ASYMPTOTIC BEHAVIOR OF $\mathcal{A}$-HARMONIC FUNCTIONS AND $p$-EXTREMAL LENGTH

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Abstract. We describe the asymptotic behavior of functions of the Royden $p$-algebra in terms of $p$-extremal length. We also prove that each bounded $\mathcal{A}$-harmonic function with finite energy on a complete Riemannian manifold is uniquely determined by the behavior of the function along $p$-almost every curve.

1. Introduction

Let $\Omega$ be an open subset of an $n$-dimensional complete Riemannian manifold $M$ and $W^{1,p}(\Omega)$ be the Sobolev space of all functions $u$ in $L^p(\Omega)$ whose distributional gradient $\nabla u$ also belongs to $L^p(\Omega)$, where $p$ is a constant such that $1 < p < \infty$. We equip $W^{1,p}(\Omega)$ with the norm $||u||_{1,p} = ||u||_p + ||\nabla u||_p$. We denote by $W^{1,p}_0(\Omega)$ the closure of $C^\infty_0(\Omega)$ in $W^{1,p}(\Omega)$. Let $F : \bigcup_{x \in \Omega} T_x M \to \mathbb{R}$ be a variational kernel satisfying following conditions:

(A1) the mapping $F_x = F|_{T_x M} : T_x M \to \mathbb{R}$ is strictly convex and differentiable for all $x$ in $\Omega$, and the mapping $x \mapsto F_x(\xi)$ is measurable whenever $\xi$ is;

(A2) for a constant $1 < p < \infty$, there exist constants $0 < C_1 \leq C_2 < \infty$ such that

$$C_1|\xi|^p \leq F_x(\xi) \leq C_2|\xi|^p$$

for all $x$ in $\Omega$ and $\xi$ in $T_x M$.

It is instructive to note that if $\mathcal{A}_x(\xi) = (A^1_x(\xi), A^2_x(\xi), \ldots, A^n_x(\xi))$, where $A^i_x(\xi) = \frac{\partial}{\partial \xi^i} F_x(\xi)$ for each $i = 1, 2, \ldots, n$, then $\mathcal{A}$ satisfies the following properties: (See [1] and [7])

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(A3) the mapping \( A_x = \mathcal{A}|_{T_x M} : T_x M \to T_x M \) is continuous for a.e. \( x \) in \( \Omega \), and the mapping \( x \mapsto \mathcal{A}_x(\xi) \) is a measurable vector field whenever \( \xi \) is;

for a.e. \( x \) in \( \Omega \) and for all \( \xi, \xi' \) in \( T_x M \),

(A4) \( \langle \mathcal{A}_x(\xi), \xi \rangle \geq C_1 |\xi|^p \); 

(A5) \( |\mathcal{A}_x(\xi)| \leq C_2 |\xi|^{p-1} \); 

(A6) \( \langle \mathcal{A}_x(\xi) - \mathcal{A}_x(\xi'), \xi - \xi' \rangle > 0 \) whenever \( \xi \neq \xi' \).

We say that a function \( u \) in \( W^{1,p}_{\text{loc}}(\Omega) \) is a solution (supersolution, respectively) of the equation

\[
- \text{div} \mathcal{A}_x(\nabla u) = 0 \quad (\geq 0, \text{ respectively})
\]

in \( \Omega \) if

\[
\int_{\Omega} \langle \mathcal{A}_x(\nabla u), \nabla \phi \rangle = 0 \quad (\geq 0, \text{ respectively})
\]

for any (nonnegative, respectively) function \( \phi \) in \( C_0^\infty(\Omega) \). A function \( v \) in \( W^{1,p}_{\text{loc}}(\Omega) \) is called a subsolution of (1.1) in \( \Omega \) if \( -v \) is a supersolution of (1.1) in \( \Omega \). We say that a function \( u \) is \( \mathcal{A} \)-harmonic (of type \( p \)) if \( u \) is a continuous solution of (1.1). In a typical case \( \mathcal{A}_x(\xi) = |\xi|^{p-2} \), \( \mathcal{A} \)-harmonic functions are called \( p \)-harmonic and, in particular, if \( p = 2 \), then we obtain harmonic functions. Suppose that \( E \) is a measurable set and that \( u \in W^{1,p}_{\text{loc}}(\Omega) \) for an open neighborhood \( \Omega \) of \( E \). Then the variational integral

\[
J(u, E) = \int_E F_x(\nabla u)
\]

is well defined. If \( J(u, M) < \infty \), then we say that \( u \) has finite energy. In fact, given \( f \in W^{1,p}(\Omega) \), each \( \mathcal{A} \)-harmonic function \( u \) with \( u - f \in W^{1,p}_0(\Omega) \) minimizes the energy functional \( J(v, \Omega) \) on the set \( \{ v \in W^{1,p}(\Omega) : v - f \in W^{1,p}_0(\Omega) \} \) (See [1]). A Green’s function \( G = G(o, \cdot) \) for \( \mathcal{A} \) on \( M \) denotes (if exists) a positive solution of the equation

\[
- \text{div} \mathcal{A}(\nabla G) = \delta_o
\]

for each \( o \) in \( M \), in the sense of distributions, i.e.,

\[
\int_M \langle \mathcal{A}(\nabla G), \nabla \phi \rangle = \phi(o)
\]

for any function \( \phi \) in \( C_0^\infty(M) \). In fact, there exists a Green’s function \( G \) satisfying (1.2) if and only if \( M \) has positive \( p \)-capacity at infinity, i.e., there exists a compact subset \( K \) of \( M \) such that

\[
\text{Cap}_p(K, \infty, M) = \inf_u \int_M |\nabla u|^p > 0,
\]

where the infimum is taken over all functions \( u \) in \( C_0^\infty(M) \) with \( u = 1 \) on \( K \).

In particular, we say that a complete Riemannian manifold \( M \) is \( p \)-parabolic if \( \text{Cap}_p(K, \infty, M) = 0 \) for every compact subset \( K \) of \( M \). Otherwise, \( M \) is called \( p \)-nonparabolic. It is well known that a complete Riemannian manifold \( M \) is...
\(p\)-nonparabolic if and only if \(M\) has the positive \(A\)-capacity, i.e., there exists a compact subset \(K\) of \(M\) such that

\[
\text{Cap}_A(K, \infty, M) = \inf_u J(u, M) > 0,
\]

where the infimum is taken over all functions \(u\) in \(C_0^\infty(M)\) with \(u = 1\) on \(K\).

We now introduce additional conditions on \(F\) as follows:

(A7) \(A_x(\lambda \xi) = \lambda|\lambda|^{p-2} A_x(\xi)\) whenever \(\lambda \in \mathbb{R} \setminus \{0\}\),

for any \(\xi, \xi'\) in \(T_xM\),

(A8) in case 2 \(\leq p < \infty\),

\[
F_x\left(\frac{\xi + \xi'}{2}\right) + F_x\left(\frac{\xi - \xi'}{2}\right) \leq \frac{1}{2} (F_x(\xi) + F_x(\xi')),
\]

in case 1 \(1 \leq p \leq 2\),

\[
F_x\left(\frac{\xi + \xi'}{2}\right)^\bar{p} + F_x\left(\frac{\xi - \xi'}{2}\right)^\bar{p} \leq \left(\frac{1}{2} (F_x(\xi) + F_x(\xi'))\right)^\bar{p},
\]

where \(\bar{p} = 1/(p - 1)\).

For \(F(\xi) = \frac{1}{2} |\xi|^p\), the condition (A8) is called the Clarkson inequality (See [3]).

Let \(BD_p(M)\) be the set of all bounded continuous functions \(u\) on a complete Riemannian manifold \(M\) whose distributional gradient \(\nabla u\) belongs to \(L^p(M)\). Then, by using the Minkowski inequality, it is easy to see that \(BD_p(M)\) forms an algebra over the real numbers with the usual addition and multiplication of functions and scalar multiplication defined pointwise. The function space \(BD_p(M)\) is called the Royden \(p\)-algebra of \(M\) (See [9]). We say that a sequence \(\{f_n\}\) of functions in \(BD_p(M)\) converges to a function \(f \in BD_p(M)\) in the \(BD_p\)-topology if

(i) \(\{f_n\}\) is uniformly bounded;

(ii) \(f_n\) converges uniformly to \(f\) on each compact subset of \(M\);

(iii) \(\lim_{n \to \infty} \int_M |\nabla (f_n - f)|^p = 0\).

It is well known that \(BD_p(M)\) is complete in the \(BD_p\)-topology. Let \(BD_{p,0}(M)\) be the closure of the set of all compactly supported smooth functions in \(BD_p(M)\). It is easy to see that \(BD_{p,0}(M)\) is not only a subalgebra but also an ideal of \(BD_p(M)\). We denote by \(\mathcal{H}BD_A(M)\) the subset of all bounded energy finite \(A\)-harmonic functions in \(BD_p(M)\), where \(A\) is an elliptic operator on \(M\) satisfying conditions (A1), (A2), (A7) and (A8). Adopting the arguments in [6], one can prove the following \(A\)-harmonic version of the Royden decomposition theorem:

**Proposition 1.1.** Let \(A\) be an elliptic operator on a \(p\)-nonparabolic complete Riemannian manifold \(M\) satisfying conditions (A1), (A2), (A7) and (A8). Then for each \(f \in BD_p(M)\), there exist unique \(u \in \mathcal{H}BD_A(M)\) and \(g \in BD_{p,0}(M)\) such that \(f = u + g\).

For a complete Riemannian manifold \(M\), there exists a locally compact Hausdorff space \(M\), called the Royden \(p\)-compactification of \(M\), which contains
$M$ as an open dense subset. In particular, every function $f \in \mathcal{B}D_0(M)$ can be extended to a continuous function, denoted again by $f$, on $\bar{M}$ and the class of such extended functions separates points in $\bar{M}$. The Royden $p$-boundary of $M$ is the set $\bar{M} \setminus M$ and will be denoted by $\partial M$. Throughout the paper, for a subset $\Omega$ of $M$, we denote by $\overline{\Omega}$ the closure of $\Omega$ in $M$ and $\hat{\Omega}$ the closure of $\Omega$ in $\bar{M}$. An important part of the Royden $p$-boundary $\partial \bar{M}$ is the $p$-harmonic boundary $\Delta \bar{M}$ defined by

$$\Delta \bar{M} = \{ x \in \partial \bar{M} : f(x) = 0 \text{ for all } f \in \mathcal{B}D_0(M)\}.$$

Let $\mathcal{F}$ be a family of locally rectifiable curves in a complete Riemannian manifold $M$. Let us fix a real number $p$ such that $1 < p < \infty$. A nonnegative Borel measurable function $\rho : M \to \mathbb{R}$ is called admissible with respect to $\mathcal{F}$ if $\int_\gamma \rho \geq 1$ for all curves $\gamma$ in $\mathcal{F}$. The $p$-extremal length of $\mathcal{F}$, denoted by $\lambda_p(\mathcal{F})$, is defined as

$$\lambda_p(\mathcal{F}) = \left( \inf_\rho \int_M \rho^p \right)^{-1},$$

where the infimum is taken over the set of all admissible functions $\rho$ with respect to $\mathcal{F}$. A property is said to hold for $p$-almost every curve in $\mathcal{F}$ if it holds for all curves in $\mathcal{F} \setminus \mathcal{F}_0$, where $\mathcal{F}_0$ is a subfamily of $\mathcal{F}$ with $p$-extremal length $\infty$.

Under the above setting, the value at the $p$-harmonic boundary of each function of the Royden $p$-algebra is completely determined by its asymptotic behavior along $p$-almost every curve as follows:

**Theorem 1.2.** Let us denote $\mathcal{G}$ to be the family of all locally rectifiable curves in a complete Riemannian manifold $M$. Let us fix a real number $p$ such that $1 < p < \infty$. A nonnegative Borel measurable function $\rho : M \to \mathbb{R}$ is called admissible with respect to $\mathcal{G}$ if $\int_\gamma \rho \geq 1$ for all curves $\gamma$ in $\mathcal{G}$. The $p$-extremal length of $\mathcal{G}$, denoted by $\lambda_p(\mathcal{G})$, is defined as

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Applying the comparison principle in Lemma 2.3 together with Theorem 1.2, one can prove that each bounded $\mathcal{A}$-harmonic function with finite energy is uniquely determined by the behavior of the function along $p$-almost every curve as follows:

**Corollary 1.3.** Let $\mathcal{G}$ be given as in Theorem 1.2. Suppose that $f, h \in \mathcal{H}BD_\mathcal{A}(M)$ and

$$\lim_{t \to \infty} f(\gamma(t)) = \lim_{t \to \infty} h(\gamma(t))$$

for $p$-almost every curve in $\mathcal{G}$. Then $f \equiv h$ on $M$. 


Corollary 1.4. Let $G$ be given as in Theorem 1.2. Suppose that $h \in \mathcal{HBD}_A(M)$ and $c \in \mathbb{R}$ with
$$\lim_{t \to \infty} h(\gamma(t)) = c$$
for $p$-almost every curve in $G$. Then $h \equiv c$ on $M$.

2. The maximum principle and the $p$-extremal length

We now study the relation between a sort of asymptotic behavior of functions in the Royden $p$-algebra $\mathcal{BD}_p(M)$ near infinity of $M$ and the values of the functions at the $p$-harmonic boundary $\Delta_M$ of $M$. We first give a characterization of the $p$-parabolicity in terms of the $p$-harmonic boundary as follows (See [6]):

Lemma 2.1. A complete Riemannian manifold $M$ is $p$-parabolic if and only if the $p$-harmonic boundary $\Delta_M$ of $M$ is empty.

Furthermore, there is a useful duality relation between $\mathcal{BD}_{p,0}(M)$ and $\Delta_M$ (See [6]):

Lemma 2.2. For any complete Riemannian manifold $M$,
$$\mathcal{BD}_{p,0}(M) = \{ f \in \mathcal{BD}_p(M) : f = 0 \text{ on } \Delta_M \}.$$ 

We now give the comparison principle and the maximum principle for $A$-harmonic functions as follows:

Lemma 2.3. Let $A$ be an elliptic operator on a $p$-nonparabolic complete Riemannian manifold $M$ satisfying conditions $(A1), (A2), (A7)$ and $(A8)$.

(C) Suppose that there exist $A$-harmonic functions $u, v \in \mathcal{HBD}_A(M)$ such that
$$v \leq u \text{ on } \Delta_M.$$ 

Then we have $v \leq u$ on $M$.

(M) Suppose that there exists an $A$-harmonic function $u \in \mathcal{HBD}_A(M)$ such that
$$a \leq u \leq b \text{ on } \Delta_M$$
for some constants $a$ and $b$ with $a \leq b$. Then we have $a \leq u \leq b$ on $M$.

Proof. Let us consider a function $\phi = \min\{u - v, 0\}$. Since $v \leq u$ on $\Delta_M$, we conclude that $\phi = 0$ on $\Delta_M$. Thus by Lemma 2.2, we conclude that $\phi$ belongs to $\mathcal{BD}_{p,0}(M)$. Since $u$ and $v$ are $A$-harmonic on $M$ and there is a sequence of compactly supported smooth functions converging to $\phi$ in $\mathcal{BD}_p(M)$, we have
$$\int_M \langle A_x(\nabla u), \nabla \phi \rangle = 0$$
and
$$\int_M \langle A_x(\nabla v), \nabla \phi \rangle = 0.$$
Let $1_{\Omega}$ be the characteristic function of the set $\Omega = \{x \in M : u(x) \leq v(x)\}$. Since $\nabla \phi = 1_{\Omega} \nabla (u - v)$ almost everywhere in $M$, we conclude that
\[
\int_{\Omega} (A_x(\nabla u) - A_x(\nabla v), \nabla (u - v)) = 0.
\]
By the condition (A6), $u - v$ is almost everywhere constant in $\Omega$. Since $u$ and $v$ are continuous, $u - v$ is constant in $\Omega$. Hence we have (C) from the continuity of $u$ and $v$.

On the other hand, since every constant function is also $A$-harmonic, we have (M) from (C).

We now introduce the notion of the $p$-capacity of a condenser: Let $\Omega \subset M$ be a nonempty open set and let $E_1$ and $E_2$ be mutually disjoint closed subsets of $\Omega$. The $p$-capacity for a triple $(E_1, E_2, \Omega)$ is defined by
\[
\text{Cap}_p(E_1, E_2, \Omega) = \inf \int_{\Omega} |\nabla v|^p,
\]
where the infimum is taken over all smooth functions $v$ on $\Omega \cup E_1 \cup E_2$ such that $0 \leq v \leq 1$ on $\Omega$, $v = 0$ on $E_1$ and $v = 1$ on $E_2$. Such a triple $(E_1, E_2, \Omega)$ is called a condenser. For an unbounded open set $\Omega \subset M$ and a nonempty compact set $E \subset \overline{\Omega}$, we define
\[
\text{Cap}_p(E, \infty, \Omega) = \lim_{r \to \infty} \text{Cap}_p(E, \overline{\Omega} \setminus B_r(o), \Omega),
\]
where $B_r(o)$ denotes the geodesic ball of radius $r > 0$ centered at a fixed point $o$ in $M$. It is needed to note that $\text{Cap}_p(E, \overline{\Omega} \setminus B_r(o), \Omega)$ is monotone decreasing in $r > 0$. On the other hand, an unbounded open set $\Omega \subset M$ is called $p$-hyperbolic if there exists a nonempty compact set $E \subset \overline{\Omega}$ such that $\text{Cap}_p(E, \infty, \Omega) > 0$.

From the properties of the $p$-capacity, it is easy to see that any open set $\Omega$ is $p$-hyperbolic if there exists a $p$-hyperbolic subset $\Omega'$ of $\Omega$. An unbounded open proper set $\Omega \subset M$ is called $A$-massive if there exists a function $u$ in $BD_p(M)$ such that
\[
\begin{cases}
A u = 0 & \text{in } \Omega; \\
u = 0 & \text{on } M \setminus \Omega; \\
sup_{\Omega} u = 1.
\end{cases}
\]
Such a function $u$ is called an inner potential of $\Omega$. In fact, for each nonconstant function $u$ in $HBD_A(M)$, the set $\{x \in M : u(x) > c\}$ is $A$-massive, where $\inf_M u < c < \sup_M u$. There is a useful property of $A$-massive sets (See [4], [5] and [6]):

**Lemma 2.4.** Let $A$ be an elliptic operator on a complete Riemannian manifold $M$ satisfying conditions (A1), (A2), (A7) and (A8). If $\Omega$ is $A$-massive, then there exists a proper $p$-hyperbolic subset $\Omega_0$ of $\Omega$ such that $\overline{\Omega_0} \subset \Omega$ and a continuous function $v$ on $\overline{\Omega}$ such that $A v = 0$ in $\Omega \setminus \overline{\Omega_0}$, $v = 0$ on $\partial \Omega$ and $v$ has finite energy, that is, $\mathcal{J}(v, M) < \infty$. 

On the other hand, the $p$-capacity of a condenser is closely related to the $p$-extremal length of a family of curves as follows: Let $\Omega$ be an unbounded open subset of $M$ and $E$ be a compact set in $\Omega$. Let $\mathcal{F}_{\Omega,E}$ be the family of all locally rectifiable curves in $\Omega$ joining $E$ to infinity of $\Omega$. This means that $\gamma$ is a curve in $\mathcal{F}_{\Omega,E}$ if $\gamma : [\alpha, \beta] \to \Omega$ ($\beta$ may be $\infty$) is locally rectifiable, $\gamma(\alpha)$ belongs to $E$, and for any compact set $K$ of $M$, there exists $t_K \in [\alpha, \beta)$ such that $\gamma(t)$ does not belong to $K$ for all $t > t_K$. Then, by results in [2], we have

$$(2.1) \quad \left( \lambda_p(\mathcal{F}_{\Omega,E}) \right)^{-1} = \text{Cap}_p(E, \infty, \Omega)$$

(See also [12] or [8]). In particular, if $\Omega$ is $A$-massive, then by Lemma 2.4, there exists a proper $p$-hyperbolic subset $\Omega_0$ of $\Omega$ such that $\Omega_0 \subset \Omega$. Since $\Omega_0$ is $p$-hyperbolic, there exists a nonempty compact subset $E \subset \Omega_0$ such that $\text{Cap}_p(E, \infty, \Omega_0) > 0$. Therefore, combining (2.1) and the monotone property of the $p$-capacity, we conclude that

$$(2.2) \quad \left( \lambda_p(\mathcal{F}_{\Omega,E}) \right)^{-1} = \text{Cap}_p(E, \infty, \Omega) > 0.$$

Let us denote $\mathcal{G}$ to be the family of all locally rectifiable curves in $M$ joining $B_R(o)$ to infinity of $M$. For an unbounded set $\Omega$ of $M$, $\mathcal{G}_\Omega$ denotes the subfamily of $\mathcal{G}$ which consists of all locally rectifiable curves in $\Omega$ joining $B_R(o) \cap \Omega$ to infinity of $\Omega$, where $R > 0$ is sufficiently large such that $B_R(o) \cap \Omega \neq \emptyset$. From now on, $\mathcal{G}$ and $\mathcal{G}_\Omega$ mean those appear in the above setting unless otherwise specified. In particular, if $\Omega$ is a $A$-massive set of $M$, we have $\lambda_p(\mathcal{G}_\Omega) < \infty$.

In fact, the $p$-extremal length of a family of curves in an unbounded set is closely related to the $p$-harmonic boundary as follows: Let us denote $e(\gamma)$ to be the end part of a curve $\gamma \in \mathcal{G}$ in $\partial \hat{M}$, this means that $e(\gamma) = \hat{\gamma} \cap \partial \hat{M}$, where $\hat{\gamma}$ denotes the closure in $\hat{M}$ of the image set under $\gamma$. The following lemmas give a tractable criterion for a family of curves in an unbounded set to have infinite $p$-extremal length:

**Lemma 2.5.** Let $\Omega$ be an unbounded open subset of $M$ such that $B_R(o) \cap \Omega \neq \emptyset$ for sufficiently large $R > 0$. Let $\mathcal{G}_0$ be a subfamily of $\mathcal{G}_{\hat{\Omega}}$ and $K$ be the closure of the set $\bigcup_{\gamma \in \mathcal{G}_0} e(\gamma)$ in $\partial \hat{M}$. Suppose that $K$ is disjoint from $\Omega \cap \Delta_M$. Then $\lambda_p(\mathcal{G}_0) = \infty$.

**Proof.** If $M$ is $p$-parabolic, then $\Delta_M = \emptyset$ and $\text{Cap}_p(B_R(o), \infty, M) = 0$. Thus by (2.1), we have

$$\left( \lambda_p(\mathcal{G}) \right)^{-1} = \text{Cap}_p(B_R(o), \infty, M) = 0.$$ 

So we may assume that $M$ is $p$-nonparabolic. By the result in [11], it suffices to show that there exists a function $\rho$ in $L^p(M)$ such that

$$\int \rho = \infty \quad \text{for each curve } \gamma \in \mathcal{G}_0.$$
Since $K$ is a nonempty compact subset in $\partial M \setminus \Delta_M$, there exists a continuous function $f$ such that $f|_K = \infty$ and $\int_M |\nabla f|^p < \infty$ (See [10], [1], and [13]). Hence the lemma follows. To be precise, from the definition of $K$, we have $e(\gamma) \in K$ for any curve $\gamma \in \mathcal{G}_0$. Thus we conclude that $f(\gamma) = \infty$ for any curve $\gamma \in \mathcal{G}_0$, where $f(\gamma) = \lim_{t \to \infty} f(\gamma(t))$. Then for any $\varepsilon > 0$, the function $\varepsilon|\nabla f|$ is admissible with respect to $\mathcal{G}_0$. Consequently,

$$\lambda_p(\mathcal{G}_0) \geq \left(\varepsilon^p \int_M |\nabla f|^p\right)^{-1}.$$ 

This completes the proof. \hfill \Box

**Lemma 2.6.** Let $\mathcal{G}_0$ be a subfamily of $\mathcal{G}$ such that $\lambda_p(\mathcal{G}_0) = \infty$ and $K$ be the closure of the set $\bigcup_{\gamma \in \mathcal{G} \setminus \mathcal{G}_0} e(\gamma)$ in $\partial M$. Then $K$ contains $\Delta_M$.

**Proof.** If $\Delta_M = \emptyset$, then nothing to prove. So we may assume that $\Delta_M \neq \emptyset$. That is, $M$ is $p$-nonparabolic. Then by (2.1), we have

$$\left(\lambda_p(\mathcal{G})\right)^{-1} = \text{Cap}_p(\overline{B}_R(0), \infty, M) > 0.$$ 

Since $\lambda_p(\mathcal{G} \setminus \mathcal{G}_0) < \infty$, by Lemma 2.5, we have $K \cap \Delta_M \neq \emptyset$. Suppose that the lemma is not true. We may assume that $x \in \Delta_M \setminus K$. Let us choose a function $f \in \mathcal{BD}_p(M)$ such that $0 < f < 1$ on $M$ and

$$\begin{cases} f(x) = 1; \\
 f|_{K \cap \Delta_M} = 0. \end{cases}$$ 

By Proposition 1.1, there exist unique $h \in \mathcal{HBD}_p(M)$ and $g \in \mathcal{BD}_{p,0}(M)$ such that $f = h + g$, where $\mathcal{A}$ is an elliptic operator on $M$ satisfying conditions (A1), (A2), (A7) and (A8). Since $g$ belongs to $\mathcal{BD}_{p,0}(M)$, by Lemma 2.2, $g = 0$ on $\Delta_M$. From this fact together with the maximum principle, one can conclude that $0 < h < 1$ on $M$ and

$$\begin{cases} h(x) = 1; \\
 h|_{K \cap \Delta_M} = 0. \end{cases}$$ 

Let us consider the set

$$\Omega = \{x \in M : h(x) > 1 - \varepsilon\},$$ 

where $\varepsilon$ is a positive constant so small that $1 - \varepsilon > 0$. Clearly, $\Omega$ is an $\mathcal{A}$-massive subset of $M$. Similarly arguing as (2.2), we have $\lambda_p(\mathcal{G}_\Omega) < \infty$. Let us denote $K_1$ to be the closure of the set $\bigcup_{\gamma \in \mathcal{G} \setminus \mathcal{G}_\Omega} e(\gamma)$ in $\partial M$. Since $\lambda_p(\mathcal{G}_\Omega \setminus \mathcal{G}_0) < \infty$, by Lemma 2.5 again, we have $K_1 \cap \Delta_M \neq \emptyset$. Since $K_1 \cap \Delta_M$ is a subset of $K \cap \Delta_M$, we conclude that

$$h|_{K_1 \cap \Delta_M} = 0.$$ 

On the other hand, from the definition of $\Omega$, we see that $h(\gamma) \geq 1 - \varepsilon$ for all curves $\gamma \in \mathcal{G}_\Omega$, where $h(\gamma) = \lim_{t \to \infty} h(\gamma(t))$. Hence we have

$$h|_{K_1} \geq 1 - \varepsilon$$ 

which is a contradiction. This completes the proof. \hfill \Box
3. The proof of Theorem 1.2

We are ready to prove the main theorem which gives a connection between a sort of asymptotic behavior of functions in $BD_p(M)$ near infinity of $M$ and the values of the functions at $\Delta_M$:

Proof of Theorem 1.2. Suppose that $f|_{\Delta_M} = 0$. By considering the positive part and the negative part of $f$ separately, we may assume that $f \geq 0$. For each positive integer $n$, let us consider the family of curves $G_n = \{ \gamma \in \mathcal{G} : f(\gamma) \geq \frac{1}{n} \}$, where $f(\gamma) = \lim_{t \to \infty} f(\gamma(t))$. Since $f|_{\Delta_M} = 0$, we conclude that $K_n \cap \Delta_M = \emptyset$ for each positive integer $n$, where $K_n$ is the closure of the set $\bigcup_{\gamma \in G_n} e(\gamma)$ in $\partial M$. Hence, by Lemma 2.5, one can conclude that $\lambda_p(G_n) = \infty$ for each positive integer $n$. Then $\lim_{t \to \infty} f(\gamma(t)) = 0$ for all curves $\gamma \in \mathcal{G} \setminus G_{\infty}$, where $G_{\infty} = \bigcup_{n=1}^{\infty} G_n$. Since $\lambda_p(G_{\infty}) = \infty$, we have $\lim_{t \to \infty} f(\gamma(t)) = 0$ for $p$-almost every curve $\gamma \in \mathcal{G}$.

On the other hand, the converse follows immediately from Lemma 2.6. This completes the proof. □

References


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