SCREEN CONFORMAL LIGHTLIKE REAL HYPERSURFACES OF AN INDEFINITE COMPLEX SPACE FORM

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Abstract. In this paper, we study the geometry of screen conformal lightlike real hypersurfaces of an indefinite Kaehler manifold. The main result is a characterization theorem for screen conformal lightlike real hypersurfaces of an indefinite complex space form.

1. Introduction

It is well known that the normal bundle $TM^\perp$ of the lightlike hypersurfaces $M$ of a semi-Riemannian manifold $\bar{M}$ is a vector subbundle of the tangent bundle $TM$ of rank 1. Then there exists a complementary non-degenerate vector bundle $S(TM)$ of $TM^\perp$ in $TM$, which called a screen distribution on $M$, such that

$$TM = TM^\perp \oplus_{\text{orth}} S(TM),$$

where $\oplus_{\text{orth}}$ denotes the orthogonal direct sum. We denote such a lightlike hypersurface by $(M, g, S(TM))$. Denote by $F(M)$ the algebra of smooth functions on $M$ and by $\Gamma(E)$ the $F(M)$ module of smooth sections of a vector bundle $E$ over $M$. We use the same notation for any other vector bundle.

We known [2] that, for any null section $\xi$ of $TM^\perp$ on a coordinate neighborhood $U \subset M$, there exists a unique null section $N$ of a unique vector bundle $\text{tr}(TM)$ of rank 1 in $S(TM)^\perp$ satisfying

$$\bar{g}(\xi, N) = 1, \quad \bar{g}(N, N) = \bar{g}(N, X) = 0$$

for any $X \in \Gamma(S(TM))$. In this case, $TM$ is decomposed as follows:

$$\bar{TM} = TM \oplus \text{tr}(TM) = \{TM^\perp \oplus \text{tr}(TM)\} \oplus_{\text{orth}} S(TM).$$

We call $\text{tr}(TM)$ and $N$ the transversal vector bundle and the null transversal vector field of $M$ with respect to $S(TM)$, respectively.

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The purpose of this paper is to prove a characterization theorem for lightlike real hypersurfaces $M$ of an indefinite complex space form $\overline{M}(c)$: If $M$ is screen conformal, then $c = 0$ (Theorem 3.7). Using this theorem, we prove several additional theorems for screen conformal lightlike real hypersurfaces $M$ of $\overline{M}(c)$.

The local Gauss and Weingarten formulas are given by

\begin{align}
\bar{\nabla}_X Y &= \nabla_X Y + B(X, Y)N, \\
\bar{\nabla}_X N &= -A_N X + \tau(X)N, \\
\nabla_X PY &= \nabla^*_X PY + C(X, PY)\xi, \\
\nabla_X \xi &= -A^*_\xi X - \tau(X)\xi
\end{align}

for any $X, Y \in \Gamma(TM)$, where $\bar{\nabla}$, $\nabla$ and $\nabla^*$ are the Levi-Civita connection of $\overline{M}$, the linear connections on $TM$ and $S(TM)$ respectively, $P$ is the projection morphism of $\Gamma(TM)$ on $\Gamma(S(TM))$ with respect to the decomposition (1.1), $B$ and $C$ are the local second fundamental forms on $TM$ and $S(TM)$ respectively, $A_N$ and $A^*_\xi$ are the shape operators on $TM$ and $S(TM)$ respectively and $\tau$ is a 1-form on $TM$. Since $\bar{\nabla}$ is torsion-free, $\nabla$ is also torsion-free and $B$ is symmetric on $TM$. From the fact that $B(X,Y) = \bar{g}(\bar{\nabla}_X Y, \xi)$ for any $X, Y \in \Gamma(TM)$, we show that the local second fundamental form $B$ is independent of the choice of a screen distribution and satisfies

\begin{equation}
B(X, \xi) = 0
\end{equation}

for any $X \in \Gamma(TM)$. The induced connection $\nabla$ of $M$ is not metric and satisfies

\begin{equation}
(\nabla g)(Y, Z) = B(X, Y) \eta(Z) + B(X, Z) \eta(Y)
\end{equation}

for any $X, Y, Z \in \Gamma(TM)$, where $\eta$ is a 1-form such that

\begin{equation}
\eta(X) = \bar{g}(X, N)
\end{equation}

for any $X \in \Gamma(TM)$. But the connection $\nabla^*$ on $S(TM)$ is metric. Two local second fundamental forms $B$ and $C$ are related to their shape operators by

\begin{align}
B(X, Y) &= g(A^*_\xi X, Y), \\
C(X, PY) &= g(A_N X, PY), \\
g(A_N X, N) &= 0
\end{align}

for any $X, Y \in \Gamma(TM)$. From (1.11), the operator $A^*_\xi$ is $\Gamma(S(TM))$-valued self-adjoint on $\Gamma(TM)$ with respect to the induced metric $g$ on $M$ such that

\begin{equation}
A^*_\xi \xi = 0.
\end{equation}

Thus $\xi$ is an eigenvector of $A^*_\xi$ corresponding to the eigenvalue 0.

We denote by $\hat{R}$, $R$ and $R^*$ the curvature tensors of the Levi-Civita connection $\nabla$ of $M$, the induced connection $\nabla$ of $M$ and the connection $\nabla^*$ on $S(TM)$, respectively. Using the Gauss-Weingarten equations for $M$ and $S(TM)$, we obtain the Gauss-Codazzi equations for $M$ and $S(TM)$ such that

\begin{equation}
\bar{g}(\hat{R}(X, Y)Z, PW) = g(R(X, Y)Z, PW) + B(X, Z)C(Y, PW) - B(Y, Z)C(X, PW),
\end{equation}
Therefore the general decompositions (1.1) and (1.3) become respectively
\[
\tilde{g} (\tilde{R}(X, Y) Z, \xi) = g(R(X, Y) Z, \xi) \\
= (\nabla_X B)(Y, Z) - (\nabla_Y B)(X, Z) \\
+ B(Y, Z) \tau(X) - B(X, Z) \tau(Y),
\]
(1.15)
\[
\tilde{g}(\tilde{R}(X, Y) Z, N) = g(R(X, Y) Z, N),
\]
(1.16)
\[
g(R(X, Y) PZ, PW) = g(R'(X, Y) PZ, PW) \\
+ C(X, PZ) B(Y, PW) \\
- C(Y, PZ) B(X, PW),
\]
(1.17)
\[
g(R(X, Y) PZ, N) = (\nabla_X C)(Y, PZ) - (\nabla_Y C)(X, PZ) \\
+ C(X, PZ) \tau(Y) - C(Y, PZ) \tau(X)
\]
(1.18)
for any \(X, Y, Z, W \in \Gamma(TM)\). The Ricci tensor \(\tilde{Ric}\) of \(\tilde{M}\) is defined by
\[
\tilde{Ric}(X, Y) = \text{trace}\{Z \to \tilde{R}(Z, X) Y\}, \quad \forall X, Y \in \Gamma(\tilde{M}).
\]
\(\tilde{M}\) is called \textit{Ricci flat} if its Ricci tensor vanishes identically. If dim\(\tilde{M} > 2\) and \(\tilde{Ric} = \gamma g\), where \(\gamma\) is a constant, then \(\tilde{M}\) is called an \textit{Einstein manifold}.

2. Hypersurfaces of indefinite Kaehler manifolds

Let \(\tilde{M} = (\tilde{M}, J, \tilde{g})\) be a real \(2m\)-dimensional indefinite Kaehler manifold, where \(\tilde{g}\) is a semi-Riemannian metric of index \(q = 2v (0 < v < m)\) and \(J\) is an almost complex structure on \(\tilde{M}\) satisfying, for all \(X, Y \in \Gamma(TM)\),
\[
J^2 = -I, \quad \tilde{g}(JX, JY) = \tilde{g}(X, Y), \quad (\nabla_X J)Y = 0.
\]
(2.1)
An indefinite complex space form, denoted by \(\tilde{M}(c)\), is a connected indefinite Kaehler manifold of constant holomorphic sectional curvature \(c\) such that
\[
\tilde{R}(X, Y) Z = \frac{c}{4} \{ \tilde{g}(Y, Z)X - \tilde{g}(X, Z)Y + \tilde{g}(JY, Z)JX \\
- \tilde{g}(JX, Z)JY + 2\tilde{g}(X, JY)JZ \}
\]
(2.2)
for all \(X, Y, Z \in \Gamma(TM)\). Suppose \((\tilde{M}, g, S(TM))\) is a lightlike real hypersurface of \(\tilde{M}\), where \(g\) is the degenerate induced metric of \(\tilde{M}\). Then the screen distribution \(S(TM)\) splits as follows [2]:

If \(\xi\) and \(N\) are local sections of \(TM^\perp\) and \(\text{tr}(TM)\) respectively, we have
\[
\tilde{g}(J\xi, \xi) = \tilde{g}(J\xi, N) = \tilde{g}(JN, \xi) = \tilde{g}(JN, N) = 0, \quad \tilde{g}(J\xi, JN) = 1.
\]
(2.3)
This shows that \(J\xi\) and \(JN\) are vector fields tangent to \(M\). Thus \(J(TM^\perp)\) and \(J(\text{tr}(TM))\) are distributions on \(M\) of rank 1 such that \(TM^\perp \cap J(TM^\perp) = \{0\}\) and \(TM^\perp \cap J(\text{tr}(TM)) = \{0\}\). Hence \(J(TM^\perp) \oplus J(\text{tr}(TM))\) is a vector subbundle of \(S(TM)\) of rank 2. Then there exists a non-degenerate almost complex distribution \(D_o\) on \(M\) with respect to \(J\), i.e., \(J(D_o) = -D_o\), such that
\[
S(TM) = \{ J(TM^\perp) \oplus J(\text{tr}(TM)) \} \oplus_{\text{orth}} D_o.
\]
(2.4)
Therefore the general decompositions (1.1) and (1.3) become respectively
\[
TM = \{ J(TM^\perp) \oplus J(\text{tr}(TM)) \} \oplus_{\text{orth}} D_o \oplus_{\text{orth}} TM^\perp,
\]
(2.5)
Let \( (2.18) \)
\[ U = (2.8) \]
\[ D = (2.7) \]
It is easy to check that
Then the tangent bundle
Differentiate (2.8) with \( = (2.12) \)
and use (2.1)-3 and (2.9)-2, we have
Thus \((2.10)\)
and the local lightlike vector fields \( U \) and \( V \) such that
\[ U = -JN; \quad V = -J\xi. \]
Denote by \( S \) the projection morphism of \( TM \) on \( D \) with respect to the decomposition \((2.7)\).
Then any vector field \( X \) on \( M \) is expressed as follows
\[ X = SX + u(X)U; \quad JX = FX + u(X)N, \]
where \( u \) and \( v \) are 1-forms locally defined on \( M \) by
\[ u(X) = g(X, V), \quad v(X) = g(X, U) \]
and \( F \) is a tensor field of type \((1, 1)\) globally defined on \( M \) by
\[ FX = JSX, \quad \forall X \in \Gamma(TM). \]
Apply \( J \) to the second equation of \((2.9)\) and using \((2.1)\) and \((2.8)\), we have
\[ F^2X = -X + u(X)U; \quad u(U) = 1. \]
Thus \( (F, u, U) \) defines an almost contact structure on \( M \). But it is not an almost contact metric structure. Because, using \((2.1)\)-2 and \((2.9)\)-2, we have
\[ g(FX, FY) = g(X, Y) - u(X)v(Y) - u(Y)v(X) \]
for all \( X, Y \in \Gamma(TM) \). By using \((2.9)\)-2 and \((2.10)\) and Gauss-Weingarten equations for a lightlike hypersurface, for any \( X, Y \in \Gamma(TM) \), we deduce
\[ (\nabla_Xu)(Y) = -u(Y)(X) - B(X, FY), \]
\[ (\nabla_Xv)(Y) = v(Y)(X) - g(A_NX, FY), \]
\[ (\nabla_XF)(Y) = u(Y)A_NX - B(X, Y)U. \]
Differentiate \((2.8)\) with \( X \) and use \((1.5)\), \((1.7)\), \((2.1)\)-3 and \((2.9)\)-2, we have
\[ B(X, U) = v(A^*_NX) = u(A_NX) = C(X, V), \quad \forall X \in \Gamma(TM), \]
\[ \nabla_XU = F(A_NX) + \tau(X)U; \quad \nabla_XV = F(A^*_NX) - \tau(X)V. \]

**Example 1.** Let \( (\mathbb{R}^6_2, g) \) be a 6-dimensional semi-Euclidean space of index 2 with signature \((-,-,+,+,+,+)\) of the canonical basis \((\partial_0, \ldots, \partial_5)\). Consider a Monge hypersurface \( M \) of \( \mathbb{R}^6_2 \) given by
\[ x_0 = u_1 + u_2 + u_3 \quad \text{and} \quad x_i = u_i (1 \leq i \leq 5). \]
Then the tangent bundle \( TM \) is spanned by
\[ \{ \partial_{u_1} = \partial_0 + \partial_1, \quad \partial_{u_2} = \partial_0 + \partial_2, \quad \partial_{u_3} = \partial_0 + \partial_3, \quad \partial_{u_4} = \partial_4, \quad \partial_{u_5} = \partial_5 \}. \]
It is easy to check that \( M \) is a lightlike hypersurface whose radical distribution \( \text{Rad}(TM) \) is spanned by
\[ \xi = \partial_0 - \partial_1 + \partial_2 + \partial_3. \]
Let \( V = \partial_0 - \partial_1 \), then \( g(V, V) = -2 \) and \( g(\xi, V) = -2 \). Then the lightlike transversal vector bundle is given by
\[
\text{tr}(TM) = \text{Span}\{N = -\frac{1}{4}(\partial_0 - \partial_1 - \partial_2 - \partial_3)\}.
\]
It follows that the corresponding screen distribution \( S(TM) \) is spanned by
\[
\{W_1 = \partial_0 + \partial_1, \ W_2 = \partial_2 - \partial_3, \ W_3 = \partial_4, \ W_4 = \partial_5\}.
\]
Since \( \mathbb{R}^5 \) has complex structure \( J \), we see that \( J\xi = W_1 - W_2 \in \Gamma(S(TM)) \)
\( JN = -\frac{1}{4}\{W_1 + W_2\} \in \Gamma(S(TM)) \), \( JW_3 = W_4 \) and \( JW_4 = -W_3 \). Thus the almost complex distribution \( D_\alpha \) is given by \( D_\alpha = \text{Span}\{W_3, W_4\} \).

**Theorem 2.1.** Let \( (M, g, S(TM)) \) be a lightlike real hypersurface of an indefinite Kaehler manifold \( M \). Then we have the following assertions.

(i) If \( F \) and \( V \) are parallel with respect to the induced connection \( \nabla \) on \( M \), then \( M \) is totally geodesic in \( M \) and the 1-form \( \tau \) vanishes.

(ii) If \( V \) and \( U \) are parallel with respect to the induced connection \( \nabla \) on \( M \), then \( S(TM) \) is totally geodesic in \( M \) and the 1-form \( \tau \) vanishes.

**Proof.** If \( V \) is parallel with respect to the induced connection \( \nabla \) on \( M \), then, from the second equation of (2.18), we have
\[
J(A^*_X N) - u(A^*_X)N + \tau(X)V = 0, \forall X \in \Gamma(TM).
\]
Apply \( J \) to the last equation and by using (2.1) and (2.8), we obtain
\[
A^*_X U = u(A^*_X)U \quad \text{and} \quad \tau(X) = 0, \forall X \in \Gamma(TM).
\]
Substituting the last equation in (2.17), we have
\[
u(A_X N) = v(A^*_U) = g(A^*_X, U) = u(A^*_X)g(U, U) = 0, \forall X \in \Gamma(TM).\]
(i) If \( F \) is parallel with respect to \( \nabla \), then, from (2.16), we have
\[
B(X, Y) = u(Y)u(A_X N), \quad \forall X, Y \in \Gamma(TM).
\]
Thus if \( V \) is also parallel, we obtain \( B = 0 \), that is, \( M \) is totally geodesic in \( M \).

(ii) If \( U \) is parallel with respect to \( \nabla \), then, from (2.18)-1, we have
\[
J(A_X N) - u(A_X N)N + \tau(X)U = 0, \quad \forall X \in \Gamma(TM).
\]
Apply \( J \) to this equation and by using (2.1) and (2.8), we obtain
\[
A_X U = u(A_X)U \quad \text{and} \quad \tau(X) = 0, \forall X \in \Gamma(TM).
\]
Thus if \( V \) is also parallel, we obtain \( A_X N = 0 \) for all \( X \in \Gamma(TM) \). Thus \( C = 0 \) due to (1.12), that is, \( S(TM) \) is totally geodesic in \( M \). \( \square \)

**Theorem 2.2.** Let \( (M, g, S(TM)) \) be a lightlike real hypersurface of an indefinite Kaehler manifold \( M \). If \( F \) is parallel with respect to the induced connection \( \nabla \), then the almost complex distribution \( D \) is parallel with respect to the induced connection \( \nabla \) and \( M \) is locally a product manifold \( L_u \times M^4 \), where \( L_u \) is a null curve tangent to \( J(\text{tr}(TM)) \) and \( M^4 \) is a leaf of \( D \).
Proof. In general, by using (1.4), (1.7), (1.11) and (2.1), we derive
\begin{equation}
(2.20) \quad g(\nabla_X \xi, J\xi) = -g(\xi, \nabla_X J\xi) = B(X, V), \quad g(\nabla_X J\xi, J\xi) = 0,
\end{equation}
\begin{equation}
\quad g(\nabla_X Y, J\xi) = g(JY, \nabla_X \xi) = -g(JY, A^*_\xi X) = -B(X, JY)
\end{equation}
for all $X \in \Gamma(TM)$ and $Y \in \Gamma(D_o)$. If $F$ is parallel with respect to the induced connection $\nabla$, then, taking $Y = V$ and $Y \in \Gamma(D_o)$ in (2.19) by turns, we have $B(X, V) = 0$ and $B(X, Y) = 0$ for all $X \in \Gamma(TM)$ respectively. It follow that $g(\nabla_X \xi, J\xi) = g(\nabla_X J\xi, J\xi) = g(\nabla_X Y, J\xi) = 0$ due to $JY \in \Gamma(D_o)$. Thus $D$ is parallel with respect to $\nabla$ and both $D$ and $J(\text{tr}(TM))$ are integrable distributions. Thus we obtain our theorem. $\square$

3. Screen conformal lightlike real hypersurfaces

A lightlike hypersurface $(M, g, S(TM))$ of a semi-Riemannian manifold $(\bar{M}, \bar{g})$ is screen conformal [1] if the shape operators $A_N$ and $A^*_\xi$ of $M$ and $S(TM)$ respectively are related by $A_N = \varphi A^*_\xi$, or equivalently
\begin{equation}
(3.1) \quad C(X, PY) = \varphi B(X, Y), \quad \forall X, Y \in \Gamma(TM),
\end{equation}
where $\varphi$ is a non-vanishing smooth function on a neighborhood $\mathcal{U}$ in $M$. In particular, if $\varphi$ is a non-zero constant, $M$ is called screen homothetic [4].

Note 1. For a screen conformal $M$, since $C$ is symmetric on $\Gamma(S(TM))$, $S(TM)$ is integrable. Thus $M$ is locally a product manifold $L_\xi \times M^*$ where $L_\xi$ is a null curve tangent to $TM^\perp$ and $M^*$ is a leaf of $S(TM)$ [2].

From (2.17) and (3.1), we obtain
\begin{equation}
(3.2) \quad B(X, U - \varphi V) = 0, \quad \forall X \in \Gamma(TM).
\end{equation}

Theorem 3.1. Let $(M, g, S(TM))$ be a screen conformal lightlike real hypersurface of an indefinite Kaehler manifold $\bar{M}$. Then the non-null vector field $U - \varphi V \neq 0$ is conjugate to any vector field on $M$. In particular, $U - \varphi V$ is an asymptotic vector field.

Corollary 1. Let $(M, g, S(TM))$ be a screen conformal lightlike real hypersurface of an indefinite Kaehler manifold $\bar{M}$. Then the second fundamental form $B$ (consequently, $C$) is degenerate on $\Gamma(S(TM))$.

Proof. Since $B(X, U - \varphi V) = 0$ for all $X \in \Gamma(S(TM))$ and $U - \varphi V \in \Gamma(S(TM))$, therefore $B$ is degenerate on $\Gamma(S(TM))$. $\square$

Theorem 3.2. Let $(M, g, S(TM))$ be a screen conformal lightlike real hypersurface of an indefinite Kaehler manifold $\bar{M}$. If $M$ or $S(TM)$ is totally umbilic, then $M$ is totally geodesic in $M$ and the leaf $M^*$ of $S(TM)$ is totally geodesic in both $M$ and $\bar{M}$.

Proof. If $M$ is a totally umbilical lightlike real hypersurface of $\bar{M}$, then there exists a smooth function $\rho$ such that
\begin{equation}
B(X, Y) = \rho g(X, Y), \quad \forall X, Y \in \Gamma(TM).
\end{equation}
From this fact and the equation (3.2), we have
\[ pg(X, U - \varphi V) = 0, \quad \forall X \in \Gamma(TM). \]
Replace X by V and U by turns in the last equation, we have \( \rho = 0 \) and \( \varphi \rho = 0 \) respectively. Thus \( B = C = 0 \), that is, \( M \) and \( S(TM) \) are totally geodesic. By the same method for totally umbilical \( S(TM) \), we have \( B = C = 0 \).

**Theorem 3.3.** Let \( (M, g, S(TM)) \) be a screen conformal lightlike real hypersurface of an indefinite Kaehler manifold \( M \). If one of the set \( \{V, U, F\} \) is parallel with respect \( \nabla \) on \( M \), then \( M \) is totally geodesic in \( M \) and \( S(TM) \) is totally geodesic in both \( M \) and \( M \). Moreover, if \( V \) or \( U \) is parallel, then \( \tau = 0 \).

**Proof.** In the proof of Theorem 2.1, if \( V \) is parallel, then \( \tau = 0 \), \( u(A_N X) = 0 \) and \( A_\xi^* X = u(A_\xi^* X)U \) for any \( X \in \Gamma(TM) \). Using the second equation of the above relations and the fact that \( A_N = \varphi A_\xi^* \), we have
\[ u(A_\xi^* X) = u(A_N X)/\varphi = 0, \quad \forall X \in \Gamma(TM). \]
From this and the fact that \( A_\xi^* X = u(A_\xi^* X)U \) for all \( X \in \Gamma(TM) \), we have \( A_\xi^* = 0 \). Also \( A_N = \varphi A_\xi^* = 0 \). Thus \( M \) and \( S(TM) \) are totally geodesic.

If \( U \) is parallel, then \( \tau = 0 \) and \( A_N X = u(A_N X)U \) for any \( X \in \Gamma(TM) \). Thus we have \( v(A_N X) = 0 \) for any \( X \in \Gamma(TM) \). Using the equation (2.17) and the fact that \( A_N = \varphi A_\xi^* \), we have
\[ u(A_N X) = v(A_N X) = v(A_N X)/\varphi = 0, \quad \forall X \in \Gamma(TM). \]
It follow that \( A_N = 0 \) and \( A_\xi^* = 0 \). Thus \( M \) and \( S(TM) \) are totally geodesic.

If \( F \) is parallel, then we have (2.19). Replace \( Y \) by \( V \) in (2.19), we have
\[ u(A_N X) = \varphi u(A_\xi^* X) = \varphi B(X, V) = 0, \quad \forall X \in \Gamma(TM). \]
Thus, from (2.19) and (3.1), we have \( B = C = 0 \). \( \Box \)

From the equation (2.20) and Theorems 3.2 and 3.3, we have:

**Theorem 3.4.** Let \( (M, g, S(TM)) \) be a screen conformal lightlike real hypersurface of an indefinite Kaehler manifold \( M \). If (i) \( M \) or \( S(TM) \) is totally umbilic, or (ii) one of the set \( \{V, U, F\} \) is parallel with respect \( \nabla \), then \( D \) is parallel with respect to \( \nabla \) and \( M \) is locally a product manifold \( L_u \times M^2 \), where \( L_u \) is a null curve tangent to \( J(tr(TM)) \) and \( M^2 \) is a leaf of \( D \).

As \( \{U, V\} \) is a basis of \( \Gamma(J(TM^\perp) \oplus J(tr(TM))) \), the vector fields
\[ \mu = U - \varphi V, \quad \nu = U + \varphi V \]
form an orthogonal basis of \( \Gamma(J(TM^\perp) \oplus J(tr(TM))) \). From (3.2), we have
\[ g(A_\xi^* \mu, X) = B(\mu, X) = 0, \quad g(A_\xi^* \mu, N) = 0, \quad A_\xi^* \mu = 0, \]
that is, \( \mu \) is an eigenvector field of \( A^*_\xi \) on \( S(TM) \) corresponding to the eigenvalue 0. Let \( G(\mu) = \text{Span}\{\mu\} \). Then \( S(\mu) = D_0 \oplus_{\text{orth}} \text{Span}\{\nu\} \) is a complementary vector subbundle to \( G(\mu) \) in \( S(TM) \) and we have the following decomposition
\[
S(TM) = G(\mu) \oplus_{\text{orth}} S(\mu).
\]

**Theorem 3.5.** Let \((M, g, S(TM))\) be a screen conformal lightlike real hypersurface of an indefinite Kaehler manifold \( M \). Then the non-null vector field \( \mu \) is parallel with respect to \( \nabla \) if and only if the 1-form \( \tau \) vanishes and the conformal factor \( \varphi \) is a constant.

**Proof.** From (2.18), (3.3) and the linearity of \( \nabla \), we have
\[
\nabla_X \mu = \tau(X) \nu - X[\varphi] V, \quad \forall X \in \Gamma(TM),
\]
due to \( A_N = \varphi A^*_\xi \). Thus we see that \( \mu \) is parallel if and only if
\[
\tau(X) U - \{X[\varphi] - \varphi \tau(X)\} V = 0, \quad \forall X \in \Gamma(TM).
\]
Taking the scalar product with \( V \) and \( U \) in turns, we get assertion. \( \Box \)

**Note 2.** From (2.18) and (3.4), we have
\[
\nabla_X \nu = 2F(A_N X) - \tau(X) \mu + X[\varphi] V, \quad \forall X \in \Gamma(TM).
\]
Thus, using the fact \( g(F(A_N X), V) = g(F(A_N X), U) = 0 \), we show that \( \nu \) is parallel if and only if \( \tau = 0 \) on \( M \), \( \varphi \) is a constant and both \( U \) and \( V \) are parallel. Moreover if \( \nu \) is parallel, then \( \mu \) is also parallel and \( B = C = 0 \).

**Theorem 3.6.** Let \((M, g, S(TM))\) be a screen conformal lightlike real hypersurface of an indefinite Kaehler manifold \( M \). If \( \mu \) is parallel with respect to \( \nabla \), then \( M \) is locally a product manifold \( L_\xi \times L_\mu \times M^2 \), where \( L_\xi \) and \( L_\mu \) are null and non-null geodesic tangent to \( TM^1 \) and \( G(\mu) \) respectively and \( M^2 \) is a leaf of \( S(\mu) \). Moreover, \( M \) is screen homothetic.

**Proof.** In general, using (3.6), for \( X \in \Gamma(S(\mu)) \) and \( Y \in \Gamma(D_0) \), we derive
\[
(3.7) \quad g(\nabla_X Y, \mu) = g(\nabla_X Y, \mu) = -g(Y, \nabla_X \mu) = 0,
\]
\[
(3.8) \quad g(\nabla_Y \nu, \mu) = -g(\nu, \nabla_Y \mu) = Y[\varphi] - 2\varphi \tau(Y).
\]
If \( \mu \) is parallel, then \( g(\nabla_X Y, \mu) = g(\nabla_X Y, \mu) = 0 \). Thus \( S(\mu) \) is an integrable distribution. From this fact and Note 1, we obtain our theorem. \( \Box \)

**Corollary 2.** Let \((M, g, S(TM))\) be a screen conformal lightlike real hypersurface of an indefinite Kaehler manifold \( M \). If \( \mu \) is parallel with respect to \( \nabla \), then \( M \) is locally a product manifold \( L_\xi \times M^2 \), where \( L_\mu \) is a non-null geodesic tangent to \( G(\mu) \) and \( M^2 \) is a leaf of \( R(\mu) = D_0 \oplus_{\text{orth}} \text{Span}\{\xi, \nu\} \).

**Proof.** From (1.1) and (3.5), we have \( TM = G(\mu) \oplus_{\text{orth}} R(\mu) \). For any \( X \in \Gamma(R(\mu)) \) and \( Y \in \Gamma(D_0) \), we get
\[
(3.9) \quad g(\nabla_Y \xi, \mu) = -g(A^*_\xi Y, \mu) = -g(Y, A^*_\xi \mu) = 0,
\]
\[
g(\nabla_Y \nu, \mu) = -g(\nu, \nabla_Y \mu) = Y[\varphi] - 2\varphi \tau(Y),
\]
Let

\[ g(\nabla_X Y, \mu) = g(\nabla_X Y, \mu) = -g(Y, \nabla_X \mu) = 0. \]

Thus the distribution \( \mathcal{R}(\mu) \) is integrable. We have our assertion. \( \square \)

**Theorem 3.7.** Let \((M, g, S(TM))\) be a screen conformal lightlike real hypersurface of an indefinite complex space form \( \hat{M}(c) \). Then we have \( c = 0 \). In particular, the ambient manifold \( \hat{M}(c) \) is a semi-Euclidean space.

**Proof.** By using \((1.15)\) and \((2.2)\), we have

\begin{align*}
(3.9) \quad \frac{c}{4} & \left\{ u(X)\tilde{g}(JY, Z) - u(Y)\tilde{g}(JX, Z) + 2u(Z)\tilde{g}(X, JY) \right\} \\
& = (\nabla_X B)(Y, Z) - (\nabla_Y B)(X, Z) + B(Y, Z)\tau(X) - B(X, Z)\tau(Y)
\end{align*}

for all \( X, Y, Z \in \Gamma(TM) \). Using this, \((1.16)\), \((1.18)\) and \((3.2)\), we obtain

\begin{align*}
(3.10) \quad \frac{c}{4} & \left\{ g(Y, PZ)\eta(X) - g(X, PZ)\eta(Y) + v(X)\tilde{g}(JY, PZ) \right. \\
& \quad - v(Y)\tilde{g}(JX, PZ) + 2v(PZ)\tilde{g}(X, JY) \bigg\} \\
& = \{ X[\varphi] - 2\varphi\tau(X) \} B(Y, PZ) - \{ Y[\varphi] - 2\varphi\tau(Y) \} B(X, PZ) \\
& \quad + \frac{c}{4} \{ u(X)\tilde{g}(JY, PZ) - u(Y)\tilde{g}(JX, PZ) + 2u(PZ)\tilde{g}(X, JY) \}.
\end{align*}

Replacing \( Y \) by \( \xi \) in \((3.10)\), we obtain

\begin{align*}
(3.11) \quad \{ \xi[\varphi] - 2\varphi\tau(\xi) \} B(X, PZ) \\
& = \frac{c}{4} \left\{ g(X, PZ) + v(X)u(PZ) + 2u(X)v(PZ) - 3\varphi u(X)v(PZ) \right\}.
\end{align*}

Taking \( X = V; \ PZ = U \) and \( X = U; \ PZ = V \), we have

\begin{align*}
(3.12) \quad \{ \xi[\varphi] - 2\varphi\tau(\xi) \} B(V, U) &= \frac{1}{2}c, \quad \{ \xi[\varphi] - 2\varphi\tau(\xi) \} B(U, V) = \frac{3}{4}c,
\end{align*}

respectively. From the two equation of \((3.12)\), we show that \( c = 0 \). Therefore, \( \hat{M}(c) \) is a semi-Euclidean space. \( \square \)

**Corollary 3.** There exist no screen conformal lightlike real hypersurfaces \( M \) of indefinite complex space form \( \hat{M}(c) \) with \( c \neq 0 \).

The type number \( t^*(p) \) of \( M \) at a point \( p \in M \) is the rank of the shape operator \( A^*_\mathbf{x} \) at \( p \). Then, from \((3.7)\) and \((3.8)\), we obtain:

**Theorem 3.8.** Let \((M, g, S(TM))\) be a screen conformal lightlike real hypersurface of an indefinite complex space form \( \hat{M}(c) \) such that \( t^*(p) > 1 \) for any \( p \in M \). Then \( M \) is locally a product manifold \( L_\xi \times L_\mu \times M^2 \), where \( L_\xi \) and \( L_\mu \) are null and non-null curve tangent to \( TM^\perp \) and \( G(\mu) \) respectively and \( M^2 \) is a leaf of \( S(\mu) \).

**Proof.** First, for any \( X \in \Gamma(S(\mu)) \) and \( Y \in \Gamma(D_\phi) \), since \( g(Y, U) = g(Y, V) = 0 \) for \( Y \in \Gamma(D_\phi) \), we show that

\begin{align*}
& g(\nabla_X Y, \mu) = g(\nabla_X Y, \mu) = -g(Y, \nabla_X \mu) = -g(Y, \nabla_X \mu) \\
& = X[\varphi]g(Y, V) - \tau(X)g(Y, \nu) = -\tau(X)\{ g(Y, U) + \varphi g(Y, V) \} = 0.
\end{align*}
Thus (3.7) holds. Next, from the equation (3.10) with \( c = 0 \), we obtain
\[
\{X[\varphi] - 2\varphi\tau(X)\}A^*_\xi Y = \{Y[\varphi] - 2\varphi\tau(Y)\}A^*_\xi X.
\]
Suppose there exists a vector field \( X_o \in \Gamma(TM) \) such that \( X_o[\varphi] - 2\varphi\tau(X_o) \neq 0 \). Then \( A^*_\xi Y = fA^*_\xi X_o \) for any \( Y \in \Gamma(TM) \), where \( f \) is a smooth function. It follows that the rank of \( A^*_\xi \) is 1. It is a contradiction as rank \( A^*_\xi > 1 \). Consequently, we have \( X[\varphi] - 2\varphi\tau(X) = 0 \) for all \( X \in \Gamma(TM) \) on \( U \). Thus (3.8) also holds. Therefore \( S(\mu) \) is integrable distribution by (3.7) and (3.8). Consequently, we have our theorem.

\[\Box\]

4. Screen conformal Einstein hypersurfaces

Let \( R^{(0,2)} \) denote the induced Ricci type tensor of \( M \) given by
\[
R^{(0,2)}(X, Y) = \text{trace}\{Z \to R(Z, X)Y\}
\]
for any \( X, Y \in \Gamma(TM) \). Consider the induced quasi-orthonormal frame field \( \{\xi; W_a\} \) on \( M \) such that \( \text{Rad}(TM) = \text{Span}\{\xi\} \) and \( S(TM) = \text{Span}\{W_a\} \). Using this quasi-orthonormal frame field and the equation (3.1), we obtain
\[
R^{(0,2)}(X, Y) = \sum_{a=1}^{m} \epsilon_a g(R(W_a, X)Y, W_a) + g(R(\xi, X)Y, N)
\]
for any \( X, Y \in \Gamma(TM) \) and \( \epsilon_a = g(W_a, W_a) \) is the sign of \( W_a \). In general, the induced Ricci type tensor \( R^{(0,2)} \), defined by the method of the geometry of the non-degenerate submanifolds [8], is not symmetric [3, 5]. Therefore \( R^{(0,2)} \) has no geometric or physical meaning similar to the Ricci curvature of the non-degenerate submanifolds and it is just a tensor quantity. Hence we need the following definition: A tensor field \( R^{(0,2)} \) of lightlike submanifolds \( M \) is called its induced Ricci tensor if it is symmetric. A symmetric \( R^{(0,2)} \) tensor will be denoted by \( \text{Ric} \). If \( M \) is a screen conformal lightlike real hypersurface of a complex space form \( M(c) \), then \( c = 0 \). Using (1.14) and (1.16), we have
\[
R^{(0,2)}(X, Y) = \varphi\{B(X, Y)\text{tr}A^*_\xi - g(A^*_\xi X, A^*_\xi Y)\}, \quad \forall X, Y \in \Gamma(TM).
\]

Theorem 4.1. Let \((M, g, S(TM))\) be a screen conformal lightlike real hypersurface of an indefinite complex space form \( M(c) \). Then the Ricci type tensor \( R^{(0,2)} \) is a symmetric Ricci tensor \( \text{Ric} \).

Note 3. Suppose the Ricci type tensor \( R^{(0,2)} \) is symmetric. Then there exists a pair \( \{\xi, N\} \) on \( U \) such that the corresponding 1-form \( \tau \) vanishes [2]. We call such a pair a distinguished null pair \([5]\) of \( M \). Although, in general, \( S(TM) \) is not unique, it is canonically isomorphic to the factor vector bundle \( S(TM)^1 = TM/\text{Rad}(TM) \) considered by Kupeli [7]. Thus all \( S(TM) \) are mutually isomorphic. For this reason, in the sequel, let \((M, g, S(TM))\) be a screen homothetic lightlike real hypersurface equipped with the distinguished null pair \( \{\xi, N\} \) of an indefinite complex space form \( (M(c), \bar{g}) \).
Theorem 4.2. Let \((M, g, S(TM))\) be a screen homothetic lightlike real hypersurface of an indefinite complex space form \(\bar{M}(c)\). Then \(M\) is locally a product manifold \(L_\xi \times L_\mu \times M^2\), where \(L_\xi\) and \(L_\mu\) are null and non-null geodesics respectively and \(M^2\) is a leaf of some non-degenerate distribution.

Proof. Since \(M\) is a screen homothetic lightlike real hypersurface equipped with a distinguished null pair \(\{\xi, N\}\), from (1.7), (1.13) and (3.6), we have \(\nabla_\xi \xi = \nabla_\mu \mu = 0\). In particular, \(\mu\) is a parallel vector field with respect to \(\nabla\) due to (3.6). Thus, by Theorem 3.6, we have our theorem. □

Theorem 4.3. Any screen conformal Einstein lightlike real hypersurface of an indefinite complex space form \(\bar{M}(c)\) is Ricci flat.

Proof. Since \(M\) is a screen conformal lightlike real hypersurface of an indefinite complex space form \(M(c)\), we get \(c = 0\). The induced tensor \(R^{(0, 2)}\) is a symmetric Ricci tensor \(Ric\) by (4.3). Let \(M\) be an Einstein manifold, that is, \(R^{(0, 2)} = \gamma g\) for some constant \(\gamma\). Then the equation (4.3) reduces to

\[
g(A_\xi^e X, A_\xi^e Y) - \alpha g(A_\xi^e X, Y) - \gamma \varphi^{-1} g(X, Y) = 0,
\]

where \(\alpha = \text{tr} A_\xi^e\) is trace of \(A_\xi^e\). Put \(X = Y = \mu\) in (4.4) and using the fact that \(A_\xi^e \mu = 0\) due to (3.4), we have \(\gamma = 0\). Thus \(M\) is Ricci flat. □

Theorem 4.4. Let \((M, g, S(TM))\) be a screen homothetic Einstein lightlike real hypersurface of an indefinite complex space form \(M(c)\) of index 2. Then \(M\) is locally a product manifold \(L_\xi \times L_\mu \times M^2\) or \(L_\xi \times L_\mu \times L_\alpha \times M^0\), where \(L_\xi\), \(L_\mu\) and \(L_\alpha\) are null geodesic, timelike geodesic and spacelike curve respectively and \(M^2\) and \(M^0\) are Euclidean spaces.

Proof. Let \(\mu = \frac{1}{\sqrt{2c^2}} \{U - \varphi V\}\), where \(\epsilon = \text{sgn} \varphi\). Then \(\mu\) is a unit timelike eigenvector field of \(A_\xi^e\) corresponding to the eigenvalue 0 by (3.4) and \(S(\mu)\) is an integrable Riemannian distribution by Theorem 4.2, due to \(q = 2\). Since \(g(A_\xi^e X, N) = 0\) and \(g(A_\xi^e X, \mu) = 0\), \(A_\xi^e\) is \(\Gamma(S(\mu))\)-valued real self-adjoint operator. Thus \(A_\xi^e\) have \((2m-3) \equiv n\) real orthonormal eigenvector fields in \(S(\mu)\) and is diagonalizable. Consider a frame field of eigenvectors \(\{\mu, e_1, \ldots, e_n\}\) of \(A_\xi^e\) on \(S(TM)\) such that \(\{e_1, \ldots, e_n\}\) is an orthonormal frame field of \(A_\xi^e\) on \(S(\mu)\). Then \(A_\xi^e e_i = \lambda_i e_i (1 \leq i \leq n)\). Put \(X = Y = e_i\) in (4.4) with \(\gamma = 0\), we show that each eigenvalue \(\lambda_i\) of \(A_\xi^e\) is a solution of the equation

\[
x(x - \alpha) = 0.
\]

The equation (4.5) has at most two distinct real solutions 0 and \(\alpha\) on \(\mathcal{H}\). Assume that there exists \(p \in \{0, \ldots, n\}\) such that \(\lambda_1 = \cdots = \lambda_p = 0\) and \(\lambda_{p+1} = \cdots = \lambda_n = \alpha\), by renumbering if necessary. Then we have

\[
\alpha = \text{tr} A_\xi^e = (n - p)\alpha.
\]

If \(\alpha = 0\), then \(A_\xi^e X = 0\) for all \(X \in \Gamma(TM)\). Also we have \(A X = 0\) for all \(X \in \Gamma(TM)\). Thus \(M\) and \(S(TM)\) are totally geodesic. From (1.14) and
(1.17), we have $R^*(X,Y)Z = R(X,Y)Z = 0$ for all $X, Y, Z \in \Gamma(S(TM))$. Thus $M$ is locally a product manifold $L_\xi \times (M^\ast = L_\mu \times M^0)$, where $L_\xi$ and $L_\mu$ are null and timelike geodesic tangent to $TM^\perp$ and $g(\mu)$ respectively and the leaf $M^\ast$ of $S(TM)$ is a Minkowski space. Since $\nabla_X \mu = 0$ and

$$g(\nabla_X Y, \mu) = -g(Y, \nabla_X \mu) = -g(Y, \nabla_X \mu) = 0$$

for all $X, Y, Z \in \Gamma(S(TM))$, we have $\nabla_X Y \in \Gamma(S(\mu))$ and $R^*(X,Y)Z \in \Gamma(S(\mu))$. This imply $\nabla_X Y = Q(\nabla_X Y)$, that is, $M^3$ is a totally geodesic and $R^*(X,Y)Z = Q(R^*(X,Y)Z) = 0$, where $Q$ is a projection morphism of $S(TM)$ on $S(\mu)$ with respect to the decomposition (3.5). Thus $M^3$ is a Euclidean space.

If $\alpha \neq 0$, then $p = n - 1$. Consider the following two distributions on $S(\mu)$:

$$\Gamma(E_0) = \{X \in \Gamma(S(\mu)) \mid A_2^X X = 0\}.$$

$$\Gamma(E_\alpha) = \{X \in \Gamma(S(\mu)) \mid A_2^X X = \alpha X\}.$$

Then we know that the distributions $E_0$ and $E_\alpha$ are mutually orthogonal non-degenerate subbundle of $S(\mu)$, of rank $(n - 1)$ and 1 respectively, satisfy $S(\mu) = E_0 \oplus_{\text{orth}} E_\alpha$. From (4.4), we get $A_2^X (A_2^X - \alpha Q) = 0$. Using this equation, we have $\text{Im} A_2^X \subset \Gamma(E_0)$ and $\text{Im} (A_2^X - \alpha Q) \subset \Gamma(E_0)$. For any $X, Y \in \Gamma(E_0)$ and $Z \in \Gamma(S(\mu))$, we get $(\nabla_X B)(Y, Z) = -g(A_2^X \nabla_X Y, Z)$. Use this and the fact $(\nabla_X B)(Y, Z) = (\nabla_Y B)(X, Z)$, we have $g(A_2^X [X, Y], Z) = 0$. If we take $Z \in \Gamma(E_\alpha)$, since $\text{Im} A_2^X \subset \Gamma(E_0)$ and $E_0$ is non-degenerate, we have $A_2^X [X, Y] = 0$. Thus $[X, Y] \in \Gamma(E_0)$ and $E_0$ is integrable. Thus $M$ is locally a product manifold $L_\xi \times (M^\ast = L_\mu \times L_\nu \times M^0)$, where $L_\nu$ is a spacelike curve and $M^0$ is an $(n - 1)$-dimensional Riemannian manifold satisfy $A_2^X = 0$. From (1.14) and (1.18), we have $R^*(X,Y)Z = R(X,Y)Z = 0$ for all $X, Y, Z \in \Gamma(E_0)$. Since $g(\nabla_X Y, \mu) = 0$ and $g(\nabla_X Y, e_\alpha) = -g(Y, \nabla_X e_\alpha) = 0$ for all $X, Y \in \Gamma(E_0)$ because $\nabla_X W \in \Gamma(E_0)$ for $X \in \Gamma(E_0)$ and $W \in \Gamma(E_\alpha)$. In fact, from (1.15) such that $c = \tau = 0$, we get

$$g(\{(A_2^X - \alpha Q)\nabla_X W - A_2^X \nabla_W X\}, Z) = 0$$

for all $X \in \Gamma(E_0)$, $W \in \Gamma(E_\alpha)$ and $Z \in \Gamma(S(\mu))$. Using the fact that $S(\mu)$ is non-degenerate distribution, we have $(A_2^X - \alpha Q)\nabla_X W = A_2^X \nabla_W X$. Since the left term of this equation is in $\Gamma(E_0)$ and the right term is in $\Gamma(E_\alpha)$ and $E_0 \cap E_\alpha = \{0\}$, we have $(A_2^X - \alpha Q)\nabla_X W = 0$ and $A_2^X \nabla_W X = -X[\varphi]W$. This imply that $\nabla_X W \in \Gamma(E_\alpha)$. Thus $\nabla_X Y = \pi \nabla_X Y$ for all $X, Y \in \Gamma(E_0)$, where $\pi$ is the projection morphism of $\Gamma(S(TM))$ on $\Gamma(E_0)$ and $\pi \nabla^* \theta$ is the induced connection on $E_\alpha$. This imply that the leaf $M^0$ of $E_0$ is totally geodesic. As $g(R^*(X,Y)Z, \mu) = 0$ and $g(R^*(X,Y)Z, e_\alpha) = 0$ for all $X, Y, Z \in \Gamma(E_0)$, we have $R^*(X,Y)Z = \pi R^*(X,Y)Z \in \Gamma(E_0)$ and the curvature tensor $\pi R^*$ of $E_0$ is flat. Thus $M^0$ is a Euclidean space. \(\square\)

References


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