SCALAR CURVATURE OF CONTACT CR-SUBMANIFOLDS IN AN ODD-DIMENSIONAL UNIT SPHERE

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Abstract. In this paper we derive an integral formula on an \((n + 1)\)-dimensional, compact, minimal contact CR-submanifold \(M\) of \((n - 1)\) contact CR-dimension immersed in a unit \((2m + 1)\)-sphere \(S^{2m+1}\). Using this integral formula, we give a sufficient condition concerning with the scalar curvature of \(M\) in order that such a submanifold \(M\) is to be a generalized Clifford torus.

1. Introduction

Let \(S^{2m+1}\) be a \((2m + 1)\)-dimensional unit sphere, that is,
\[
S^{2m+1} = \{ z \in \mathbb{C}^{m+1} : \|z\| = 1 \}.
\]
For any point \(z \in S^{2m+1}\) we put \(\xi = Jz\), where \(J\) denotes the almost complex structure of \(\mathbb{C}^{m+1}\). We consider the orthogonal projection \(\pi : T_z\mathbb{C}^{m+1} \rightarrow T_zS^{2m+1}\). Putting \(\phi = \pi \circ J\), we can see that the set \((\phi, \xi, \eta, g)\) is a Sasakian structure on \(S^{2m+1}\), where \(\eta\) is a 1-form dual to \(\xi\) and \(g\) the standard metric tensor field on \(S^{2m+1}\). So \(S^{2m+1}\) can be considered as a Sasakian manifold of constant \(\phi\)-sectional curvature 1, that is, of constant curvature 1 (cf. [1, 2, 12]).

Let \(M\) be an \((n+1)\)-dimensional submanifold tangent to the structure vector field \(\xi\) of \(S^{2m+1}\) and denote by \(D_x\) the \(\phi\)-invariant subspace \(\phi T_xM \cap T_x M\) of the tangent space \(T_x M\) of \(M\) at \(x \in M\). Then \(\xi\) cannot be contained in \(D_x\) at any point \(x \in M\).

When the \(\phi\)-invariant subspace \(D_x\) has constant dimension for any \(x \in M\), \(M\) is called a contact CR-submanifold and the constant is called contact CR-dimension of \(M\) (cf. [5, 6, 9, 10]).

On an \((n + 1)\)-dimensional contact CR-submanifold of \((n - 1)\) contact CR-dimension, there is a non-zero vector \(U\) which is orthogonal to \(\xi\) and contained...
in the complementary orthogonal subspace $D^\perp_x$ of $D_x$ in $T_xM$. In this case $N =: \phi U$ must be normal to $M$ and thus $M$ can be dealt with a contact CR-submanifold in the sense of Yano-Kon ([12]).

In this paper we shall study $(n+1)$-dimensional contact CR-submanifolds of $(n-1)$ contact CR-dimension immersed in $S^{2m+1}$ and prove the following theorem as a Sasakian version corresponding to the results provided in [3] and [7].

**Theorem.** Let $M$ be an $(n+1)$ $(\geq 3)$-dimensional compact, minimal, contact CR-submanifold of $(n-1)$ contact CR-dimension in $S^{2m+1}$. If the scalar curvature of $M$ is greater or equal to $n^2/2 - 1$, then

$$M = S^{2t+1}(r_1) \times S^{2s+1}(r_2), \quad t + s = \frac{n+1}{2} - 1,$$

where $r_1^2 + r_2^2 = 1$.

**Remark.** The above main theorem was provided in [9] under the condition that the distinguished normal vector field $N$ is parallel with respect to the normal connection $\nabla^\perp$. For the complex and the quaternionic analogues corresponding to the above theorem, see [3] and [7], respectively.

Manifolds, submanifolds, geometric objects and mappings we discuss in this paper will be assumed to be connected, differentiable and of class $C^\infty$.

### 2. Fundamental properties of contact CR-submanifolds

Let $\overline{M}$ be a $(2m+1)$-dimensional almost contact metric manifold with structure $(\phi, \xi, \eta, g)$. Then, by definition, it follows that

\begin{align*}
\phi^2 X &= -X + \eta(X)\xi, \quad \phi \xi = 0, \quad \eta(\phi X) = 0, \quad \eta(\xi) = 1, \\
g(\phi X, \phi Y) &= g(X, Y) - \eta(X)\eta(Y), \quad \eta(X) = g(X, \xi)
\end{align*}

(2.1)

for any vector fields $X, Y$ tangent to $\overline{M}$.

Let $M$ be a contact CR-submanifold of $(n-1)$ contact CR-dimension in $\overline{M}$, where $n - 1$ must be even. Then, as was already mentioned in §1, the structure vector $\xi$ is always contained in $D^\perp_x$ and $\phi D^\perp_x \subset T_xM^\perp$ at any point $x \in M$, where $T_xM^\perp$ denotes the normal space of $M$ at $x \in M$. Further, by definition $\dim D^\perp_x = 2$ at any point $x \in M$, and so there exists a unit vector field $U$ contained in $D^\perp_x$ which is orthogonal to $\xi$. Since $\phi D^\perp_x \subset T_xM^\perp$ at any point $x \in M$, $\phi U$ is a unit normal vector field to $M$, which will be denoted by $N$, that is,

$$N := \phi U.
$$

Moreover, it is clear that $\phi TM \subset TM \oplus \text{Span}\{N\}$. Hence we have, for any tangent vector field $X$ and for a local orthonormal basis $\{N_\alpha\}_{\alpha=1,\ldots,p}$ ($N_1 := N$, $p := 2m - n$) of normal vectors to $M$, the following decomposition in tangential and normal components:

$$\phi X = F X + u(X)N,$$

(2.3)
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\( \phi N_\alpha = PN_\alpha, \quad \alpha = 2, \ldots, p. \)

It is easily shown that \( F \) is a skew-symmetric linear endomorphism acting on \( T_2 M \). Since the structure vector field \( \xi \) is tangent to \( M \), (2.1) and (2.3) imply

\( F \xi = 0, \quad FU = 0, \quad g(U, X) = u(X), \quad u(\xi) = g(U, \xi) = 0, \quad u(U) = 1. \)

Next, applying \( \phi \) to (2.3) and using (2.1), (2.3) and (2.5), we also have

\( F^2 X = -X + \eta(X) \xi + u(X) U, \quad u(FX) = 0. \)

On the other hand, it is clear from (2.1), (2.2) and (2.5) that

\( \phi N = -U, \)

which and (2.4) yield the existence of a local orthonormal basis \( \{ N, N_a, N_a^* \}_{a=1, \ldots, q} \) of normal vectors to \( M \) such that

\( N_a^* := \phi N_a, \quad a = 1, \ldots, q := (p - 1)/2. \)

We denote by \( \nabla \) and \( \nabla \) the Levi-Civita connection on \( M \) and \( M \), respectively, and by \( \nabla^\perp \) the normal connection induced from \( \nabla \) on the normal bundle \( TM^\perp \) of \( M \). Then Gauss and Weingarten formulae are given by

\( \nabla X Y = \nabla X Y + h(X, Y), \)

\( \nabla X N = -AX + \nabla^\perp X N = -AX + \sum_{a=1}^{q} \{ s_a(X) N_a + s_{a^*}(X) N_{a^*} \}, \)

\( \nabla X N_a = -A_a X - s_a(X) N + \sum_{b=1}^{q} \{ s_{ab}(X) N_b + s_{a^* b^*}(X) N_{b^*} \}, \)

\( \nabla X N_{a^*} = -A_{a^*} X - s_{a^*}(X) N + \sum_{b=1}^{q} \{ s_{a^* b}(X) N_b + s_{a^* b^*}(X) N_{b^*} \} \)

for any vector fields \( X, Y \) tangent to \( M \), where \( s \)'s are coefficients of the normal connection \( \nabla^\perp \). Here \( h \) denotes the second fundamental form and \( A, A_a, A_{a^*} \) the shape operators corresponding to the normals \( N, N_a, N_{a^*} \), respectively. They are related by

\( h(X, Y) = g(A X, Y) N + \sum_{a=1}^{q} \{ g(A_a X, Y) N_a + g(A_{a^*} X, Y) N_{a^*} \}. \)

From now on we specialize to the case of an ambient Sasakian manifold \( \overline{M} \), that is,

\( \nabla X \xi = \phi X, \)

\( (\nabla X \phi) Y = -g(X, Y) \xi + \eta(Y) X. \)
Since $\xi$ is tangent to $M$, from (2.1), (2.3), (2.7), (2.8), (2.10)$_2$, (2.10)$_3$ and (2.13), we can easily verify that

$$ (2.14) \quad A_a X = -FA_a \cdot X + s_a \cdot (X) U, \quad A_a \cdot X = FA_a X - s_a (X) U, $$

$$ (2.15) \quad s_a (X) = -u(A_a \cdot X), \quad s_a \cdot (X) = u(A_a X), \quad a = 1, \ldots, q. $$

Since $F$ is skew-symmetric, (2.14) implies

$$ (2.16)_1 \quad g((FA_a + A_a F) X, Y) = s_a (X) u(Y) - s_a (Y) u(X), $$

$$ (2.16)_2 \quad g((FA_a + A_a F) X, Y) = s_a \cdot (X) u(Y) - s_a \cdot (Y) u(X). $$

On the other hand, since $FD_x = D_x$ at each point $x \in M$, we take an orthonormal basis $\{e_i\}_{i=1, \ldots, n+1}$ of tangent vectors to $M$ such that

$$ (2.17) \quad e_{l+1} := Fe_1, \ldots, e_{2l} := Fe_l, \quad e_n := U, \quad e_{n+1} := \xi, $$

where we have put $l = (n - 1)/2$. Replacing $X$ by $Fe_i$ in the first equation of (2.15) and using (2.5), we have

$$ s_a (Fe_i) = -g(A_a \cdot Fe_i, U), $$

which together with (2.5) and (2.16)$_2$ yields

$$ s_a (Fe_i) = -s_a \cdot (e_i), \quad i = 1, \ldots, l. $$

Similarly, replacing $X$ by $Fe_i$ in the second equation of (2.15) and using (2.5) and (2.16)$_1$, we have

$$ s_a (Fe_i) = -s_a \cdot (e_i), \quad s_a \cdot (Fe_i) = s_a (e_i), \quad i = 1, \ldots, l. $$

Differentiating (2.3) and (2.7) covariantly along $M$ and comparing the tangential with normal parts, we have

$$ (2.19) \quad (\nabla_Y F) X = -g(Y, X) \xi + \eta(X) Y - g(A Y, X) U + u(X) A Y, $$

$$ (2.20) \quad (\nabla_Y u) X = g(F A Y, X), $$

$$ (2.21) \quad \nabla_X U = F A X $$

with the aid of (2.3), (2.8), (2.9), (2.10)$_1$, (2.11) and (2.13). On the other hand, since $\xi$ is tangent to $M$, from (2.9) and (2.12), it follows that

$$ \phi X = \nabla_X \xi = \nabla_X \xi + h(X, \xi), $$

which together with (2.3) and (2.11) gives

$$ (2.22) \quad \nabla_X \xi = F X, $$

$$ (2.23) \quad g(A \xi, X) = u(X), \quad \text{i.e.,} \quad A \xi = U, $$

$$ (2.24) \quad A_a \xi = 0, \quad A_a \cdot \xi = 0, \quad a = 1, \ldots, q. $$

If the ambient manifold $M$ is a $(2m + 1)$-dimensional unit sphere $S^{2m+1}$ as a Sasakian manifold of constant curvature 1, then its curvature tensor $\overline{R}$ satisfies

$$ \overline{R}(X, Y) Z = g(Y, Z) X - g(X, Z) Y $$
for any vector fields $X, Y, Z$ tangent to $\mathcal{M}$. Therefore, by means of the equation of Gauss, we can easily see that the Ricci tensor $\text{Ric}(Y, Z)$ has the form
\begin{equation}
\text{Ric}(Y, Z) = n g(Y, Z) + (\text{tr} A) g(AY, Z) - g(A^2 Y, Z)
+ \sum_{a=1}^{q} \{ (\text{tr} A_a) g(A_a Y, Z) + (\text{tr} A_a^*) g(A_a^* Y, Z) \\
- g(A_a^2 Y, Z) - g(A_a^{2*} Y, Z) \}
\end{equation}
and consequently the scalar curvature $\rho$ is given by
\begin{equation}
\rho = n(n+1) + (\text{tr} A)^2 - \text{tr} A^2
+ \sum_{a=1}^{q} \{ (\text{tr} A_a)^2 + (\text{tr} A_a^*)^2 - \text{tr} A_a^2 - \text{tr} A_a^{2*} \}.
\end{equation}
Moreover, from the equation of Codazzi, we also have
\begin{equation}
(\nabla_X A) Y - (\nabla_Y A) X = \sum_{a=1}^{q} \{ s_a(X) A_a Y - s_a(Y) A_a X \\
+ s_a^*(X) A_a^* Y - s_a^*(Y) A_a^* X \}
\end{equation}
for any vector fields $X, Y$ tangent to $M$ (cf. [1, 2, 12]).

3. An integral formula on the compact contact $CR$-submanifold

Let $M$ be an $(n+1)$-dimensional contact $CR$-submanifold of $(n-1)$ contact $CR$-dimension immersed in a $(2m+1)$-dimensional unit sphere $S^{2m+1}$.

We now put $T := \nabla_U U + (\text{div} U)(U)$ and take the same orthonormal basis $\{e_i\}_{i=1,\ldots,n+1}$ of tangent vectors to $M$ as given in (2.17). Then it follows from (2.21) that
\begin{equation}
T = FAU
\end{equation}
since $\text{div} U = \sum_{i=1}^{n+1} g(e_i, \nabla e_i U) = \text{tr}(FA) = 0$.

From now on, for later use we shall compute $\text{div} T = \sum_{i=1}^{n+1} g(e_i, \nabla e_i T)$ (for a general formula of $\text{div} T$, see [11]).

Differentiating (3.1) covariantly and using (2.5), (2.19), (2.21) and (2.23), we have
\begin{equation}
\nabla_X T = - g(X, AU) \xi + X - g(A^2 U, X) U + u(AU)AX \\
+ FAFAX + F(\nabla_X A) U,
\end{equation}
from which, taking account of (2.5), (2.6) and (2.23), it follows that

$$\text{div} T = n - u(A^2U) + (\text{tr} A)u(AU) + \sum_{i=1}^{n+1} g(FAFAe_i, e_i)$$

(3.3)

$$- \sum_{i=1}^{t} g((\nabla e_i, A)Fe_i - (\nabla Fe_i, A)e_i, U).$$

On the other hand, using (2.5), (2.6), (2.15), (2.18) and (2.27), we can easily obtain that

$$\sum_{i=1}^{t} g((\nabla e_i, A)Fe_i - (\nabla Fe_i, A)e_i, U)$$

(3.4)

$$= \sum_{i=1}^{t} \sum_{a=1}^{q} \{s_a(e_i)^2 + s_a(Fe_i)^2 + s_{a^*}(e_i)^2 + s_{a^*}(Fe_i)^2\}$$

because of $2l = n - 1$. Inserting (3.4) back into (3.3), the equation (3.3) turns out to be

$$\text{div} T = n + \sum_{i=1}^{n+1} g(FAFAe_i, e_i) - u(A^2U)$$

(3.5)

$$- \sum_{i=1}^{t} \sum_{a=1}^{q} \{s_a(e_i)^2 + s_a(Fe_i)^2 + s_{a^*}(e_i)^2 + s_{a^*}(Fe_i)^2\}.$$
Lemma 3.1. Let $M$ be an $(n+1)$-dimensional compact contact CR-submanifold of $(n-1)$ contact CR-dimension immersed in $S^{2m+1}$. Then the following equality is valid:

$$
\frac{1}{2} \int_M \|FA - AF\|^2 + \rho - (n^2 - 1) + (\text{tr}A)u(AU) - (\text{tr}A)^2
$$

(3.7)

$$
- \sum_{a=1}^q \{(\text{tr}A_a)^2 + (\text{tr}A_a^*)^2\} + \sum_{a=1}^q (\text{tr}A_a^2 + \text{tr}A_a^2)
$$

$$
- \sum_{i=1}^l \sum_{a=1}^q \{s_a(e_i)^2 + s_a(Fe_i)^2 + s_a^*(e_i)^2 + s_a^*(Fe_i)^2\} \ast 1 = 0.
$$

4. The proof of main theorem

In order to prove the main theorem stated in §1, we prepare:

Lemma 4.1. Let $M$ be an $(n+1)$-dimensional compact, minimal, contact CR-submanifold of $(n-1)$ contact CR-dimension in $S^{2m+1}$. If the scalar curvature of $M$ is greater or equal to $n^2 - 1$, then

(4.1) $FA - AF = 0$

and the distinguished normal vector field $N$ is parallel with respect to the normal connection $\nabla^\perp$. Moreover, we have

(4.2) $A_a = 0, \ A_a^* = 0, \ a = 1, \ldots, q.$

Proof. We first notice that (2.15) and (2.24) yield

$$
\sum_{i=1}^l \{s_a(e_i)^2 + s_a(Fe_i)^2\} = u(A_a^2 U) - u(A_a U)^2,
$$

$$
\sum_{i=1}^l \{s_a^*(e_i)^2 + s_a^*(Fe_i)^2\} = u(A_a^2 U) - u(A_a U)^2.
$$

Inserting these equations back into (3.7) and taking account of (2.24), we have

$$
\frac{1}{2} \int_M \|FA - AF\|^2 + \rho - (n^2 - 1) + (\text{tr}A)u(AU) - (\text{tr}A)^2
$$

(4.3)

$$
- \sum_{a=1}^q \{(\text{tr}A_a)^2 + (\text{tr}A_a^*)^2\} + \sum_{a=1}^q (u(A_a U)^2 + u(A_a^* U)^2)
$$

$$
+ \sum_{a=1}^q \sum_{i=1}^l \{g(A_a^2 e_i, e_i) + g(A_a^2 Fe_i, Fe_i) + g(A_a^2, e_i) + g(A_a^2, Fe_i)\} \ast 1 = 0.
$$
If \( \rho \) is greater or equal to \( n^2 - 1 \), our assumptions yield (4.1) and
\[
A_a e_i = A_a F e_i = 0, \quad A_a \cdot e_i = A_a \cdot F e_i = 0,
\]
(4.4)
\[u(A_a U) = 0, \quad u(A_a \cdot U) = 0, \quad a = 1, \ldots, q, \quad i = 1, \ldots, l,
\]
which and (2.15) imply
\[s_a(e_i) = s_a(F e_i) = 0, \quad s_a \cdot (e_i) = s_a \cdot (F e_i) = 0,
\]
\[s_a(U) = 0, \quad s_a \cdot (U) = 0, \quad a = 1, \ldots, q, \quad i = 1, \ldots, l.
\]
Since \( s_a(\xi) = s_a \cdot (\xi) = 0 \) because of (2.24), we have \( s_a = s_a \cdot = 0 \) \((a = 1, \ldots, q)\) which means that the distinguished normal vector field \( N \) is parallel with respect to the normal connection by means of (2.10). Also it is clear from (2.15) that \( A_a U = A_a \cdot U = 0 \), which combined with (2.24) and (4.4) implies (4.2).

**Proof of main theorem.** By means of Lemma 4.1, for the submanifold \( M \) given in the main theorem, we can easily see that its first normal space is contained in \( \text{Span}\{N\} \) which is invariant under parallel translation with respect to the normal connection. Thus we may apply Erbacher’s reduction theorem ([4]) and so we can see that there exists an \((n + 2)\)-dimensional totally geodesic unit sphere \( S^{n+2} \) such that \( M \subset S^{n+2} \). Here we note that \((n + 2)\) is odd. Moreover, since the tangent space \( T_x S^{n+2} \) of the totally geodesic submanifold \( S^{n+2} \) at \( x \in M \) is \( T_x M \oplus \text{Span}\{N\} \), \( S^{n+2} \) is an invariant submanifold of \( S^{2m+1} \) with respect to \( \phi \) (for definition, see [1, 12]) because of (2.2) and (2.3). Hence the submanifold \( M \) can be regarded as a real hypersurface of \( S^{n+2} \) which is a totally geodesic invariant submanifold of \( S^{2m+1} \).

Tentatively we denote \( S^{n+2} \) by \( M' \), and by \( i_1 \) the immersion of \( M \) into \( M' \) and \( i_2 \) the totally geodesic immersion of \( M' \) onto \( S^{2m+1} \). Then, from the Gauss formula (2.9), it follows that
\[
\nabla'_{i_1} i_1 Y = i_1 \nabla_X Y + h'(X, Y) = i_1 \nabla_X Y + g(A'X, Y)N',
\]
(4.5)
where \( h' \) is the second fundamental form of \( M \) in \( M' \) and \( A' \) is the corresponding shape operator to a unit normal vector field \( N' \) to \( M \) in \( M' \). Since \( i = i_2 \circ i_1 \), making use of (4.5), we have
\[
\nabla_{(i_2 \circ i_1) X} (i_2 \circ i_1 Y) = i_2(\nabla'_{i_1} X) i_1 Y = i_2(i_1 \nabla_X Y + g(A'X, Y)N'),
\]
(4.6)
because \( M' \) is totally geodesic in \( S^{2m+1} \). Comparing (2.9) with (4.6), we easily see that
\[
N = i_2 N', \quad A = A'.
\]
(4.7)
Since \( M' \) is an invariant submanifold of \( S^{2m+1} \), for any \( X' \in TM' \),
\[
\phi i_2 X' = i_2 \phi' X'
\]
(4.8)
is valid, where $\phi'$ is the induced Sasakian structure of $M' = S^{n+2}$. Thus it follows from (2.3), (4.7) and (4.8) that
\[
\phi i X = \phi (i_2 \circ i_1) X = i_2 \phi' i_1 X = i_2 (i_1 F' X + u' (X) N') = i F' X + u' (X) N.
\]
Comparing this equation with (2.3), we have $F = F'$ and $u = u'$. By means of Lemma 4.1, it is clear that $M$ is a real hypersurface of $S^{n+2}$ which satisfies $F' A' = A' F'$. Thus, applying a theorem due to Kon ([8]), we may complete the proof of our main theorem. $\blacksquare$

References


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