SOME TOEPLITZ OPERATORS ON WEIGHTED BERGMAN SPACES

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Abstract. We consider the problem to determine when a Toeplitz operator is bounded on weighted Bergman spaces. We show that Toeplitz operators induced by elements of some set are bounded and each element of the set is related with a Carleson measure on the weighted Bergman space.

1. Introduction

Let $dA$ denote normalized Lebesgue area measure on the unit disk $\mathbb{D}$. For $\alpha > -1$, the weighted Bergman space $A^2_\alpha$ consists of the analytic functions in $L^2(\mathbb{D}, dA_\alpha)$, where $dA_\alpha(z) = (\alpha + 1)(1 - |z|^2)^\alpha dA(z)$. Then $A^2_\alpha$ is closed in $L^2(\mathbb{D}, dA_\alpha)$ and for each $z \in \mathbb{D}$, there is a reproducing kernel $K^\alpha_z$ in $A^2_\alpha$ such that $f(z) = \langle f, K^\alpha_z \rangle$ for all $f \in A^2_\alpha$, in fact, $K^\alpha_z(w) = \frac{1}{(1 - zw)^\frac{1}{\alpha + 2}}$ and the normalized reproducing kernel $k^\alpha_z$ is the function $\frac{K^\alpha_z}{\|K^\alpha_z\|_{2,\alpha}}$, that is, $k^\alpha_z(w) = \frac{(1 - |z|^2)^{1 + \frac{\alpha}{2}}}{(1 - zw)^{\frac{1}{\alpha + 2}}}$, where the norm $\| \|_{2,\alpha}$ and the inner product are taken in the space $L^2(\mathbb{D}, dA_\alpha)$.

A linear operator $S$ on $A^2_\alpha$ induces a function $\hat{S}$ on $\mathbb{D}$ given by $\hat{S}(z) = \langle Sk^\alpha_z, k^\alpha_z \rangle$, $z \in \mathbb{D}$. The function $\hat{S}$ is called the Berezin transform of $S$.

For $u \in L^1(\mathbb{D}, dA)$, the Toeplitz operator $T^\alpha_u$ with symbol $u$ is the operator on $A^2_\alpha$ defined by $T^\alpha_u(f) = P_\alpha(uf)$, $f \in A^2_\alpha$, where $P_\alpha$ is the orthogonal projection from $L^2(\mathbb{D}, dA_\alpha)$ onto $A^2_\alpha$ and let $\hat{u}$ denote $T^\alpha_u$. Then the Toeplitz operator $T^\alpha_u$ is bounded whenever $u \in L^\infty(\mathbb{D}, dA)$ but every element of $L^1(\mathbb{D}, dA)$ does not imply the boundedness of the Toeplitz operator $T^\alpha_u$. Since $L^\infty(\mathbb{D}, dA)$ is dense in $L^1(\mathbb{D}, dA)$, $T^\alpha_u$ is densely defined on $A^2_\alpha$. We note that Berezin transforms and Carleson measures are useful tools in the study of Toeplitz operators ([2], [4], [5]). Using those tools, many mathematicians working in operator theory are characterized the boundedness and compactness of Toeplitz operators.

Received May 26, 2009; Revised February 25, 2010.
2010 Mathematics Subject Classification. 47A15 and 47B35.
Key words and phrases. weighted Bergman spaces, Toeplitz operators, Berezin transforms, integral operators.
This research was partially supported by Sookmyung Women’s University Research Grants 2009.
In this paper, we prove that Toeplitz operators with special symbols are bounded and \( \|uk_\alpha^o\|_{p,\alpha} \) having vanishing property implies the compactness of the Toeplitz operator \( T^o_u \).

Section 3 contains some properties of special symbols, that is, each element of some set implies a Carleson measure and we deal with appropriate products of Toeplitz operators and Hankel operators.

2. Unitary operator and example

Let \( \text{Aut}(\mathbb{D}) \) denote the set of all bianalytic maps of \( \mathbb{D} \) onto \( \mathbb{D} \). By Schwarz’s lemma, each element of \( \text{Aut}(\mathbb{D}) \) is a linear fractional transformation of the form \( \lambda \varphi_z, |\lambda| = 1, \) where \( \varphi_z(w) = \frac{z-w}{1-\overline{z}w}, w \in \mathbb{D}. \) Then \( \varphi_z \circ \varphi_z \) is the identity function on \( \mathbb{D} \) and \( \text{Aut}(\mathbb{D}) \) is called the Möbius group under composition. For \( \alpha > -1 \) and \( z \in \mathbb{D}, \) we define \( U^o_\alpha f(w) = f \circ \varphi_z(w) \left( \frac{1-|z|^2}{1-\overline{z}w} \right)^{\frac{\alpha}{2}+1}, f \in L^2(\mathbb{D}, dA_\alpha), w \in \mathbb{D}. \)

A simple compactation shows that \( U^o_\alpha \) is an isometry. Since \( (1 - \overline{z}\varphi_z(w))^\alpha = \left( \frac{1-|z|^2}{1-\overline{z}w} \right)^{\alpha+2}, U^o_\alpha U^*_\alpha = \text{identity operator and hence } (U^o_\alpha)^* = (U^o_\alpha)^{-1} = U^o_\alpha; \) that is, \( U^o_\alpha \) is a self-adjoint unitary operator on \( A^2_\alpha. \) Moreover, \( U^o_\alpha(A^2_\alpha) = A^2_\alpha \) and \( U^o_\alpha \) is also denoted by \( U^o_\alpha \) and \( U^o_\alpha 1 = k^o_z(w). \)

For a linear operator \( S \) on \( A^2_\alpha, \) we define the conjugate operator \( S_z \) by \( U^o_\alpha SU^o_\alpha. \)

Now we are ready to state useful properties.

Lemma 2.1. For \( u \in L^1(\mathbb{D}, dA) \) and \( z \in \mathbb{D}, \) \( (T^o_u)_z = T^o_{u \circ \varphi_z}. \)

Proof. Since \( (T^o_u)_z = U^o_z T^o_u U^o_z \) and \( (U^o_\alpha)^{-1} = U^o_\alpha, \) it is enough to show that \( U^o_\alpha T^o_u = T^o_{u \circ \varphi_z} U^o_\alpha. \) Take any \( f \in A^2_\alpha \) and any \( w \in \mathbb{D}. \) Then

\[
U^o_\alpha T^o_u (f)(w) = U^o_\alpha P_\alpha (uf)(w)
\]

\[
= P_\alpha (uf)(\varphi_z(w)) \frac{(1-|z|^2)^{\frac{\alpha}{2}+1}}{(1-\overline{z}w)^{\alpha+2}}
\]

\[
= (a+1) \int_\mathbb{D} \frac{u(t)f(t)(1-|t|^2)^{\alpha}}{(1-\varphi_z(t)^2)^{\frac{\alpha}{2}+1}} dA(t) \frac{(1-|z|^2)^{\frac{\alpha}{2}+1}}{(1-\overline{z}w)^{\alpha+2}}
\]

\[
= (a+1) \int_\mathbb{D} \frac{u(t)f(t)(1-|t|^2)^{\alpha}}{(1-\varphi_z(t)^2)^{\frac{\alpha}{2}+1}} \frac{(1-\overline{z}w)^{\alpha+2}}{(1-\overline{z}w-zt+wt)^{\frac{\alpha}{2}+1}} dA(t)
\]

\[
= (a+1) \int_\mathbb{D} \frac{u(t)f(t)(1-|t|^2)^{\alpha}}{(1-\varphi_z(t)^2)^{\frac{\alpha}{2}+1}} \frac{(1-\overline{z}w-zt+wt)^{\frac{\alpha}{2}+1}}{(1-\overline{z}w-zt+wt)^{\frac{\alpha}{2}+1}} dA(t)
\]

\[
= (a+1) \int_\mathbb{D} u \circ \varphi_z(s)f \circ \varphi_z(s)(1-|\varphi_z(s)|^2)^{\alpha} \frac{(1-|z|^2)^{\frac{\alpha}{2}+1}}{(1-\overline{z}w-z\varphi_z(s)+w\varphi_z(s))^{\frac{\alpha}{2}+1}}
\]
\[(\alpha + 1) \int_D u \circ \varphi_z(s) f \circ \varphi_z(s) \frac{(1 - |z|^2)\alpha(1 - |s|^2)\alpha}{|1 - zs|^{2\alpha}} (1 - |z|^2)\frac{z}{|1 - zs|^2} \]
\[\times \frac{(1 -zs)^{2+\alpha}dA(s)}{(1 - |z|^2)^{2+\alpha}(1 - w\bar{s})^{2+\alpha}}\]
\[(\alpha + 1) \int_D u \circ \varphi_z(s) f \circ \varphi_z(s) \frac{(1 - |s|^2)\alpha(1 - |z|^2)\frac{z}{|1 - zs|^2}}{(1 -zs)^{2+\alpha}(1 - w\bar{s})^{2+\alpha}} (1 - |z|^2)^{2+\alpha}dA(s)\]
\[(\alpha + 1) \int_D u \circ \varphi_z(s) f \circ \varphi_z(s) \frac{(1 - |z|^2)\frac{z}{|1 - zs|^2}}{(1 -zs)^{2+\alpha}(1 - w\bar{s})^{2+\alpha}}dA(s)\]
\[= \int_D u \circ \varphi_z(s)U_z^\alpha f(s) \frac{dA_\alpha(s)}{(1 - w\bar{s})^{2+\alpha}}\]
\[= P_\alpha(u \circ \varphi_z U_z^\alpha)(w).\]

Thus \((T_u^\alpha)^{-1} = T_u^\alpha_{\varphi_z}.\) □

**Corollary 2.2.** If \(u_1, \ldots, u_n \in L^1(\mathbb{D}, dA),\) then
\[U_z^\alpha T_{u_1} \cdots T_{u_n} U_z^\alpha = T_{u_1 \circ \varphi_z} \cdots T_{u_n \circ \varphi_z}.\]

**Proof.** It follows from the fact that \(U_z^\alpha U_z^\alpha = \text{is the identity}.\) □

**Proposition 2.3.** For \(u \in L^1(\mathbb{D}, dA)\) and \(z \in \mathbb{D},\) \(\bar{T}_u^\alpha \circ \varphi_z = (\bar{T}_u^\alpha)_z.\)

**Proof.** Since \((\bar{T}_u^\alpha)_z = \bar{T}_u^\alpha_{\varphi_z},\) it is enough to show that \(\bar{T}_u^\alpha_{\varphi_z} = \bar{T}_u^\alpha \circ \varphi_z.\)

Take any \(t \in \mathbb{D}.\) Then
\[\bar{T}_u^\alpha \circ \varphi_z(t)\]
\[= (T_u^\alpha f_{\varphi_z(t)}, k_{\varphi_z(t)})\]
\[= (P_\alpha(uk_{\varphi_z(t)}^\alpha), k_{\varphi_z(t)}^\alpha)\]
\[= (uk_{\varphi_z(t)}^\alpha, k_{\varphi_z(t)}^\alpha)\]
\[= \int_D u(x) k_{\varphi_z(t)}^\alpha(x) \bar{k}_{\varphi_z(t)}^\alpha(x) dA_\alpha(x)\]
\[= \int_D u(x) \left(1 - |x|^2\right)^{1+\frac{\alpha}{2}} \left(\frac{1}{1 - \bar{z}t} \frac{1 - zt}{1 - \bar{z}t - \bar{x}t + \bar{x}z}\right)^{2+\alpha} \left(\frac{1 - |x|^2}{1 - |x|^2}\right)\]
\[\times \left(\frac{1 - \bar{z}t}{1 - \bar{z}t - \bar{x}t + \bar{x}z}\right)^{2+\alpha} (\alpha + 1)(1 - |x|^2)^\alpha dA(x)\]
\[= \int_D u(x) \left(1 - |x|^2\right)^{1+\frac{\alpha}{2}} \left(\frac{1}{1 - \bar{z}t(1 - \bar{z}t)}\right)\]
\[\left(\frac{1 - zt}{1 - \bar{z}t - \bar{x}t + \bar{x}z}\right)^{2+\alpha} (\alpha + 1)(1 - |x|^2)^\alpha dA(x)\]
\[\times \left(\frac{1 - |x|^2}{1 - |x|^2}\right)^{1+\frac{\alpha}{2}} \left(\frac{1}{1 - \bar{z}t(1 - \bar{z}t)}\right)^{2+\alpha}\]
This completes the proof. □

We will show that the Toeplitz operators with special symbols are bounded. To do so, we need the following proposition, in fact, the following proposition holds for every linear operator on $A^2_b$.

**Proposition 2.4.** If $S$ is a linear operator on $A^2_b$ and $z, w \in \mathbb{D}$, then $(S^*)^* = (S^*)_z$ and $SK^\alpha_w(z)$. 

**Proof.** Take any $f, g$ in $A^2_b$. Since $\langle Sz, g \rangle = \langle U^\alpha_z SU^\alpha_z f, g \rangle = \langle f, U^\alpha_z S^* U^\alpha_z g \rangle = \langle f, (S^*)_z g \rangle$, $(S^*)_z = (S^*)^*$. For the 2nd equality, $SK^\alpha_w(z) = \langle SK^\alpha_w, K^\alpha_z \rangle = \langle K^\alpha_w, SK^\alpha_z \rangle = SK^\alpha_z(z)$. □

Since $K^\alpha_w(z) = \frac{1}{(1 - zw)^{1 + \alpha}} = \sum_{n=0}^{\infty} \frac{\Gamma(n+2+\alpha)}{\Gamma(2+\alpha)} z^n w^n$, $k^\alpha_z(w) = (1 - |z|^2)^{1+\alpha} \sum_{n=0}^{\infty} \frac{\Gamma(n+2+\alpha)}{\Gamma(2+\alpha)} z^n w^n$. We define $S(\sum a_n w^n) = \sum a_n (-w)^n$. Then $S(K^\alpha_w(z)) = \sum_{n=0}^{\infty} \frac{\Gamma(n+2+\alpha)}{\Gamma(2+\alpha)} z^n (-1)^n w^n = K^\alpha_z(-w)$ and $S$ has an infinite-dimensional range and $S$ is an isometry and invertible, that is, $S^* = S^{-1} = S$. Thus $S$ is not compact. Since $\tilde{S}(z) = \langle SK^\alpha_w, k^\alpha_z \rangle = (1 - |z|^2)^{2+\alpha} \langle SK^\alpha_w, K^\alpha_z \rangle = (1 - |z|^2)^{2+\alpha} K^\alpha_z(-z) = \left( \frac{1-|z|^2}{1+|z|^2} \right)^{2+\alpha}$, $\lim_{z \to \partial \mathbb{D}} \tilde{S}(z) = 0$ and hence the vanishing property does not imply the compactness of operators.

### 3. Toeplitz operators with special symbols

This section deals with Toeplitz operators with special symbols. We begin by constructing some set and show that each element of the set implies a bounded linear operator. Recall that $P_s$ is the orthogonal projection from $L^2(\mathbb{D}, dA_\theta)$ onto $A^2_b$ and for $z \in \mathbb{D}$ and $f \in L^2(\mathbb{D}, dA_\theta)$, $P_s(f)(z) = \langle P_s(f), K^\alpha_z \rangle = \int_\mathbb{D} f(w) K^\alpha_w(z) dA_\theta(w)$. Moreover, we extend the domain of $P_s$ to $L^1(\mathbb{D}, dA_\theta)$ and for $f \in A^2_b$, $f(z) = \int_\mathbb{D} f(w) dA_\theta(w) (1 - |z|^2)^{1+\alpha}$, $z \in \mathbb{D}$ (see [5]).
Let $MK = \{ u \in L^1(\mathbb{D}, dA) : \sup_{\lambda \in \mathbb{D}} \| u \xi^\lambda \|_{p,\alpha} < \infty \text{ for every } p \in (1, \infty) \}$, where $\| \cdot \|_{p,\alpha}$ is the norm on $L^p(\mathbb{D}, dA_\alpha)$. Since $\| uk_\lambda \|_{p,\alpha} = \| \overline{u}k_\lambda \|_{p,\alpha}$, $MK$ is closed under the formation of conjugations.

**Lemma 3.1.** For any $u \in MK$, $(T_u^\alpha)^* = T_{\overline{u}}^\alpha$.

**Proof.** Take any $f, g$ in $A^2_\alpha$. Since $(T_u^\alpha f, g) = \langle P_\alpha (uf), g \rangle = \langle uf, g \rangle = \langle f, P_\alpha(\overline{g}) \rangle = \langle f, T_{\overline{u}}^\alpha g \rangle$, $(T_u^\alpha)^* = T_{\overline{u}}^\alpha$. $\Box$

**Lemma 3.2.** Suppose $u \in MK$, $z \in \mathbb{D}$, and $p \in (1, \infty)$. Then there is a constant $c > 0$ such that $\| (T_u^\alpha z)_1 \|_{p,\alpha} \leq c \| uk_\alpha \|_{p,\alpha}$.

**Proof.** Since $(T_u^\alpha z)_z = U_z^\alpha T_u^\alpha U_z^\alpha$ and $U_z^\alpha 1 = k_z^\alpha$, $\| (T_u^\alpha z)_1 \|_{p,\alpha} = \| U_z^\alpha T_u^\alpha U_z^\alpha 1 \|_{p,\alpha} = \| U_z^\alpha P_\alpha(uk_\alpha) \|_{p,\alpha} \leq \| P_\alpha \| \| uk_\alpha \|_{p,\alpha}$ and hence $\| (T_u^\alpha z)_1 \|_{p,\alpha} \leq c \| uk_\alpha \|_{p,\alpha}$ for some $c$.

Suppose $f \in A^2_\alpha$ and $w \in \mathbb{D}$. Then $(T_u^\alpha f)(w) = (T_u^\alpha f, K^\alpha_w) = \int_\mathbb{D} f(z) T_{\overline{u}}^\alpha K^\alpha_w(z) dA_\alpha(z) = \int_\mathbb{D} f(z) (T_{\overline{u}}^\alpha K^\alpha_w)(w) dA_\alpha(z)$, here the last equality follows from Proposition 2.4. Thus $T_u^\alpha$ is the integral operator with kernel $(T_{\overline{u}}^\alpha K^\alpha_w)(w)$.

**Lemma 3.3.** Suppose $0 < t$, $s \in (1, \infty)$ and $u \in MK$. Then there is a constant $c > 0$ such that $\int_\mathbb{D} \frac{(T_u^\alpha)^s |\phi_z|^t |k_z^\alpha|^s}{(1-|z|^2)^{t+s}} dA_\alpha(z) \leq c \| uk_\alpha \|_{p,\alpha}$ for all $z \in \mathbb{D}$ and $\int_\mathbb{D} \frac{|(T_u^\alpha)^s |\phi_z|^t |k_z^\alpha|^s}{(1-|z|^2)^{t+s}} dA_\alpha(z) \leq c \| uk_\alpha \|_{p,\alpha}$ for all $w \in \mathbb{D}$.

**Proof.** Since $k_z^\alpha = (1-|z|^2)^{\frac{1}{2} + \frac{t}{2}} K^\alpha_z$, $T_u^\alpha K^\alpha_z = \frac{U_z^\alpha (T_u^\alpha 1)}{(1-|z|^2)^{t+s}} = \frac{(T_u^\alpha)^s |\phi_z|^t |k_z^\alpha|^s}{(1-|z|^2)^{t+s}}$ and hence

$$\int_{\mathbb{D}} \frac{|(T_u^\alpha)^s |\phi_z|^t |k_z^\alpha|^s}{(1-|z|^2)^{t+s}} dA_\alpha(w)$$

$$= \int_{\mathbb{D}} \frac{|(T_u^\alpha z)_1 \cdot \phi_z(w) (1-|z|^2)^{\frac{t}{2} + 1}}{(1-|z|^2)^{t+s}} dA_\alpha(w)$$

$$= \int_{\mathbb{D}} \frac{|(T_u^\alpha z)_1 (\lambda) (1-|z|^2)^{\frac{t}{2} + 1}}{(1-|z|^2)^{t+s}} dA_\alpha(w)$$

$$= \int_{\mathbb{D}} \frac{|(T_u^\alpha z)_1 (\lambda) (1-|\phi_z(\lambda)|^\alpha)^{\frac{t}{2} + 1}}{(1-|\phi_z(\lambda)|^\alpha)^{t+s}} dA_\alpha(\lambda)$$

$$= \int_{\mathbb{D}} \frac{|(T_u^\alpha z)_1 (\lambda) (1-|\phi_z(\lambda)|^\alpha)^{\frac{t}{2} + 1}}{(1-|\phi_z(\lambda)|^\alpha)^{t+s}} dA_\alpha(\lambda)$$

$$= \frac{1}{(1-|z|^2)^t} \int_{\mathbb{D}} \frac{1}{1-|z|^2} \frac{1}{(1-|z|^2)^{t+s}} dA_\alpha(\lambda)$$

$$\leq \frac{1}{(1-|z|^2)^t} \int_{\mathbb{D}} \frac{1}{1-|z|^2} \frac{1}{(1-|z|^2)^{t+s}} dA_\alpha(\lambda)$$
where \( s \) and \( s' \) are conjugate exponents.

By Lemma 3.2, there is a constant \( c_1 \), such that \( \| (T^\alpha_w)_{z,1} \|_{s,\alpha} \leq c_1 \| u k^\alpha _w \|_{s,\alpha} \).

Let \( c = c_1 \left( \int_{|z| \geq |\lambda|^{(2-2t+\alpha)^r} \left( 1 - |\lambda|^2 \right)^{1-t}} dA(\lambda) \right)^{\frac{1}{r}} \). If \( c \) is infinity, then trivially the inequality holds and hence \( \int_{|z|} \| (T^\alpha_w K^\alpha_s)(z) \|_{s,\alpha} dA_\alpha(z) \leq c \| u k^\alpha _w \|_{s,\alpha} \). Proposition 2.4 and Lemma 3.1 and \( MK \subseteq \mathcal{M}K \) imply that there is a constant \( c \) such that \( \int_{|z|} \| (T^\alpha_w K^\alpha_s)(z) \|_{s,\alpha} dA_\alpha(z) \leq c \| u k^\alpha _w \|_{s,\alpha} \). This completes the proof.

Suppose \( \alpha \neq 0 \). Then there is \( t > 0 \) such that \( s' = \frac{2\alpha}{4t-2-\alpha} > 1 \) and

\[
\int_{|z|} \frac{1}{(1 - |\lambda|^{(2-2t+\alpha)^r} \left( 1 - |\lambda|^2 \right)^{1-t}} dA(\lambda) = \int_{|z|} \frac{1}{(1 - |\lambda|^{(2-2t+\alpha)^r} \left( 1 - |\lambda|^2 \right)^{1-t}} dA(\lambda).
\]

Axler’s paper ([1], Lemma 4) asserts that the last integral is finite. In Lemma 3.3, \( c \) is finite.

**Theorem 3.4.** For each \( u \in \mathcal{M}K \), \( T^\alpha_w \) is bounded.

**Proof.** Let \( h(\lambda) = \left( \frac{1}{(1 - |\lambda|^2) \lambda^2} \right)^{\frac{1}{r}} \). By the above observation and Lemma 3.3,

\[
\int_{|z|} \frac{\| (T^\alpha_w K^\alpha_s)(w) \|_{s,\alpha}}{(1 - |w|^2) \lambda^2} dA_\alpha(w) \leq c_1 h(z) \text{ and } \int_{|z|} \frac{\| (T^\alpha_w K^\alpha_s)(z) \|_{s,\alpha}}{(1 - |w|^2) \lambda^2} dA_\alpha(z) \leq c_2 h(w),
\]

where \( c_1 = c \sup \| u k^\alpha _w \|_{s,\alpha} \) and \( c_2 = c \sup \| u k^\alpha _w \|_{s,\alpha} \).

The Schur’s test (see page 126 of [3]) implies that \( T^\alpha_w \) is bounded and \( \| T^\alpha_w \| \leq \sqrt{c_1c_2} \).

Recall that \( T^\alpha_w \) is the integral operator with kernel \( (T^\alpha_w K^\alpha_s)(w) \), that is,

\( (T^\alpha_w f)(w) = \int_{D} f(z) (T^\alpha_w K^\alpha_s)(w) dA(z) \). For \( 0 < r < 1 \), we define an operator \( T^\alpha_w \) on \( \mathcal{A}^2_\alpha \) by \( (T^\alpha_w f)(w) = \int_{D} f(z) T^\alpha_w K^\alpha_s(w) dA(z) \). Since \( \int_{D} \int_{D} |T^\alpha_w K^\alpha_s(w)\chi_D(z)|^2 dA_\alpha(w) dA_\alpha(z) = \int_{D} \int_{D} |T^\alpha_w K^\alpha_s(z)\chi_D(w)|^2 dA_\alpha(w) dA_\alpha(z) \leq \| T^\alpha_w \|_2^2 \int_{D} \| K^\alpha_s \|_2^2 dA(z) < \infty \), \( T^\alpha_w K^\alpha_s(w)\chi_D(z) \in L^2(D \times D, dA_\alpha \times dA_\alpha) \) and hence \( T^\alpha_w \) is a Hilbert-Schmidt operator. Thus each \( T^\alpha_w \) is compact. Let \( h(\lambda) = \left( \frac{1}{(1 - |\lambda|^2) \lambda^2} \right)^{\frac{1}{r}} \). By Lemma 3.3 and Theorem 3.4,

\[
\int_{D} |T^\alpha_w K^\alpha_s(w)\chi_{D \setminus \partial D}(z)| h(w) dA_\alpha(w) \leq c_1 h(z) \text{ and } \int_{D} |T^\alpha_w K^\alpha_s(z)\chi_{D \setminus \partial D}(z)| h(d) dA_\alpha(z) \leq c_2 h(w),
\]

where \( c_1 = c \sup \| u k^\alpha _w \|_{s,\alpha} \) and \( c_2 = c \sup \| u k^\alpha _w \|_{s,\alpha} \). The Schur’s test implies that \( \| T^\alpha_w - T^\alpha_w \| \leq c_1 c_2 \). If \( \lim_{\|w\|_{s,\alpha} \to 0} \| u k^\alpha _w \|_{s,\alpha} = 0 \), then \( \lim_{\|w\|_{s,\alpha} \to 0} c_1 = 0 \) and hence \( \lim_{\|w\|_{s,\alpha} \to 0} c_2 = 0 \). Since each \( T^\alpha_w \) is compact, \( T^\alpha_w \) is also a compact operator. Thus we have the following theorem.
Theorem 3.5. Let $u \in MK$. If $\lim_{z \to \partial D} \|uk_z^\alpha\|_{p,\alpha} = 0$ for every $p \in (1, \infty)$, then $T_u^\alpha$ is a compact operator.

We note that $A_u^\alpha$ consists of the analytic functions in $L^p(D, d\mu)$. Suppose $\mu$ is a finite positive Borel measure on $\mathbb{D}$ and $p > 1$. Recall that if $i_p : A_u^\alpha \to L^p(D, d\mu)$ is bounded, then $\mu$ is called a Carleson measure on the Bergman space $A_u^\alpha$ and Carleson measures are very useful tools in operator theory.

Proposition 3.6. For $u \in MK$, $|u|dA_\alpha$ is a Carleson measure on $A_u^\alpha$.

Proof. It is enough to show that $\widetilde{u}$ is bounded. For $w \in D$, $|\widetilde{u}(w)| = |\langle T_u^\alpha k_w^\alpha, k_w^\alpha \rangle| = |\langle P_u(uk_w^\alpha), k_w^\alpha \rangle| = |\langle uk_w^\alpha, k_w^\alpha \rangle| \leq \|uk_w^\alpha\|_{2,\alpha} < \infty$. Thus $|u|dA_\alpha$ is a Carleson measure on $A_u^\alpha$.

Corollary 3.7. For $u \in MK$, $T_u^\alpha$ is a bounded linear operator.

Proof. It follows from the fact that $|u|dA_\alpha$ is a Carleson measure.

Using the concept of a Carleson measure, we can give another proof for Theorem 3.5.

Proposition 3.8. If $\|uk_z^\alpha\|_{p,\alpha} \to 0$ as $z \to \partial D$ for every $p \in (1, \infty)$ and $u \in MK$, then $T_u^\alpha$ is compact.

Proof. Let’s show that $|u|dA_\alpha$ is a vanishing Carleson measure on the Bergman space $A_u^\alpha$. To do so, it is enough to show that $\lim_{|z| \to 1^-} |\widetilde{u}(z)| = 0$. For $z \in D$, $|\widetilde{u}(z)| = |\langle T_u^\alpha k_z^\alpha, k_z^\alpha \rangle| = |\langle uk_z^\alpha, k_z^\alpha \rangle| \leq \|uk_z^\alpha\|_{2,\alpha} \|k_z^\alpha\|_{2,\alpha} = \|uk_z^\alpha\|_{2,\alpha}$. The property $\lim_{|z| \to 1^-} \|uk_z^\alpha\|_{2,\alpha} = 0$ implies that $|u|dA_\alpha$ is a vanishing Carleson measure. Thus $T_u^\alpha$ is compact. Since $\int_D |f|_D^2 dA_\alpha \leq \int_D |u|_D^2 |u|dA_\alpha$, $T_u^\alpha$ is also compact.

We define an operator $H_u^\alpha : A_u^\alpha \to (A_u^\alpha)^\perp$ by

$H_u^\alpha(g) = (I - P_u)(ug), \ g \in A_u^\alpha$.

Then $H_u^\alpha$ is called the Hankel operator on the weighted Bergman space with symbol $u$. Clearly $H_u^\alpha$ is densely defined for any $u \in L^1(D, dA)$ and if $u \in L^\infty(D, dA)$, then $H_u^\alpha$ is bounded with $\|H_u^\alpha\| \leq \|u\|_\infty$.

Proposition 3.9. If $u^2 \in MK$, then $H_u^\alpha$ is bounded.

Proof. Take any $f$ in $A_u^\alpha$. Then $\|H_u^\alpha(f)\|_{2,\alpha}^2 = \|(I - P_u)(uf)\|_{2,\alpha}^2 \leq \|uf\|_{2,\alpha}^2 = \int_D |f(z)|^2 |u(z)|^2 dA_\alpha(z)$. By Proposition 3.6, $|u|^2 dA_\alpha$ is a Carleson measure on $A_u^\alpha$ and hence there is a constant $c < \infty$ such that $\int_D |f(z)|^2 |u(z)|^2 dA_\alpha(z) \leq c \int_D |f(z)|^2 dA_\alpha(z)$. Thus $H_u^\alpha$ is bounded.

Corollary 3.10. (1) Suppose $u^2 \in MK$. Then $(H_u^\alpha)^{1/2}$ and $H_u^\alpha k_z^\alpha$ are in $L^2(D, dA_\alpha)$ for every $z \in \mathbb{D}$.

(2) Suppose $u^2 \in MK$ and $z \in \mathbb{D}$. Then $H_u^\alpha \circ \varphi_z$ is bounded.
Proof. (1) We note that \( \|H_u^\alpha U_z\|_{2,\alpha} = \|H_u^\alpha k_z\|_{2,\alpha} \) and hence \( \|(H_u^\alpha)_z\|_{2,\alpha} = \|H_u^\alpha k_z\|_{2,\alpha} \). Since \( \|(H_u^\alpha)_z\|_{2,\alpha} = \|U_z^t H_u^\alpha U_z^t\|_{2,\alpha} = \|H_u^\alpha k_z\|_{2,\alpha} \leq \|H_u^\alpha\| \), we have the results.

(2) By Lemma 2.1, \( (T_u^\alpha)_z = T_u^\alpha \). Then \( (H_u^\alpha)_z = (I - T_u^\alpha)_z = I - T_u^\alpha = H_u^\alpha \). For \( f \in A^3_\alpha \), \( \|H_u^\alpha U_z^\alpha (f)\|_{2,\alpha} = \|(H_u^\alpha)_z(f)\|_{2,\alpha} = \|U_z^t H_u^\alpha U_z^t(f)\|_{2,\alpha} = \|H_u^\alpha\| \|U_z(f)\|_{2,\alpha} \). Thus \( \|H_u^\alpha U_z^\alpha\| \leq \|H_u^\alpha\| \). Since \( H_u^\alpha \) is bounded, \( H_u^\alpha \) is bounded. □

Consider some products of Toeplitz operators and Hankel operators. The simple calculation implies that \( H_u^\alpha H_u^\beta = T_{u^\alpha \beta} - T_{u^\beta u} \). Suppose \( u \in L^1(\mathbb{D}, dA) \) and \( f \in A^3_\alpha \). Since \( H_u^\alpha f(z) = u(z)f(z) - P_\alpha(u)f(z) = u(z)\langle f, K_z^\alpha \rangle = \langle (u(z) - u)f, K_z^\alpha \rangle \) for \( g \in (A^3_\alpha)^{1} \),

\[
\langle H_u^\alpha f, g \rangle = \int_\mathbb{D} (u(z) - u(w))\overline{K_w^\alpha(f)(w)}dA_w(w)\overline{g(z)}dA_z(z) = \int_\mathbb{D} f(w)\overline{K_w^\alpha(z)}dA_w(w)\overline{g(z)}dA_z(z) = \int_\mathbb{D} f(w)(-H_w^\alpha\overline{g(w)})dA_w(w) = \langle f, -H_w^\alpha g \rangle
\]

and hence \( H_u^\alpha = -H_u^\alpha \).

Suppose \( u, v, u^2, v^2 \) are in \( MK \). If \( H_u^\alpha \) is compact, then the following are compact:

(i) \( T_{u^\alpha} - T_{u^\beta} \) \( T_{u^\alpha} \) (ii) \( T_{v^\alpha} - T_{v^\beta} \) \( T_{v^\alpha} \) (iii) \( H_v^\alpha T_{v^\alpha} \) (iv) \( H_v^\alpha H_u^\alpha = H_v^\alpha H_u^\alpha \).

Corollary 3.11. Suppose \( u_1, \ldots, u_n \in MK \) and \( z \in \mathbb{D} \). Then

\[
U_z H_{u_1}^\alpha \cdots H_{u_n}^\alpha U_z = H_{u_1 \circ \varphi_z}^\alpha \cdots H_{u_n \circ \varphi_z}^\alpha.
\]

Proof. It follows from the fact that \( U_z^\alpha U_z^\alpha \) is the identity operator. □

Corollary 3.12. Suppose \( u_1, u_2 \in L^1(\mathbb{D}, dA) \). If \( u_1 = u_2 \circ \varphi_z \) for some \( z \in \mathbb{D} \), then \( H_{u_1}^\alpha \) and \( H_{u_2}^\alpha \) are unitary equivalent.

Proof. By Corollary 3.10, \( H_{u_1}^\alpha \) and \( H_{u_2}^\alpha \) are unitary equivalent. □

References

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