THE DISJOINT CURVE PROPERTY AND BRIDGE SURFACES

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Abstract. We show that every bridge surface of certain types of $(1,1)$ prime knot has the disjoint curve property. Also we determine when a bridge surface of a pretzel knot of type $(-2,3,n)$ has the disjoint curve property.

1. Introduction

Let $M$ denote a compact orientable 3-manifold and let $(W_1, W_2; H)$ be a genus $g$ Heegaard splitting of $M$. In 1960s, W. Haken [4] introduced a condition of Heegaard splittings which is called the reducibility. A splitting $(W_1, W_2; H)$ is said to be reducible if there are essential disks $D_i \subset W_i$ ($i = 1, 2$) with $\partial D_1 = \partial D_2$. Otherwise, $(W_1, W_2; H)$ is said to be irreducible. He proved that all splittings of a reducible manifold are themselves reducible. Strongly irreducibility was introduced by A. Casson and McA. Gordon [3] as a generalization of irreducibility. A splitting $(W_1, W_2; H)$ is said to be weakly reducible if there are essential disks $D_i \subset W_i$ ($i = 1, 2$) with $\partial D_1 \cap \partial D_2 = \emptyset$. Otherwise, $(W_1, W_2; H)$ is said to be strongly irreducible. They showed that all splittings of a non-Haken manifold are either reducible or strongly irreducible. A. Thompson [17] introduced the notion of disjoint curve property as a further generalization of reducibility. A splitting $(W_1, W_2; H)$ admits the disjoint curve property if there are essential disks $D_i \subset W_i$ ($i = 1, 2$) and an essential loop $c \subset H$ with $(\partial D_1 \cup \partial D_2) \cap c = \emptyset$. A splitting is full if it does not have the disjoint curve property. A. Thompson proved that all splittings of a toroidal 3-manifold have the disjoint curve property in [17]. And J. Hempel has shown that each splitting of a Seifert fibred space has the disjoint curve property using the classification of splittings of Seifert fibered spaces in [8]. Thus, in any 3-manifold which is reducible, toroidal, or a Seifert fibred space, all Heegaard splittings have the disjoint curve property.
However, it is certainly not the case that all splittings of all manifolds have the disjoint curve property. J. Hempel in [8] adapts an argument of T. Kobayashi [11] to produce examples of splittings which are full and are in fact arbitrarily far from having the disjoint curve property.

In this paper, we show that every bridge surface of certain types of (1, 1)-prime knot has the disjoint curve property. T. Saito has shown that a bridge surface of (1, 1)-hyperbolic knot is full, except for certain type of knots in [15]. Also we determine that every bridge surface of the pretzel knot of type (−2, 3, n) has the disjoint curve property.

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2. Preliminaries

In this section, we recall standard notation and review several notions from the theory of \(g\)-genus \(n\)-bridge Heegaard splittings. We would like to refer [5] for more detail about relationship between definitions.

Let \(W\) be a 3-manifold with non-empty boundary \(\partial W\) and \(K = \{k_1, k_2, \ldots, k_n\}\) be a set of disjoint arcs properly embedded in \(W\), that is, \(k_i \cap \partial W = \partial k_i\) for every \(1 \leq i \leq n\).

**Definition 2.1.** \(K\) is **trivial** in \((W, K)\) if there is a set \(\{D_1, \ldots, D_n\}\) of disjoint discs embedded in \(W\) such that \(D_i \cap (\cup k_i) = \partial D_i \cap k_i = k_i\) and so that \(D_i \cap \partial W\) is the arc \(\text{cl}(\partial D_i - k_i)\). We call \(D_i\) a cancelling disc of \(k_i\).

When \(W\) is a ball and \(K\) is trivial, the pair \((W, K)\) is called a **trivial** \(n\)-string 

**Definition 2.2.** \(F\) is **incompressible** in \((W, T)\) if there is a disc \(D\) embedded in \(W\) such that \(D \cap \partial W = \partial D\) and that \(\partial D\) does not bound a disc disjoint from \(T\) on \(F\). Such \(D\) is called a **compressing disc** of \(F\). We call \(F\) a **compressible** if it is not **incompressible**.

**Definition 2.3.** A 2-manifold \(F\) is **meridionally incompressible** in \((W, T)\) if there is a disc \(D\) embedded in \(W\) such that \(\text{int} D\) intersects \(T\) transversely in a single point, that \(D \cap F = \partial D \cap (F - T) = \partial D\) and that \(\partial D\) in \(F\) does not bound a disc whose interior intersects \(T\) transversely in a single point. Such \(D\) is called a meridionally compressing disc of \(F\). \(F\) is **meridionally incompressible** if it is not meridionally compressible.

We define a **compressible** 2-submanifold and a meridionally compressible 2-submanifold of \(\partial W\) similarly. Assume that either \(F\) is a 2-manifold properly embedded in \(W\) such that \(F\) is transverse to \(T\), or \(F\) is a 2-submanifold of \(\partial W\)
with $\partial F \cap T = \emptyset$. A simple loop $l$ on $F$ is said to be $T$-essential if it is disjoint from $T$ and if it does not bound a disc which intersects $T$ transversely in zero or one point.

Let $M$ be a closed orientable 3-manifold and $L$ a link in $M$. Let $H$ be a genus $g$ Heegaard splitting surface of $M$, that is, $H$ divides $M$ into two handlebodies $W_1$ and $W_2$ of genus $g$. Suppose that $H$ is transverse to $L$.

**Definition 2.4.** $H$ is a $g$-genus $n$-bridge splitting, $(g,n)$-splitting, of $(M, L)$ if $L$ intersects $W_i$ in a set of trivial $n$ arcs for $i = 1, 2$.

A link $L$ is called a $g$-genus $n$-bridge link, $(g,n)$-link, if it admits a $g$-genus $n$-bridge splitting. A link in $S^3$ is simply called an $n$-bridge link in $S^3$ if it has a 0-genus $n$-bridge splitting.

Let $K$ be a knot in a closed 3-manifold $M$. Let $(M, K) = (W_1, k_1) \cup_H (W_2, k_2)$ be a bridge splitting of a 3-manifold $(M, K)$.

**Definition 2.5.** $H$ is $K$-reducible if $W_1$ and $W_2$ contain $K$-compressing or meridionally compressing discs $D_1$ and $D_2$ of $H$ respectively such that $\partial D_1 = \partial D_2 = \emptyset$. $H$ is $K$-irreducible if it is not $K$-reducible.

Note that if $H$ is $K$-reducible, then $K$ is the trivial knot bounding a disc composed of two cancelling discs as shown in [6].

**Definition 2.6.** $H$ is weakly $K$-reducible if $W_1$ and $W_2$ contain $K$-compressing or meridionally compressing discs $D_1$ and $D_2$ of $H$ respectively such that $\partial D_1 \cap \partial D_2 = \emptyset$. $H$ is strongly $K$-irreducible if it is not weakly $K$-reducible.

**Definition 2.7.** $H$ has the disjoint curve property if there exist essential simple closed curves $c$, $\partial D_1$, and $\partial D_2$ on $H$ such that $c \cap (\partial D_1 \cup \partial D_2) = \emptyset$ and $D_1$ and $D_2$ are $K$-compressing or meridionally compressing discs in $W_1$ and $W_2$ respectively.

By $E(K)$, we mean the exterior of $K$ in $M$, i.e., $E(K) = \text{cl}(M - N(K))$, where $N(K)$ is a regular neighborhood of $K$ in $M$.

**Definition 2.8.**
1. $K$ is a trivial knot if it bounds a disc imbedded in $M$.
2. $K$ is a core knot if $K$ is non-trivial and $M$ admits a genus one Heegaard splitting $(V_i, V_2; P)$ such that $K$ is isotopic to the core of $V_i$ for $i = 1$ or 2.
3. $K$ is a 2-bridge knot if there is a genus zero Heegaard splitting $(B_1, B_2 : F_0)$ of $S^3$ such that $(B_i, B_i \cap K)$ $(i = 1, 2)$ is a 2-string trivial tangle.
4. $K$ is split if $M$ contains a sphere $S$ which decomposes $M$ into a punctured lens space and a ball containing $K$ in its interior. This sphere $S$ is called a splitting sphere.
5. $K$ is a composite knot if $M$ contains a 2-sphere $S$ which intersects $K$ transversely in 2 points and $S \cap E(K)$ is $\partial$-incompressible in $E(K)$. We call this 2-sphere $S$ a decomposing sphere. A knot is said to be prime if it is not composite.
It is proved in [7] that if $H$ is weakly $K$-reducible, then $K$ is the trivial knot or a 2-bridge knot when $M = S^3$, and $K$ is a core knot or a composite knot of a core knot and a 2-bridge knot when $M$ is a lens space.

**Definition 2.9.** A knot $K$ is called a *torus knot* if $K$ is isotopic to a simple loop on a genus one Heegaard surface of $M$ and is not a core knot. The torus knot $T_{p,q}$ of type $(p, q)$ is the knot which wraps around the standard solid torus $p$ times in the longitudinal direction, and $q$ times in the meridional direction.

![Figure 1. Satellite knot](image)

**Definition 2.10.** A knot $K$ is said to be *satellite* if $E(K)$ contains an incompressible torus $T$ which is not parallel to $\partial E(K)$.

**Definition 2.11.** A *hyperbolic knot* is a knot whose complement can be endowed with a metric of constant curvature $-1$.

The seminal work of W. Thurston demonstrates that every knot in $S^3$ is either a torus knot, a satellite knot or a hyperbolic knot. These three categories are mutually exclusive.

### 3. Heegaard splitting

In Section 3, we may consider 3-manifolds with nonempty boundaries such as knot complements. In those cases, we use the notion of Heegaard splitting of compression body.

J. Hempel has shown that if a closed orientable 3-manifold $M$ is reducible, Seifert fibered or toroidal, then any splitting of $M$ has the disjoint curve property in [8].

J. Hempel’s result is generalized to the Proposition 3.1 by T. Saito [16]. He introduced a notion called the disjoint $(A, D)$-pair property. Here, a Heegaard splitting $(W_1, W_2; H)$ admits the *disjoint $(A, D)$-pair property* if there are an essential annulus $A_i$ normally embedded in $W_i$ and an essential disk $D_j$ in $W_j$, $((i, j) = (1, 2)$ or $(2, 1))$ such that $\partial A_i \cap \partial D_j = \emptyset$. 

Proposition 3.1 ([16]). Let $M$ be a compact orientable 3-manifold. If $M$ is reducible, Seifert fibered or toroidal, then any genus $g \geq 2$ Heegaard splitting of $M$ admits the disjoint $(A,D)$-pair property. Moreover, if a Heegaard splitting admits the disjoint $(A,D)$-pair property, then it admits the disjoint curve property.

We can deduce the following corollary from Proposition 3.1.

Corollary 3.2. If $K$ is a torus knot in $S^3$, then every genus $g \geq 2$ Heegaard splitting of $E(K)$ has the disjoint curve property.

Proof. Since $K$ is a torus knot, $E(K)$ is a Seifert fibered manifold. Let $E(K) = W_1 \cup_H W_2$ for $g \geq 2$. By Proposition 3.1, $H$ has the disjoint curve property. □

A 3-manifold is toroidal if it contains an essential torus, namely an incompressible torus which is not parallel to a boundary component.

Corollary 3.3. If $K$ is a satellite knot in $S^3$, then every genus $g \geq 2$ Heegaard splitting of $E(K)$ has the disjoint curve property.

Proof. Since $K$ is a satellite knot, $E(K)$ is toroidal. Let $E(K) = W_1 \cup_H W_2$ for $g \geq 2$. By Proposition 3.1, $H$ has the disjoint curve property. □

4. Bridge surface

The next two propositions guarantee the existence of cancelling discs satisfying certain conditions for a Heegaard surface which is in (1,1) position with respect to the given knot.

Proposition 4.1 ([13, Theorem 3]). Let $L$ be a lens space and $K$ a 1-bridge knot in $L$, and let $(W_1,W_2;H)$ be a Heegaard splitting of genus one of $L$ which gives a 1-bridge representation of $K$ i.e., $\alpha_i = W_i \cap K$ is a single trivial arc in $W_i$ ($i = 1, 2$).

Suppose that $K$ is a non-trivial torus knot and is not a core of $L$. Then for $i = 1, 2$, there exists a disk $\Delta_i$ in $W_i$ such that $\partial W_i \cap \Delta_i = \beta_i$ is an arc in $\partial W_i$, $\partial \Delta_i = \alpha_i \cup \beta$, and $\beta_1 \cap \beta_2 = \partial \beta_1 = \partial \beta_2$.

Proposition 4.2 ([7, Theorem III]). Let $M$ be the $S^3$ or a lens space (not homeomorphic to $S^2 \times S^1$). Let $H$ be a genus 1 Heegaard surface of $M$. This surface $H$ divides $M$ into two solid tori $W_1$ and $W_2$. Suppose a knot $K$ is in 1-genus 1-bridge position with respect to $H$ and $H$ is neither $K$-reducible nor weakly $K$-reducible.

If $K$ is a satellite knot, then there is an annulus $Z$ on $H$ such that there is a cancelling disc $C_i$ of $t_i$ (trivial arc in $V_i$) with $\partial C_i \cap H \subset Z$ for $i = 1, 2$. Moreover, the incompressible torus is isotopic to $\partial N(C_1 \cup Z \cup C_2)$ in $E(K)$.

Using Propositions 4.1 and 4.2, we can deduce the disjoint curve property for bridge surfaces of non trivial torus knots and satellite knots.
Theorem 4.3. Let $M$ be the $S^3$ or a lens space and $K$ be a 1-bridge knot in $M$. If $K$ is a non-trivial torus knot in $M$, then the bridge surface $H$ has the disjoint curve property.

Proof. Any 1-bridge representation of a torus knot in a lens space is trivial. Let $(M, K) = (W_1, \alpha_1) \cup_H (W_2, \alpha_2)$ and $C_i$ be a cancelling disc of $\alpha_i$. By Proposition 4.1, $H \cap C_i = \beta_i$ is an arc in $H$. Then $A = N(\beta_1 \cup \beta_2)$ is an annulus in $H$. Now the disc $D_i = \text{cl}(\partial N(C_i, W_i) - H)$ is a properly embedded and $K$-compressing disc in $W_i$ for $i = 1, 2$. $\text{cl}(H - \text{int}(A))$ is an annulus in $H$. Moreover one can find an essential curve $c$ in $\text{cl}(H - \text{int}(A))$ such that $\partial D_1 \cap c = \emptyset$ and $\partial D_2 \cap c = \emptyset$. Therefore $H$ has the disjoint curve property. We describe explicitly how to get essential curves satisfying the disjoint curve property for a torus knot in Figure 2.

Theorem 4.4. Let $M$ be the $S^3$ or a lens space (not homeomorphic to $S^2 \times S^1$). Let $H$ be a genus 1 Heegaard surface of $M$. This surface $H$ divides $M$ into two solid tori $W_1$ and $W_2$. Suppose a knot $K$ is in 1-genus 1-bridge position with respect to $H$. If $K$ is a satellite knot, then $H$ has the disjoint curve property.
Proof. If $H$ is $K$-reducible, then $K$ is a trivial knot by [15]. If $H$ is weakly $K$-reducible, then $M$ is $S^2 \times S^1$ and $K$ is a core knot by [15]. By assumption, we may assume that $H$ is neither $K$-reducible nor weakly $K$-reducible. Let $(M, K) = (W_1, t_1) \cup_H (W_2, t_2)$, and $C_i$ be a cancelling disc of $t_i$. By Proposition 4.2, there exists an annulus $Z$ in $H$ such that there is a cancelling disc $C_i$ of $t_i$ with $(\partial C_i \cap H) \subset Z$ for $i = 1, 2$. If we take $D_i = \text{cl}(\partial N(C_i, W_i) - H)$, then it is a properly embedded and $K$-compressing disc in $W_i$ for $i = 1, 2$. Because $\text{cl}(H - \text{int}(Z))$ is an annulus, one can find an essential curve $c$ in $\text{cl}(H - \text{int}(Z))$ such that $\partial D_1 \cap c = \emptyset$ and $\partial D_2 \cap c = \emptyset$. We describe explicitly how to get essential curves which satisfy the disjoint curve property for the case when $\partial C_2 \cap H$ is meridinal in Figure 3, the case when $\partial C_2 \cap H$ is longitudinal in Figure 4 and the case when $\partial C_2 \cap H$ is neither meridinal nor longitudinal in Figure 5. The dotted line in each figure indicates the knot in a splitted solid torus. Therefore $H$ has the disjoint curve property. \[\square\]

Now we consider hyperbolic knots. A knot $K$ in an orientable closed 3-manifold $M$ is called a $(1, 1)$-knot if $(M, K) = (W_1, k_1) \cup_H (W_2, k_2)$. To define the distance of a $(1, 1)$-splitting, we
use the twice punctured torus $\Sigma = H - K$. For notation and definition, we refer to [15].

For $i = 1$ or 2, let $\mathcal{K}(W_i)$ be the maximal subcomplex of $C(\Sigma)$ consisting of simplexes $\langle [e_0], [e_1], \ldots, [e_k] \rangle$ such that an essential loop representing $[e_j]$ ($j = 0, 1, \ldots, k$) bounds a disk in $W_i - k_i$.

**Definition.** We define the distance of a $(1, 1)$-splitting $(W_1, W_2; H)$ by $d(W_1, W_2) = \min \{ d([x], [y]) : [x] : a \text{ vertex in } \mathcal{K}(W_1), [y] : a \text{ vertex in } \mathcal{K}(W_2) \}$.

For a pair $\alpha \geq 4$ and $\beta$ of co-prime integers and an element $\gamma \in \mathbb{Q} \cup \{1/0\}$, $K(\alpha, \beta; \gamma)$ denotes the knot $K_2$ in $K_1(\gamma)$, where $K_1 \cup K_2$ is the 2-bridge link of type $(\alpha, \beta)$ (cf. Chapter 10 of [14]) and $K_1(\gamma)$ is the manifold obtained by $\gamma$-surgery on $K_1$. By definition, every $K(\alpha, \beta; \gamma)$ is a $(1, 1)$-knot.

**Proposition 4.5** ([15]). Let $K$ be a $(1, 1)$-knot in $M$. Suppose that $(M, K)$ is not equivalent to $K(\alpha, \beta; \gamma)$ for any $\alpha, \beta$ and $\gamma$, and that the bridge index of $K$ is at least three if $M \cong S^3$. Then $K$ is a hyperbolic knot if and only if it has a $(1, 1)$-splitting with distance $\geq 3$.

By Proposition 4.5, one can see that a $(1, 1)$-knot is hyperbolic if a $(1, 1)$-splitting does not have the disjoint curve property and a $(1, 1)$-splitting with distance neither 0 nor 1.

Let $L(p_1, p_2, \ldots, p_n)$ be an $n$-pretzel link in $S^3$ where $p_i \in \mathbb{Z}$ represents the number of half twists. In particular, if $n = 3$, it is called a classical pretzel link, denoted by $L(p, q, r)$. For notation and definition for pretzel knots, we refer to [10].

If $n$ is odd, then an $n$-pretzel link $L(p_1, p_2, \ldots, p_n)$ is a knot if and only if none of two $p_i$’s are even. If $n$ is even, then $L(p_1, p_2, \ldots, p_n)$ is a knot if and only if one of the $p_i$’s is even. A pretzel knot is denoted by $K(p_1, p_2, \ldots, p_n)$.

A link $L$ is almost alternating if it is not alternating and there is a diagram $D_L$ of $L$ such that one crossing change makes the diagram alternating; we call $D_L$ an almost alternating diagram. By [10], classical pretzel links are prime and either alternating or almost alternating. W. Menasco has shown that prime alternating knots are either hyperbolic or torus knots [12]. It has been generalized by C. Adams that prime almost alternating knots are either hyperbolic or torus knots [2]. It is known that no satellite knot is an almost alternating knot [9]. By the following theorem [10], we can classify classical pretzel knots completely into hyperbolic or torus knots.

**Proposition 4.6** ([10]). The following are the only nontrivial pretzel knots which are torus knots.

1. $K(p, \pm 1, \mp 1)$ are unknots for all $p$.
2. $K(\pm 1, \pm 1, \pm 3)$ are $(2, \pm 3)$ torus knots.
3. $K(\pm 2, \mp 1, \pm 3)$ are $(2, \pm 3 \pm 2)$ torus knots.
4. $K(\pm 2, \pm 3, \pm 3), K(\mp 2, \pm 3, \pm 5)$ are $(3, \pm 4), (3, \pm 5)$ torus knots, respectively.
Now we consider a pretzel knot of type \((-2,3,n)\). Every \(K(-2,3,n)\) pretzel knot, \(n \in \mathbb{Z}\), can be described by the diagram as in Figure 6.

\[
\text{Figure 6. } (-2,3,n) \text{ pretzel knot diagram}
\]

**Theorem 4.7.** For pretzel knots of type \((-2,3,n)\), the following holds.

1. If \(n = 1, 3, 5\), then the genus one 1-bridge surface which has the disjoint curve property.
2. If \(n \neq 1, 3, 5\), then there is a genus two 1-bridge surface which has the disjoint curve property.

**Proof.** If \(n = 1, 3, 5\), then \(K(-2,3,n)\) is a nontrivial torus knot by Proposition 4.6. According to Theorem 4.3, genus one 1-bridge surface has the disjoint curve property.

If \(n \neq 1, 3, 5\), then each of these knots has a hyperbolic structure on its complement. Each pretzel knot of type \(K(-2,3,n)\) admits \((2,1)\)-splitting as follows. The given knot can be realized as a curve on the surface of a double torus. \(S^3\) has a genus two Heegaard splitting. Namely, \(S^3\) can be thought of as two double tori whose boundaries have been identified. Actually, \(S^3\) has a unique Heegaard splitting of each genus.

\[
\text{Figure 7. } (2,1)\text{-splitting}
\]

Let \((W_1, W_2; H)\) be a Heegaard splitting of genus 2 of \(S^3\) and the knot can be divided into two arcs namely, \(K(-2,3,n) = K_1 \cup K_2\). Let \(t_i\) be a trivial arc
in $W_i$ obtained from $K_i$ by pushing interior of $K_i$ into $\text{int} W_i$ for $i = 1, 2$. Then $(M, K) = (W_1, t_1) \cup_H (W_2, t_2)$ is a $(2, 1)$-splitting, and $C_i$ is a cancelling disc of $t_i$ (see Figure 7 for details).

If we take $D_i = \text{cl}(\partial N(C_i, W_i) - H)$, then it is a properly embedded $K$-compressing disc in $W_i$ for $i = 1, 2$.

Because $\partial(N(C_1, H) \cup N(C_2, H))$ is an annulus in $H$, one can find an essential curve $c$ in $\text{cl}(H - \partial(N(C_1, H) \cup N(C_2, H)))$ such that $\partial D_1 \cap c = \emptyset$ and $\partial D_2 \cap c = \emptyset$. Therefore $H$ has the disjoint curve property. This completes the proof of Theorem 4.7.

**Example.** We now describe how to construct $(2, 1)$-splitting and how to find 3 essential circles which satisfy the disjoint curve property for the $(-2, 3, 7)$-pretzel knot in Figure 8 and Figure 9.

**Figure 8.** $(2,1)$-splitting for a $(-2,3,7)$ pretzel knot

**Figure 9.** Essential curves satisfying the disjoint curve property for a $(-2,3,7)$ pretzel knot
References


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