fact,
$$\left\| \frac{f(3^n x)}{3^n} - \frac{f(3^{n-1} x)}{3^{n-1}} \right\|$$
$$= \left\| \frac{1}{3^n} \left[ f(3^{n-1} (3x)) - f(3^n x) \right] \right\|$$
$$\leq \frac{1}{3^{n\beta_2}} \sum_{k=1}^{m-n} 3^{k(\beta_p - \beta_q)} \cdot 2 \cdot 3^{-\beta_p} \varepsilon_1 \|3^n x\|^p + \sum_{k=1}^{m-n} 3^{k(\beta_q - \beta_2)} \cdot 3^{-\beta_2} \cdot 3^{k(\beta_2 - \beta_2)} \varepsilon_2 \|3^n x\|^q$$
$$\leq \frac{1}{3^{\beta_p - \beta_p}} \left[ \frac{2 \varepsilon_1}{3^{\beta_p - \beta_p}} \|x\|^p + \frac{(1 + 3^{\beta_q}) \varepsilon_2}{3^{\beta_p - \beta_p}} \|x\|^q \right]$$

for any \( m > n, \ m, n \in \mathbb{N} \). By virtue of \( \beta_p > \beta_p > 0 \), it follows that
$$\lim_{n \to \infty} \left\| \frac{f(3^n x)}{3^n} - \frac{f(3^n x)}{3^n} \right\| = 0.$$  

Thus \( \{ f(3^n x) \} \) is a Cauchy sequence. However the \( F \)-space is complete, thus \( \{ f(3^n x) \} \) converges. It follows that \( T(x) = \lim_{n \to \infty} \frac{f(3^n x)}{3^n} \) exists. Hence by letting \( n \to \infty \) in (5), one obtains
$$\|T(x) - f(x)\| \leq \frac{2 \varepsilon_1}{3^{\beta_2 - \beta_p}} \|x\|^p + \frac{(1 + 3^{\beta_q}) \varepsilon_2}{3^{\beta_2 - \beta_p}} \|x\|^q.$$
Now we shall deal with the additivity of $T$. On account of (6), one has

$$T(3^m x) = \lim_{n \to \infty} \frac{f(3^n(3^m x))}{3^n} \cdot 3^m = 3^m T(x).$$

And employing the condition (2), we set

$$T \left( \frac{x + y}{2} \right) = \lim_{n \to \infty} 2 \cdot \frac{1}{3^n} \left( \frac{3^n x + 3^n y}{2} \right) - \frac{1}{3^n} f(3^n x) - \frac{1}{3^n} f(3^n y)$$

$$\leq \lim_{n \to \infty} \left( \frac{\varepsilon_1}{3^n|\beta_2 - \beta_1 p|} \|x\|^p + \frac{\varepsilon_2}{3^n|\beta_2 - \beta_1 q|} \|y\|^q \right)$$

$$= 0.$$

By (3), (6), and (7), it follows

$$\|2T(2x) - 4T(x)\| = \|2T(2x) - T(3x) - T(x)\|$$

$$= 3^{-n} \left[ \|2T(3^n \cdot 2x) - T(3^n \cdot 3x) - T(3^n x)\| \right]$$

$$\leq 3^{-n \beta_2} \left( \|2T(3^n \cdot 2x) - 2f(3^n \cdot 2x)\| + \|T(3^n \cdot 3x) - f(3^n \cdot 3x)\| \right)$$

$$+ 3^{-n \beta_2} \left( \|T(3^n x) - f(3^n x)\| + \|2f \left( \frac{3^n(3x + x)}{2} \right) - f(3^n \cdot 3x) - f(3^n x) \| \right)$$

$$\leq \frac{2\varepsilon_1 \|x\|^p}{3^n|\beta_2 - \beta_1 p|} \left( 2^{\beta_1 p + \beta_2 + 3^{\beta_1 p} + 1} \right)$$

$$+ \frac{(1 + 3^{\beta_1 q})\varepsilon_2 \|x\|^q}{3^n|\beta_2 - \beta_1 q|} \left( 2^{\beta_1 q + \beta_2 q + 3^{\beta_1 q} + 1} \right)$$

$$+ \frac{3^{\beta_1 p} \cdot \varepsilon_1}{3^n|\beta_2 - \beta_1 p|} \|x\|^p + \frac{\varepsilon_2}{3^n|\beta_2 - \beta_1 q|} \|y\|^q.$$

Clearly, $\|2T(2x) - 4T(x)\| \to 0$ as $n \to \infty$. Thus, $2T(2x) = 4T(x)$. From (8), we get

$$T(x + y) = \frac{1}{2} \left( T(2x) + T(2y) \right) = T(x) + T(y).$$

We will prove the uniqueness of $T$. Suppose that $H : G \to E$ is another additive mapping satisfying (3) for all $x \in G$. It follows that

$$\|T(x) - H(x)\| = \frac{1}{n^{\beta_2}} \|T(nx) - H(nx)\|$$

$$= \frac{1}{n^{\beta_2}} \|T(nx) - f(nx) - H(nx) + f(nx)\|$$

$$\leq \frac{1}{n^{\beta_2}} \left( \|T(nx) - f(nx)\| + \|H(nx) - f(nx)\| \right)$$

$$\leq \frac{4\varepsilon_1 \|x\|^p}{n^{\beta_2 - \beta_1 p} \cdot (3^{\beta_2 - 3^{\beta_1 p}})} + \frac{2(1 + 3^{\beta_1 q})\varepsilon_2 \|x\|^q}{n^{\beta_2 - \beta_1 q} \cdot (3^{\beta_2 - 3^{\beta_1 q}})}.$$
When $\beta_2 < p, q < \frac{1}{\beta_1}$, we claim that
\[
\|3^n f(3^{-n}) - f(x)\| \\
\leq \sum_{k=1}^{n} 3^{k(\beta_2 - \beta_1 p)} \cdot 2 \cdot 3^{-\beta_2 \varepsilon_1} \|x\|^p \\
+ \sum_{k=1}^{n} (3^{k(\beta_2 - \beta_1 q)} \cdot 3^{-\beta_2} + 3^{(k-1)(\beta_2 - \beta_1 q)} \varepsilon_2) \|x\|^q.
\]

Note that substituting $3^{-n}x$ by $x$ in (5) and later multiplying both sides by $3^n \beta_2$, we can yield the above formula (9).

Define $T(x) = \lim_{n \to \infty} 3^n f(3^{-n}x)$. The rest of the proofs follows as that in the case of $\beta_2 < p, q < \beta_2 \beta_1$, and therefore we omit it.

Consequently, we obtain
\[
\|T(x) - f(x)\| \leq \frac{2\varepsilon_1}{3^{\beta_1 p} - 3^{\beta_2}} \|x\|^p + \frac{(1 + 3^{\beta_1 q})\varepsilon_2}{3^{\beta_1 q} - 3^{\beta_2}} \|x\|^q.
\]

Moreover, if for each fixed $x \in G$, there exists a real number $\delta_x > 0$, such that $f(tx)$ is continuous on $[0, \delta_x]$, we claim that $f(tx)$ is bounded on $[0, \delta_x]$. Otherwise, if this were not the case then for any $n \in \mathbb{N}$, there exists $t_n \in [0, \delta_x]$ such that $\|f(t_n x)\| \geq n$. For the bounded sequence $\{t_n\}$, we could apply the Bolzano-Weierstass theorem to find a convergent subsequence $\{t_{n_k}\}$ and $t_0 \in [0, \delta_x]$ such that $\lim_{k \to \infty} t_{n_k} = t_0$. It follows that $\lim_{k \to \infty} t_{n_k} x = t_0 x$ for each fixed $x \in G$. Since $f(tx)$ is continuous in $t_0$, we can conclude that $\lim_{k \to \infty} f(t_{n_k} x) = f(t_0 x)$. Thus, we get a contradiction to $\lim_{k \to \infty} \|f(t_{n_k})\| = \infty$. The remaining proof follows a similar argument as in the proof of [16], hence we obtain that $T(x)$ is linear. Thus, claim is given.

\[\square\]

Remark 1. Let $G$ and $E$ be a $\beta_1$-homogeneous $F^*$-space and a $\beta_2$-homogeneous $F$-space, respectively. Suppose that $f : G \to E$ satisfies
\[
\left\| 2f\left( \frac{x + y}{2} \right) - f(x) - f(y) \right\| \leq \delta.
\]

Then there exists a unique additive mapping $T : G \to E$ such that
\[
\|T(x) - f(x)\| \leq \frac{2\delta}{3^{\beta_2} - 1}
\]
for all $x \in G$.

Now we construct an $F$-norm satisfying the condition that there exists $0 < \beta < 1$ such that $\frac{\|x\|}{\|y\|} \leq \frac{\|x\|}{\|y\|}$ but not $\beta$-homogeneity. So, the condition of spaces $G$ and $E$ in theorem can be weakened.
Example 1. We define the non-negative function $\| \cdot \|$ in $\mathbb{R}$ by

$$
\| x \| = \begin{cases} 
|x|^\beta & |x| \leq 1 \\
|x| & |x| > 1 
\end{cases} \quad (\forall x \in \mathbb{R}).
$$

Then $\| \cdot \|$ is an $F$-norm with the property that $\| x^n \| \leq \frac{\| x \|}{n^\beta}$ ($n \in \mathbb{N}$), but not the $\beta$-homogeneity.

Proof. We have only to show that $\| \cdot \|$ satisfies the triangle inequality. To establish one, we shall consider three cases. In the case where $|x| > 1, |y| > 1$, one has

$$
\| x + y \| = |x + y| \leq |x| + |y| = \| x \| + \| y \|.
$$

In the case where $|x| < 1, |y| < 1$, and likewise $|x + y| \leq 1$,

$$
\| x + y \| = |x + y|^\beta \leq (|x| + |y|)^\beta \leq |x|^\beta + |y|^\beta = \| x \| + \| y \|,
$$

or $|x| < 1, |y| < 1$ and likewise $|x + y| > 1$, and therefore

$$
\| x + y \| = |x + y| \leq |x| + |y| \leq |x|^\beta + |y|^\beta = \| x \| + \| y \|.
$$

While in the case where $|x| > 1, |y| < 1$ or $|x| < 1, |y| > 1$, we might as well suppose that $|x| > 1, |y| < 1$. Then if $|x + y| \leq 1$ holds, we obtain

$$
\| x + y \| = |x + y|^\beta \leq |x|^\beta + |y|^\beta \leq |x| + |y|^\beta = \| x \| + \| y \|.
$$

However, if $|x + y| > 1$ then,

$$
\| x + y \| = |x + y| \leq |x| + |y| \leq |x| + |y|^\beta = \| x \| + \| y \|.
$$

Therefore $\| \cdot \|$ is an $F$-norm.

Now we will prove that $\left\| \frac{x}{n} \right\| \leq \frac{\| x \|}{n^\beta}$ for any $n \in \mathbb{N}$. Indeed, when $|x| \leq n$, then

$$
\left\| \frac{x}{n} \right\| = \left| \frac{x}{n} \right|^\beta = \frac{1}{n^\beta} |x|^\beta = \frac{1}{n^\beta} \| x \| \| x \|.
$$

and when $|x| > n$, one has

$$
\left\| \frac{x}{n} \right\| = \frac{|x|}{n^\beta} \leq \frac{|x|}{n^\beta} = \frac{\| x \|}{n^\beta}.
$$

It follows that $\left\| \frac{x}{n} \right\| \leq \frac{\| x \|}{n^\beta}$ for any $x \in \mathbb{R}$.

It is easy to see that the $\| \cdot \|$ is not $\beta$-homogeneous.

Therefore the proof is completed. $\square$

3. Instability of Eq.(1)

We will first cite the counterexample constructed by Z. Gajda [3].

Example 2. For a fixed $\varepsilon > 0$ and $\mu = \frac{\varepsilon}{6}$, define a function $f : \mathbb{R} \to \mathbb{R}$ by

$$
f(x) = \sum_{n=0}^{\infty} \frac{\phi(2^n x)}{2^n} \quad x \in \mathbb{R},
$$

where $\phi$ is a non-negative function on $\mathbb{R}$ such that $\phi(x) = 0$ for $|x| > 1$.
where the function \( \phi : \mathbb{R} \to \mathbb{R} \) is given by

\[
\phi(x) = \begin{cases} 
\mu x & x \leq 1, \\
\mu^2 x & -1 < x < 1, \\
-\mu x & x \leq -1.
\end{cases}
\]

**Theorem 3.1.** The function \( f \) defined above satisfies

\[
|f(x + y) - f(x) - f(y)| \leq \varepsilon(|x| + |y|^{\frac{1}{2}})
\]

for all \( x, y \in \mathbb{R} \). However

\[
\sup\left\{ \frac{|f(x) - T(x)|}{|x|} : x \in \mathbb{R}\setminus\{0\} \right\} = \infty
\]

for each additive mapping \( T : \mathbb{R} \to \mathbb{R} \).

**Proof.** The inequality (10) is trivially fulfilled if \( x = y = 0 \).

Now, we assume that \( |x| + |y|^{\frac{1}{2}} < 1 \). Then \( |x| < 1, |y|^{\frac{1}{2}} < 1 \). There exists an \( N \in \mathbb{N} \) such that

\[
2^{N-1}(|x| + |y|^{\frac{1}{2}}) < 1, \quad 2^N(|x| + |y|^{\frac{1}{2}}) \geq 1.
\]

Since \( |x| + |y| \leq |x| + |y|^{\frac{1}{2}} \), we get \( 2^{N-1}(|x| + |y|) \leq 2^{N-1}(|x| + |y|^{\frac{1}{2}}) < 1 \). Hence,

\[
|2^{N-1}(x + y)| \leq 2^{N-1}(|x| + |y|) < 1 \quad \text{and} \quad |2^{N-1}x| < 1, \quad |2^{N-1}y| < 1,
\]

which means that for each \( n \in \{0, 1, 2, \ldots, N-1\} \), \( 2^{n-1}x, 2^{n-1}y, 2^{n-1}(x + y) \in (-1, 1) \). Since \( \phi \) is a linear mapping on the interval, we infer that

\[
\phi(2^n(x + y)) = \phi(2^n x) + \phi(2^n y)
\]

for \( n = 0, 1, \ldots, N-1 \). As a result, we obtain

\[
\frac{|f(x + y) - f(x) - f(y)|}{|x| + |y|^{\frac{1}{2}}} \leq \sum_{n=0}^{\infty} \frac{|\phi(2^n(x + y)) - \phi(2^n x) - \phi(2^n y)|}{2^n(|x| + |y|^{\frac{1}{2}})}
\]

\[
= \sum_{n=N}^{\infty} \frac{|\phi(2^n(x + y)) - \phi(2^n x) - \phi(2^n y)|}{2^n(|x| + |y|^{\frac{1}{2}})}
\]

\[
\leq \sum_{k=0}^{\infty} \frac{3\mu}{2^{k} \cdot 2^N(|x| + |y|^{\frac{1}{2}})} \leq \sum_{k=0}^{\infty} \frac{3\mu}{2^k} = 6\mu.
\]

Finally, assume that \( |x| + |y|^{\frac{1}{2}} \geq 1 \). Then because of the boundedness of \( f \), we have

\[
\frac{|f(x + y) - f(x) - f(y)|}{|x| + |y|^{\frac{1}{2}}} \leq 6\mu = \varepsilon,
\]

since

\[
|f(x)| \leq \sum_{n=0}^{\infty} 2\mu, \quad x \in \mathbb{R}.
\]

Thus, we conclude that \( f \) satisfies (10) for all \( x, y \in \mathbb{R} \). The proof of the last assertion in the theorem follows the same argument as in [3]. \( \square \)
Remark 2. Let the function $f$ be as before.

(i) If $G = (\mathbb{R}, \| \cdot \|_1)$ with the Euclidean metric $\| \cdot \|_1 = | \cdot |$ and $E = (\mathbb{R}, \| \cdot \|_2)$ with the $\beta$-homogeneous norm $\| \cdot \|_2 = | \cdot |^\beta$, then

$$\| f(x + y) - f(x) - f(y) \|_2 \leq \varepsilon (|x|^{\beta_1} + |y|^{\beta_2})$$

for any $x, y \in \mathbb{R}$, however

$$\sup \left\{ \frac{\| f(x) - T(x) \|_2}{\| x \|_1^{\beta_1}} : x \in \mathbb{R} \setminus \{0\} \right\} = \infty$$

for each additive mapping $T: G \rightarrow E$.

(ii) If $G = (\mathbb{R}, \| \cdot \|_1)$ with the $\beta$-homogeneous norm $\| \cdot \|_1 = | \cdot |^\beta$ and $E = (\mathbb{R}, \| \cdot \|_2)$ with the Euclidean metric $\| \cdot \|_2 = | \cdot |$, then

$$\| f(x + y) - f(x) - f(y) \|_2 \leq \varepsilon (\|x\|_1^{\beta_1} + \|y\|_1^{\beta_2})$$

for any $x, y \in \mathbb{R}$, however

$$\sup \left\{ \frac{\| f(x) - T(x) \|_2}{\| x \|_1^{\beta_1}} : x \in \mathbb{R} \setminus \{0\} \right\} = \infty$$

for each additive mapping $T: G \rightarrow E$.

(iii) If $G = (\mathbb{R}, \| \cdot \|_1)$ with the $\beta_1$-homogeneous norm $\| \cdot \|_1 = | \cdot |^{\beta_1}$ and $E = (\mathbb{R}, \| \cdot \|_2)$ with the $\beta_2$-homogeneous norm $\| \cdot \|_2 = | \cdot |^{\beta_2}$, then

$$\| f(x + y) - f(x) - f(y) \|_2 \leq \varepsilon^{\beta_2} (\|x\|_1^{\beta_1} + \|y\|_1^{\beta_2})$$

for any $x, y \in \mathbb{R}$, however

$$\sup \left\{ \frac{\| f(x) - T(x) \|_2}{\| x \|_1^{\beta_1}} : x \in \mathbb{R} \setminus \{0\} \right\} = \infty$$

for each additive mapping $T: G \rightarrow E$.

Remark 3. Set $\mu = \frac{\varepsilon}{8}$. By using a similar proof as in Theorem 3.1 for Jensen’s equation, we can also get

$$\left| 2f \left( \frac{x + y}{2} \right) - f(x) - f(y) \right| \leq \varepsilon (|x| + |y|^{\frac{1}{2}}),$$

however

$$\sup \left\{ \frac{|f(x) - T(x)|}{|x|} : x \in \mathbb{R} \setminus \{0\} \right\} = \infty$$

for each additive mapping $T: \mathbb{R} \rightarrow \mathbb{R}$.

Thus, we can obtain a conclusion similar to remark 2 relating to Jensen’s equation. This leads to the fact that the stability of Jensen’s equation does not hold as long as one of the numbers $p, q$ equals $\beta$, $\frac{1}{p}$ or $\frac{2}{\beta_1}$ ($0 < \beta_1, \beta_2 \leq 1$).
In summary, under the condition that $G$ and $E$ are $F$-spaces with certain property, one is interested to prove that the Hyers-Ulam-Rassias stability is fulfilled in three cases: $(\Delta_1) p, q < \beta_2$ (see [24]), $(\Delta_2) p, q > \frac{1}{\beta_1}$ (see [24]) and $(\Delta_3) \beta_2 < p, q < \frac{1}{\beta_1}$, $(p, q \neq \frac{\beta_2}{\beta_1})$, but this fails as long as $p$ or $q$ is equal to $\beta_2$, $\frac{1}{\beta_1}$ or $\frac{\beta_2}{\beta_1}$ ($0 < \beta_1, \beta_2 \leq 1$).

References


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