A COMPLEX SURFACE OF GENERAL TYPE
WITH $p_g = 0$, $K^2 = 3$ AND $H_1 = \mathbb{Z}/2\mathbb{Z}$

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Abstract. As the sequel to our previous work [4], we construct a minimal complex surface of general type with $p_g = 0$, $K^2 = 3$ and $H_1 = \mathbb{Z}/2\mathbb{Z}$ by using a rational blow-down surgery and Q-Gorenstein smoothing theory.

1. Introduction

This paper is a continuation of our previous work [4], in which the authors constructed a simply connected minimal complex surface of general type with $p_g = 0$ and $K^2 = 3$. Motivated by Y. Lee and the second author’s recent construction [3] of a surface of general type with $p_g = 0$, $K^2 = 2$ and $H_1 = \mathbb{Z}/2\mathbb{Z}$, we extend the result to the $K^2 = 3$ case in this paper. That is, we construct a new non-simply connected minimal surface of general type with $p_g = 0$, $K^2 = 3$ and $H_1 = \mathbb{Z}/2\mathbb{Z}$ using a rational blow-down surgery and a Q-Gorenstein smoothing theory.

The key ingredient of this paper is to find a right rational surface $Z$ which makes it possible to get such a complex surface. Once we have a right candidate $Z$ for $K^2 = 3$, the remaining argument is similar to that of $K^2 = 3$ case appeared in our previous work [4]. That is, by applying a rational blow-down surgery and a Q-Gorenstein smoothing theory developed in Lee and Park [2] to $Z$, we obtain a minimal complex surface of general type with $p_g = 0$ and $K^2 = 3$. Then we show that the surface has $H_1 = \mathbb{Z}/2\mathbb{Z}$. Since almost all the proofs are parallel to the case of the main construction in the our previous work [4, §3], we only explain how to construct such a minimal complex surface. The main result of this paper is the following

**Theorem 1.** There exists a minimal complex surface of general type with $p_g = 0$, $K^2 = 3$ and $H_1 = \mathbb{Z}/2\mathbb{Z}$. 

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Remark. D. Cartwright and T. Steger [1] also constructed minimal surfaces of general type with \( p_g = 0 \), \( K^2 = 3 \) and \( \pi_1 = \mathbb{Z}/2\mathbb{Z} \) using a completely different method.

2. Main construction

We start with a special elliptic fibration \( Y := \mathbb{CP}^2 \# 9 \mathbb{CP}^2 \) which is used in the main construction of this paper. Let \( L_1, L_2, L_3 \) and \( A \) be lines in \( \mathbb{CP}^2 \) and let \( B \) be a smooth conic in \( \mathbb{CP}^2 \) intersecting as in Figure 1. We consider a pencil of cubics \( \{ \lambda(L_1 + L_2 + L_3) + \mu(A + B) \mid [\lambda : \mu] \in \mathbb{CP}^1 \} \) in \( \mathbb{CP}^2 \) generated by two cubic curves \( L_1 + L_2 + L_3 \) and \( A + B \), which has 5 base points, say, \( p, q, r, s \) and \( t \). In order to obtain an elliptic fibration over \( \mathbb{CP}^1 \) from the pencil, we blow up three times at \( q \) and twice at \( s \) and \( t \), respectively, including infinitely near base-points at each point. We perform two further blowing-ups at the base points \( p \) and \( r \). By blowing-up nine times, we resolve all base points (including infinitely near base-points) of the pencil and we then get an elliptic fibration \( Y = \mathbb{CP}^2 \# 9 \mathbb{CP}^2 \) over \( \mathbb{CP}^1 \) (Figure 2). We denote by \( E_i \) (or \( \tilde{E}_i \)), \( i = 1, \ldots, 9 \), the exceptional divisors (or their proper transforms in \( Y \), respectively) induced by the nine blowing-ups. Note that there are five sections of the elliptic fibration \( Y \) corresponding to the five base points \( p, q, r, s \), and \( t \), which are denoted by \( E_5, E_6, \ldots, E_9 \), respectively. Furthermore, the elliptic fibration \( Y \) has an \( I_7 \)-singular fiber consisting of the proper transforms \( \tilde{L}_i \) of \( L_i \) \( (i = 1, 2, 3) \), \( \tilde{E}_1, \tilde{E}_2, \tilde{E}_3 \) and \( \tilde{E}_4 \). Also \( Y \) has an \( I_2 \)-singular fiber consisting of the proper transforms \( \tilde{A} \) and \( \tilde{B} \) of \( A \) and \( B \), respectively. According to the list of Persson [5], we may assume that \( Y \) has three more nodal singular fibers by choosing generally \( L_i \)’s, \( A \) and \( B \). Among the three nodal singular fibers, we use only two nodal singular fibers, say \( F_1 \) and \( F_2 \), for the main construction (Figure 2). Next, by blowing-up several times on \( Y \), we construct a rational surface \( Z \) which contains special configurations of linear chains of \( \mathbb{CP}^1 \)’s. At first we blow up five times at the marked point \( \bigcirc \) on \( F_2 \cap E_5 \). We also blow up two times at

Figure 1. A pencil of cubics
A COMPLEX SURFACE WITH $p_g = 0$, $K^2 = 3$ AND $H_1 = \mathbb{Z}/2\mathbb{Z}$

Figure 2. An elliptic fibration $Y$

the marked point $\bigcirc$ on $F_2 \cap E_7$. Finally we blow up at the six marked points • on each fiber. We then get a rational surface $Z = Y # 13 \mathbb{CP}^2$. We denote by $e_i$ (or $\tilde{e}_i$), $i = 1, \ldots, 13$, the exceptional divisors (or their proper transforms in $Z$, respectively) induced by the 13 blow-ups and we also denote by $\widetilde{F}_i$ ($i = 1, 2$) the proper transforms of $F_i$. Then there exist two disjoint linear chains of $\mathbb{CP}^1$'s in $Z$:

$C_{110,67} = -2 -3 -5 -7 -2 -3 -2 -3 -2 -3 -3$ (which consists of $\tilde{e}_{12}, \tilde{E}_7, \tilde{F}_1, \tilde{E}_5, \tilde{L}_3, \tilde{E}_1, \tilde{L}_1, \tilde{E}_2, \tilde{E}_3, \tilde{E}_6, \tilde{B}$) and $C_{6,1} = -8 -2 -2 -2 -2$ (which consists of $\tilde{F}_2, \tilde{e}_8, \tilde{e}_7, \tilde{e}_6, \tilde{e}_5$) (Figure 3). Here $C_{p,q} = -b_k - b_{k-1} - \ldots - b_1$ is a small neighborhood linear chains of $\mathbb{CP}^1$ such that $p > q$, gcd($p,q) = 1$, $b_i \geq 2$, and $[b_k, \ldots, b_1]$ forms a continued fraction with

$$\frac{p^2}{pq - 1} = b_k - \frac{1}{b_{k-1} - \frac{1}{\ddots - \frac{1}{b_1}}}$$

Next, by applying $\mathbb{Q}$-Gorenstein smoothing theory as in our previous work [4], we construct the minimal complex surface appeared in the main theorem. That is, we first contract two disjoint chains $C_{110,67}$ and $C_{6,1}$ of $\mathbb{CP}^1$'s from $Z$ so that it produces a normal projective surface $X$ with two permissible singular points. And then, by using a similar technique in our previous work [4], we can conclude that $X$ has a $\mathbb{Q}$-Gorenstein smoothing and a general fiber $X_t$ of
the $\mathbb{Q}$-Gorenstein smoothing of $X$ is a minimal complex surface of general type with $p_g = 0$ and $K^2 = 3$. Let us denote a general fiber of the $\mathbb{Q}$-Gorenstein smoothing of $X$ by $X_t$. Finally it remains to show that $H_1(X_t; \mathbb{Z}) = \mathbb{Z}/2\mathbb{Z}$.

2.1. Proof of $H_1(X_t; \mathbb{Z}) = \mathbb{Z}/2\mathbb{Z}$

Let $Z_{110,6}$ be a rational blow-down 4-manifold obtained from $Z$ by replacing two disjoint configurations $C_{110,67}$ and $C_{6,1}$ with the corresponding rational balls $B_{110,67}$ and $B_{6,1}$, respectively. Then, since a general fiber $X_t$ of the $\mathbb{Q}$-Gorenstein smoothing of $X$ is diffeomorphic to the rational blow-down 4-manifold $Z_{110,6}$, we have $H_1(X_t; \mathbb{Z}) = H_1(Z_{110,6}; \mathbb{Z})$. Hence it suffices to show that $H_1(Z_{110,6}; \mathbb{Z}) = \mathbb{Z}/2\mathbb{Z}$.

**Proposition 2.** $H_1(Z_{110,6}; \mathbb{Z}) = \mathbb{Z}/2\mathbb{Z}$.

**Proof.** One can prove this proposition using the similar technique in Section 2 of Lee and Park [3]. Here we present another way to prove it as follows: First note that the rational surface $Z = Y^\sharp 13\mathbb{CP}^2$ can be decomposed into $Z = Z_0 \cup \{C_{110,67} \cup C_{6,1}\}$ and the rational blow-down 4-manifold $Z_{110,6}$ can be decomposed into $Z_{110,6} = Z_0 \cup \{B_{110,67} \cup B_{6,1}\}$. Let $W = Z_0 \cup B_{110,67}$ and consider the following exact homology sequence for a pair $(W, \partial W)$:

$$\cdots \to H_2(W, \partial W; \mathbb{Z}) \xrightarrow{\partial} H_1(\partial W; \mathbb{Z}) \xrightarrow{i^*} H_1(W; \mathbb{Z}) \to H_1(W, \partial W; \mathbb{Z}) = 0.$$

Here the last term is zero because the punctured exceptional curve $e_{11}$ lies in $Z_0 \cup C_{6,1}$ (refer to Figure 3) and $Z_0 \cup C_{6,1}$ is simply connected. So $Z_0 \cup C_{6,1}$
A complex surface with $p_g = 0$, $K^2 = 3$ and $H_1 = \mathbb{Z}/2\mathbb{Z}$ is also simply connected by Van Kampen Theorem and $H_1(W, \partial W; \mathbb{Z}) \cong H_1(Z_6 \cup C_{6,1} \cup B_{110,67}, C_{6,1}; \mathbb{Z}) = 0$. Note that $\partial W = \partial B_{6,1} = L(36, -5)$ and a generator of $H_1(\partial W; \mathbb{Z}) = \mathbb{Z}/36\mathbb{Z}$ can be represented by a normal circle, say $\alpha$, of a disk bundle $C_{6,1}$ over the $(-8)$-curve $\tilde{F}_2$. Then we have
\[
\partial_*(\{e\}_W) = 2\alpha \in H_1(\partial W; \mathbb{Z}) = \mathbb{Z}/36\mathbb{Z}.
\]
Furthermore, by choosing a suitable basis $B$ of $H_2(W, \partial W; \mathbb{Z})$ and by evaluating $B$ under $\partial_*$, we can conclude that the generator $\alpha \in H_1(\partial W; \mathbb{Z})$ is not in the image of $\partial_*$. Hence it follows from the exact sequence above that we have $H_1(W; \mathbb{Z}) = \mathbb{Z}/2\mathbb{Z}$ and $H_1(\partial W; \mathbb{Z})$ is generated by the element $i_*(\alpha)$.

Next, we consider the Mayer-Vietoris sequence for a triple $(Z_{110,6}; W, B_{6,1})$:
\[
\begin{align*}
H_2(Z_{110,6}; \mathbb{Z}) & \xrightarrow{\partial_2} H_1(L(36, -5); \mathbb{Z}) \xrightarrow{i_* \oplus j_*} H_1(W; \mathbb{Z}) \oplus H_1(B_{6,1}; \mathbb{Z}) \\
& \xrightarrow{i_*} H_1(Z_{110,6}; \mathbb{Z}) \rightarrow 0.
\end{align*}
\]
Since the map $i_* \oplus j_*$ sends the generator $\alpha$ to (a generator, a generator), we finally have $H_1(Z_{110,6}; \mathbb{Z}) = \mathbb{Z}/2\mathbb{Z}$.

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