RIGIDNESS AND EXTENDED ARMENDARIZ PROPERTY

Muhittin Başer, Fatma Kaynarca, and Tai Keun Kwak

Abstract. For a ring endomorphism $\alpha$ of a ring $R$, Krempa called $\alpha$ a rigid endomorphism if $\alpha \alpha(a) = 0$ implies $a = 0$ for $a \in R$, and Hong et al. called $R$ an $\alpha$-rigid ring if there exists a rigid endomorphism $\alpha$. Due to Rege and Chhawchharia, a ring $R$ is called Armendariz if whenever the product of any two polynomials in $R[x]$ over $R$ is zero, then so is the product of any pair of coefficients from the two polynomials. The Armendariz property of polynomials was extended to one of skew polynomials (i.e., $\alpha$-Armendariz rings and $\alpha$-skew Armendariz rings) by Hong et al. In this paper, we study the relationship between $\alpha$-rigid rings and extended Armendariz rings, and so we get various conditions on the rings which are equivalent to the condition of being an $\alpha$-rigid ring. Several known results relating to extended Armendariz rings can be obtained as corollaries of our results.

Throughout this paper, all rings are associative with identity. Given a ring $R$, the polynomial ring over $R(x)$ is denoted by $R[x]$. Recall that a ring $R$ is called reduced if it has no nonzero nilpotent elements. Armendariz [1, Lemma 1] showed that for a reduced ring $R$, if any polynomial $f(x) = a_0 + a_1 x + \cdots + a_m x^m$, $g(x) = b_0 + b_1 x + \cdots + b_n x^n \in R[x]$ satisfies $f(x)g(x) = 0$, then $a_ib_j = 0$ for each $i, j$. Since then, Rege and Chhawchharia [12] called $R$ an Armendariz ring if it satisfies this condition. Many properties of Armendariz rings have been studied by many authors [2, 3, 5, 6, 8, 10, 11, 12].

The reducedness and Armendariz property of a ring were extended as follows. For a ring $R$ with a ring endomorphism $\alpha : R \rightarrow R$, a skew polynomial ring (also called an Ore extension of endomorphism type) $R[x; \alpha]$ of $R$ is the ring obtained by giving the polynomial ring over $R$ with the new multiplication $xr = \alpha(r)x$ for all $r \in R$. Recall that an endomorphism $\alpha$ of a ring $R$ is called rigid [9] if $\alpha \alpha(a) = 0$ implies $a = 0$ for $a \in R$, and a ring $R$ is called $\alpha$-rigid [4] if there exists a rigid endomorphism $\alpha$ of $R$. Note that any rigid endomorphism of a ring is a monomorphism, and $\alpha$-rigid rings are reduced rings [4, Proposition 5]. On the other hand, the Armendariz property with respect to polynomials was extended to one of skew polynomials. A ring $R$ is called $\alpha$-Armendariz (resp.,
\(\alpha\)-skew Armendariz) [6, Definition 1.1] (resp., [5, Definition]) if for \(p(x) = a_0 + a_1 x + \cdots + a_m x^m\) and \(q(x) = b_0 + b_1 x + \cdots + b_n x^n\) in \(R[x; \alpha]\), \(p(x)q(x) = 0\) implies \(a_i b_j = 0\) (resp., \(a_i \alpha^j(b_j) = 0\)) for all \(0 \leq i \leq m\) and \(0 \leq j \leq n\). It can be easily checked that every subring \(S\) with \(\alpha(S) \subseteq S\) of an \(\alpha\)-Armendariz ring (resp., an \(\alpha\)-skew Armendariz ring) is also \(\alpha\)-Armendariz (resp., \(\alpha\)-skew Armendariz). Note that every \(\alpha\)-rigid ring is \(\alpha\)-Armendariz [6, Proposition 1.7], and every \(\alpha\)-Armendariz ring is \(\alpha\)-skew Armendariz [6, Theorem 1.8], but the converses do not hold by [6, Example 1.6] and [6, Example 1.9], respectively. Moreover \(R\) is an \(\alpha\)-rigid ring if and only if \(R[x; \alpha]\) is reduced [5, Proposition 3]. In [3], Chen and Tong showed the relationship between \(\alpha\)-rigid rings and \(\alpha\)-skew Armendariz rings. Motivated by their results, in this paper, we continue the study of \(\alpha\)-Armendariz rings, improving several results in [3] and [6], and moreover we obtain various rings which are equivalent to \(\alpha\)-rigid rings. Several known results relating to Armendariz rings can be obtained as corollaries of our results.

In [12, Remark 3.1], Rege and Chhawchharia showed that every \(n \times n\) full matrix ring over any ring \(R\) is not \(I_R\)-Armendariz for \(n \geq 2\) where \(I_R\) is an identity endomorphism of \(R\). We also know that there exists a \(2 \times 2\) full (and also upper triangular) matrix ring \(R\) with an endomorphism \(\alpha\) such that \(R\) is not \(\alpha\)-Armendariz by [6, Theorem 1.8] and [5, Example 13] in general. Hence, we consider the following.

A ring \(R\) can be extended to a ring

\[
S_3(R) = \left\{ \begin{pmatrix} a & b & c \\ 0 & a & d \\ 0 & 0 & a \end{pmatrix} \mid a, b, c, d \in R \right\}
\]

and an endomorphism \(\alpha\) of \(R\) can also be extended to the endomorphism \(\hat{\alpha}: S_3(R) \to S_3(R)\) defined by \(\hat{\alpha}((a_{ij})) = (\alpha(a_{ij}))\). Recall that the trivial extension \(T(R, M) = R \oplus M\) of \(R\) by \(M\) is isomorphic to the ring of all matrices \((a_{i,j})\), where \(r \in R\) and \(m \in M\) and the usual matrix operations are used. Hong et al. [6, Proposition 2.1] proved that, if \(R\) is an \(\alpha\)-rigid ring, then \(S_3(R)\) is \(\hat{\alpha}\)-Armendariz and so the trivial extension \(T(R, R)\) of \(R\) is \(\hat{\alpha}\)-Armendariz. Now, we show that these are equivalent. First we state the following lemma.

**Lemma 1.** Let \(\alpha\) be an endomorphism of a ring \(R\).

(1) [6, Proposition 1.3(ii)] If \(R\) is an \(\alpha\)-Armendariz ring, then \(\alpha\) is a monomorphism.

(2) [6, Proposition 2.4] \(R\) is an \(\alpha\)-rigid ring if and only if for each \(a \in R\), \(\alpha^2(a)a = 0\) implies \(a = 0\).

**Theorem 2.** Let \(\alpha\) be an endomorphism of a ring \(R\). Then the following are equivalent:

(1) \(R\) is an \(\alpha\)-rigid ring.

(2) \(S_3(R) = \left\{ \begin{pmatrix} a & b & c \\ 0 & a & d \\ 0 & 0 & a \end{pmatrix} \mid a, b, c, d \in R \right\}\) is an \(\hat{\alpha}\)-Armendariz ring.

(3) The trivial extension \(T(R, R)\) of \(R\) is an \(\hat{\alpha}\)-Armendariz ring.
Proof. Note that $T(R, R)$ is isomorphic to the subring
\[
\left\{ \begin{array}{ccc}
  a & b & 0 \\
  0 & a & 0 \\
  0 & 0 & a \\
\end{array} \right| a, b \in R
\]
of a ring $S_3(R)$ and each subring of an $\alpha$-Armendariz ring is also $\alpha$-Armendariz. Hence, it is enough to show that $(3) \Rightarrow (1)$. Let $T(R, R)$ be $\alpha$-Armendariz. Assume on the contrary that $R$ is not $\alpha$-rigid. By Lemma 1, there exists $0 \neq a \in R$ such that $\alpha(a)a = 0$ and $\alpha(a) \neq 0$. For $p(x) = \left( \begin{array}{cc}
  \alpha(a) & 0 \\
  0 & \alpha(a) \\
\end{array} \right) + \left( \begin{array}{cc}
  0 & 1 \\
  0 & 0 \\
\end{array} \right) x$, $q(x) = \left( \begin{array}{cc}
  0 & 0 \\
  0 & 1 \\
\end{array} \right) + \left( \begin{array}{cc}
  0 & 0 \\
  0 & 0 \\
\end{array} \right) x \in T(R, R)[x; \alpha]$, we have $p(x)q(x) = 0$, but $\left( \begin{array}{cc}
  \alpha(a) & 0 \\
  0 & \alpha(a) \\
\end{array} \right) \left( \begin{array}{cc}
  0 & 0 \\
  0 & 0 \\
\end{array} \right) \neq 0$; which is a contradiction. Thus $R$ is $\alpha$-rigid. \qed

If we take $\alpha$ as the identity endomorphism $I_R$ of a ring $R$, then we have the following corollary which generalizes the results in [8, Proposition 2] and [10, Theorem 2.3].

Corollary 3. For a ring $R$, the following are equivalent:

(1) $R$ is a reduced ring.

(2) $S_3(R) = \left\{ \begin{array}{ccc}
  a & b & c \\
  0 & a & d \\
  0 & 0 & a \\
\end{array} \right| a, b, c, d \in R \}$ is an Armendariz ring.

(3) The trivial extension $T(R, R)$ of $R$ is an Armendariz ring.

Hong et al. [5, p. 261] showed that the ring
\[
S_n(R) = \left\{ \begin{array}{cccc}
  a & a_{11} & \cdots & a_{1n} \\
  0 & a_{21} & \cdots & a_{2n} \\
  \vdots & \vdots & \ddots & \vdots \\
  0 & 0 & \cdots & a \\
\end{array} \right| a, a_{ij} \in R
\]
cannot be $\alpha$-Armendariz for $n \geq 4$, even if $R$ is an $\alpha$-rigid ring. However, we obtain subrings of $S_n(R)$ for $n \geq 4$ which are $\alpha$-Armendariz as follows.

From [11], $RA = \{ rA \mid r \in R \}$ for any $A \in \text{Mat}_n(R)$ where $\text{Mat}_n(R)$ is the $n \times n$ full matrix ring and for $n \geq 2$, let $V = \sum_{i=1}^{n-1} E_{ij(i+1)}$ where $E_{ij}$'s are the matrix units. For an even number $n = 2k(\geq 2)$, let
\[
A_n^e(R) = \sum_{i=1}^{k} \sum_{j=k+i}^{n} RE_{ij}, \quad \text{and} \quad B_n^e(R) = \sum_{i=1}^{k+1} \sum_{j=k+i}^{n} RE_{ij};
\]
and for an odd number $n = 2k+1(\geq 3)$, let
\[
A_n^o(R) = \sum_{i=1}^{k+1} \sum_{j=k+i}^{n} RE_{ij}, \quad \text{and} \quad B_n^o(R) = \sum_{i=1}^{k+2} \sum_{j=k+i}^{n} RE_{ij}.
\]
In addition, for $n \geq 2$ put
\[
A_n(R) = RI_n + RV + \cdots + RV^{k-1} + A_n^e(R) \quad \text{and} \quad B_n(R) = RI_n + RV + \cdots + RV^{k-2} + B_n^e(R) \quad \text{for} \ n = 2k;
\]
Assume that 

\[ A_n(R) = R I_n + RV + \cdots + RV^{k-1} + A_n^\circ(R) \quad \text{and} \quad B_n(R) = R I_n + RV + \cdots + RV^{k-2} + B_n^\circ(R) \]

for \( n = 2k + 1 \), where \( I_n \) is the unit matrix of \( \text{Mat}_n(R) \).

**Proposition 4.** Let \( \alpha \) be an endomorphism of a ring \( R \). The following are equivalent:

1. \( R \) is an \( \alpha \)-rigid ring.
2. \( A_n(R) \) is an \( \alpha \)-Armendariz ring for \( n = 2k + 1 \geq 3 \).
3. \( A_n(R) + RE_{1k} \) is an \( \alpha \)-Armendariz ring for \( n = 2k \geq 4 \).
4. \( V_n(R) = R I_n + RV + RV^2 + \cdots + RV^{n-1} \) is an \( \alpha \)-Armendariz ring for \( n \geq 2 \).

**Proof.** Assume that \( R \) is an \( \alpha \)-rigid ring. First, let \( S = A_n(R) \) if \( n = 2k + 1 \geq 3 \) and \( S = A_n(R) + RE_{1k} \) if \( n = 2k \geq 4 \), and let \( P(x) = C_2 + C_1 x + \cdots + C_0 x^n \) and \( Q(x) = D_0 + D_1 x + \cdots + D_n x^n \) in \( S[x; \alpha] / P(x)Q(x) = 0 \). We show that \( C_i D_j = 0 \) for \( 0 \leq i \leq u, 0 \leq j \leq v \). Let \( p_{st}(x) = c_{st}^{(0)} + c_{st}^{(1)} x + \cdots + c_{st}^{(u)} x^u \) and \( q_{st}(x) = d_{st}^{(0)} + d_{st}^{(1)} x + \cdots + d_{st}^{(v)} x^v \), where \( c_{st}^{(i)} \) and \( d_{st}^{(j)} \) are the \((s, t)\)-entries of \( C_i \) and \( D_j \), respectively for \( 0 \leq i \leq u \) and \( 0 \leq j \leq v \). Then, we can write that \( P(x) = (p_{st}(x)) \) and \( Q(x) = (q_{st}(x)) \) for \( 1 \leq s, t \leq n \) and then \( (p_{st}(x))(q_{st}(x)) = 0 \) in \( S \). Since \( R \) is \( \alpha \)-rigid, \( R[x; \alpha] \) is reduced by [5, Proposition 3]. From [3, Lemma 2.4], we have \((p_{st}(x))(q_{st}(x)))_{st} = 0 \) for \( 1 \leq s, t \leq n \). So \( p_{st}(x) = 0 \) in \( R[x; \alpha] \) for \( 1 \leq l \leq n \). That is,

\[
(c_{st}^{(0)} + c_{st}^{(1)} x + \cdots + c_{st}^{(u)} x^u)(d_{st}^{(0)} + d_{st}^{(1)} x + \cdots + d_{st}^{(v)} x^v) = 0.
\]

Since \( R \) is an \( \alpha \)-rigid, \( R \) is \( \alpha \)-Armendariz by [6, Proposition 1.7]. Hence \( c_{st}^{(i)} d_{st}^{(j)} = 0 \) for \( 1 \leq s, t, l \leq n, 0 \leq i \leq u \) and \( 0 \leq j \leq v \), and so \( C_i D_j = (c_{st}^{(i)})(d_{st}^{(j)}) = 0 \). Therefore \( S \) is an \( \alpha \)-Armendariz. This proves that \( (1) \Rightarrow (2) \) and \( (1) \Rightarrow (3) \).

Next, assume that \( R \) is an \( \alpha \)-rigid ring. For \( n = 2, 3 \), \( V_n(R) \) is an \( \alpha \)-Armendariz by Theorem 2 and for \( n \geq 4 \), \( V_n(R) \) is also an \( \alpha \)-Armendariz since \( V_n(R) \) is a subring of \( A_n(R) \) or \( A_n(R) + RE_{1k} \) and \( \alpha(V_n(R)) \subseteq V_n(R) \).

The converses follow the proof of Theorem 2, respectively. \qed

**Corollary 5.** The following are equivalent for a ring \( R \).

1. \( R \) is a reduced ring.
2. \( A_n(R) \) is an Armendariz ring for \( n = 2k + 1 \geq 3 \).
3. \( A_n(R) + RE_{1k} \) is an Armendariz ring for \( n = 2k \geq 4 \).
4. \( V_n(R) = R I_n + RV + RV^2 + \cdots + RV^{n-1} \) is an Armendariz ring for \( n \geq 2 \).

If we define \( \rho : V_n(R) \to R[x]/\langle x^n \rangle \) by \( \rho(a_0 I_n + a_1 V + \cdots + a_{n-1} V^{n-1}) = a_0 + a_1 x + \cdots + a_{n-1} x^{n-1} + \langle x^n \rangle \), then \( \rho \) is a ring isomorphism, where \( \langle x^n \rangle \) is an ideal of \( R[x] \) generated by \( x^n \) and \( n \geq 2 \). So we have the following corollary.
Proof. Assume that $R[x; \alpha]$ is Armendariz. Let $p(x)q(x) = 0$, where $p(x) = a_0 + a_1 x + \cdots + a_m x^m$ and $q(x) = b_0 + b_1 x + \cdots + b_n x^n$ in $R[x; \alpha]$. We show that $a_i \alpha^i(b_j) = 0$ for all $i, j$. Set $f(y) = a_0 + (a_1 x)y + \cdots + (a_m x^m)y^m$ and $g(y) = b_0 + (b_1 x)y + \cdots + (b_n x^n)y^n$ in $(R[x; \alpha])[y]$. Then $f(y)g(y) = 0$, since $p(x)q(x) = 0$ and $y$ commutes with $x$. Since $R[x; \alpha]$ is Armendariz, we have $a_i x^i b_j x^j = 0$, and so $a_i \alpha^i(b_j) = 0$ for all $i, j$. Therefore $R$ is an $\alpha$-skew Armendariz ring.

Corollary 10 ([5, Corollary 4]). Let $\alpha$ be an endomorphism of a ring $R$. If $R$ is an Armendariz ring, then $R$ is an $\alpha$-skew Armendariz ring.

Proof. Let $R$ be an $\alpha$-rigid ring. Then $R[x; \alpha]$ is reduced by [5, Proposition 3] and so $R[x; \alpha]$ is Armendariz. Therefore $R$ is an $\alpha$-skew Armendariz ring by Proposition 9.

Observe that the conclusion of Proposition 9 cannot be replaced by the condition “$R$ is $\alpha$-Armendariz” by the next example.

Example 11. Let $R$ be the polynomial ring $\mathbb{Z}_2[x]$ over $\mathbb{Z}_2$, the ring of integers modulo 2, and let the endomorphism $\alpha$ of $R$ be defined by $\alpha(f(x)) = f(0)$ for $f(x) \in \mathbb{Z}_2[x]$. Then $R$ is a reduced $\alpha$-skew Armendariz ring by [5, Example 5], but $R$ is not $\alpha$-Armendariz by [6, Example 1.9]. Now, we show that $S = R[y; \alpha]$ is an Armendariz ring. Let $f(T) = f_0 + f_1 T + \cdots + f_m T^m$ and $g(T) = g_0 + g_1 T + \cdots + g_n T^n \in S[T]$ with $f(T)g(T) = 0$. We also let $f_i = \sum_{t=0}^{m} f_{i,t}(x)y^t$ and $g_j = \sum_{t=0}^{n} g_{j,t}(x)y^t$ where $f_{i,t}(x), g_{j,t}(x) \in \mathbb{Z}_2[x]$ for each $0 \leq i \leq m$ and $0 \leq j \leq n$. Without loss of generality, assume that $f_i(x) \neq 0$ and $g_j(x) \neq 0$ for at least one $i$ and $j$. Then $f_i(x)g_j(x) = 0$ in $S$. Therefore $S$ is an $\alpha$-skew Armendariz ring.
Proposition 12. Let $m$ be a rigid ring. Then we have the following system of equations:

\begin{align*}
(0) & \quad f_0 g_0 = 0 ; \\
(1) & \quad f_0 g_1 + f_1 g_0 = 0 ; \\
& \vdots \\
(k) & \quad f_0 g_k + f_1 g_{k-1} + \cdots + f_{k-1} g_1 + f_k g_0 = 0 ; \\
(k+1) & \quad f_0 g_{k+1} + f_1 g_k + \cdots + f_k g_1 + f_{k+1} g_0 = 0 ; \\
& \vdots \\
(m+n) & \quad f_m g_n = 0 .
\end{align*}

We claim that $f_0(x) = f_1(x) = \cdots = f_m(x) = 0$ and each $g_j(x)$ has no constant term for $0 \leq t \leq v_j$ and $0 \leq j \leq n$. We proceed by induction on $i+j$. Since $R$ is a-skew Armendariz and $f_0 g_0 = 0$ from Eq. (0), we obtain $f_0(x) \alpha^s(g_0(x)) = 0$ for all $0 \leq s \leq u_0$ and $0 \leq t \leq v_0$, and so $f_0(x) = 0$ and $g_0(0) = 0$ for $0 \leq t \leq v_0$. Thus $g_0(x)$ is hol constant term for $0 \leq t \leq v_0$. This proves for $i+j=0$. Now suppose that our claim is true for $i+j < k-1$. By the induction hypothesis and Eq. (k), we get $0 = f_0 g_k + f_k g_0 = (f_0(x) y + f_0(x) g_{x}(y) + \cdots + f_0(x) y^{m_0}(y) y^{m_0})(g_{x}(x) + g_{x}(x) y + \cdots + g_{x}(x) y^{m_0}(y) y^{m_0}) + f_0(x) g_{x}(x) + g_{x}(x) y + \cdots + g_{x}(x) y^{m_0}(y) y^{m_0}) = f_0(x) g_{x}(x) + [f_0(x) g_{x}(x) + f_0(x) \alpha(g_{x}(x)) y + [f_0(x) g_{x}(x) + f_0(x) \alpha(g_{x}(x)) y + f_0(x) \alpha^2(g_{x}(x)) + f_0(x) \alpha^3(g_{x}(x)) y^2 + \cdots + f_0(x) \alpha^{m_0}(g_{x}(x)) y^{m_0+v_0}]$. Then $f_0(x) = 0$, and so we have the following:

\begin{align*}
(i) & \quad f_0(x) \alpha(g_{x}(x)) = 0 ; \\
(ii) & \quad f_0(x) \alpha(g_{x}(x)) + f_0(x) \alpha^2(g_{x}(x)) = 0 ; \\
& \vdots \\
(iii) & \quad f_0(x) \alpha(g_{x}(x)) + f_0(x) \alpha^2(g_{x}(x)) + \cdots + f_0(x) \alpha^i(g_{x}(x)) = 0 ; \\
& \vdots \\
(iv) & \quad f_0(x) \alpha^{m_0}(g_{x}(x)) = 0 .
\end{align*}

Hence, $g_{x}(0) = g_{x}(0) = \cdots = g_{x}(0) = 0$, and so $g_{x}(x)$ has no constant term for all $0 \leq t \leq v_n$. Thus $f_j = \sum_{x=1}^{m_0} f_{x}(x) y^x$, $g_j = \sum_{y=0}^{m_0} g_{x}(x) y^y$ and each $g_{x}(x)$ has no constant term for $0 \leq i \leq m_0$, $0 \leq t \leq v_j$ and $0 \leq j \leq n$, and so $f_j g_j = 0$ for all $i, j$. Therefore $S = R[y; \alpha] = (\mathbb{Z}_2[x])[y; \alpha]$ is Armendariz.

The following extends the result in [3, Lemma 3.8].

**Proposition 12.** Let $\alpha$ be an endomorphism of a ring $R$. If $S$ is a ring and $\sigma : R \rightarrow S$ is a ring isomorphism, then we have the following.

1. $R$ is an $\alpha$-rigid ring if and only if $S$ is a $\sigma \alpha^{-1}$-rigid ring.
2. $R$ is an $\alpha$-Armendariz ring if and only if $S$ is a $\sigma \alpha^{-1}$-Armendariz ring.
3. $R$ is an $\alpha$-skew Armendariz ring if and only if $S$ is a $\sigma \alpha^{-1}$-skew Armendariz ring.
Proof. (1) For \( a \in R \), there exists \( a' \in S \) such that \( \sigma(a) = a' \) since \( \sigma \) is bijective, and so \( a\sigma(a) = 0 \) if and only if \( \sigma(a)(\sigma\sigma^{-1})(\sigma(a)) = 0 \) if and only if \( a(\sigma\sigma^{-1})(a') = 0 \). This yields that \( R \) is \( \alpha \)-rigid if and only if \( S \) is \( \sigma\sigma^{-1} \)-rigid.

(2) and (3) Similarly, \( p(x) = \sum_{i=0}^{m} a_{i}x^{i}, \) \( q(x) = \sum_{j=0}^{n} b_{j}x^{j} \in R[x;\alpha] \) if and only if \( p'(x) = \sum_{i=0}^{m} a_{i}x^{i}, \) \( q'(x) = \sum_{j=0}^{n} b_{j}x^{j} \in S[x;\sigma\sigma^{-1}] \), letting \( \sigma(a_{i}) = a_{i}', \sigma(b_{j}) = b_{j}' \) for all \( i,j \) since \( \sigma \) is bijective. Then \( p(x)q(x) = 0 \) in \( R[x;\alpha] \) if and only if \( \sum_{i+j=k} a_{i}a_{i}'(b_{j}) = 0 \) for each \( 0 \leq k \leq m+n \) and if and only if \( \sum_{i+j=k} \sigma(a_{i})(\sigma\sigma^{-1})(b_{j}) = 0 \) for each \( 0 \leq k \leq m+n \), since \( (\sigma\sigma^{-1})^{t} = \sigma\sigma^{-1} \) for any positive integer \( t \) if and only if \( \sum_{i+j=k} a_{i}'(\sigma\sigma^{-1})(b_{j}) = 0 \) for each \( 0 \leq k \leq m+n \) if and only if \( p'(x)q'(x) = 0 \) in \( S[x;\sigma\sigma^{-1}] \). Hence, for all \( i,j, \ a_{i}b_{j} = 0 \) if and only if \( a_{i}'b_{j}' = 0 \); and \( a_{i}a_{i}'(b_{j}) = 0 \) if and only if \( a_{i}'(\sigma\sigma^{-1})(b_{j}) = 0 \) if and only if \( a_{i}'(\sigma\sigma^{-1})(b_{j}') = 0 \). The proof is completed.

Recall that if \( \alpha \) is an endomorphism of a ring \( R \), then the map \( \bar{\alpha} : R[x] \rightarrow R[x] \) defined by \( \bar{\alpha}(\sum_{i=0}^{m} a_{i}x^{i}) = \sum_{i=0}^{m} \alpha(a_{i})x^{i} \) is an endomorphism of the polynomial ring \( R[x] \) and clearly this map extends \( \alpha \). The Laurent polynomial ring \( R[x,x^{-1}] \) with an indeterminate \( x \), consists of all formal sums \( \sum_{i=k}^{n} a_{i}x^{i} \), where \( a_{i} \in R \) and \( k, n \) are (possibly negative) integers. The map \( \bar{\alpha} : R[x,x^{-1}] \rightarrow R[x,x^{-1}] \) defined by \( \bar{\alpha}(\sum_{i=k}^{n} a_{i}x^{i}) = \sum_{i=k}^{n} \alpha(a_{i})x^{i} \) extends \( \alpha \) and also is an endomorphism of \( R[x,x^{-1}] \).

**Theorem 13.** Let \( \alpha \) be an endomorphism of a ring \( R \). The following are equivalent:

1. \( R \) is an \( \alpha \)-rigid ring.
2. \( R[x] \) is an \( \bar{\alpha} \)-rigid ring.
3. \( R[x,x^{-1}] \) is an \( \bar{\alpha} \)-rigid ring.

**Proof.** (1)\( \Rightarrow \) (2) Assume that \( R \) is \( \alpha \)-rigid, but \( R[x] \) is not \( \bar{\alpha} \)-rigid. Then there exists a nonzero \( f(x) = \sum_{i=0}^{n} a_{i}x^{i} \in R[x] \) such that \( f(x)\bar{\alpha}(f(x)) = 0 \). Suppose that \( a_{k} \neq 0 \) and \( a_{0} = \cdots = a_{k-1} = 0 \) where \( 0 \leq k \leq n \). Then \( 0 = f(x)\bar{\alpha}(f(x)) = (a_{k}x^{k} + \cdots + a_{n}x^{n})(\alpha(a_{k})x^{k} + \cdots + \alpha(a_{n})x^{n}) \) yields \( a_{k}\alpha(a_{k}) = 0, \) and so \( a_{k} = 0, \) which is a contradiction. Thus \( R[x] \) is \( \alpha \)-rigid.

(2)\( \Rightarrow \) (3) Let \( f(x) \in R[x,x^{-1}] \) with \( f(x)\bar{\alpha}(f(x)) = 0 \). Then there exists a positive integer \( n \) such that \( f_{1}(x) = f(x)x^{n} \in R[x], \) and so \( f_{1}(x)\bar{\alpha}(f_{1}(x)) = 0. \) Since \( R[x] \) is \( \alpha \)-rigid, we obtain \( f_{1}(x) = 0, \) and hence \( f(x) = 0. \) Thus \( R[x,x^{-1}] \) is \( \bar{\alpha} \)-rigid.

(3)\( \Rightarrow \) (1) \( R \) is \( \alpha \)-rigid as a subring of \( R[x,x^{-1}] \) when \( R \) is \( \bar{\alpha} \)-rigid.

**Corollary 14.** (1) Let \( R \) be a reduced ring with an endomorphism \( \alpha. \) Then \( R \) is \( \alpha \)-Armendariz if and only if \( R[x] \) is \( \bar{\alpha} \)-Armendariz.

(2) [2, Proposition 6] Let \( R \) be a reduced ring and \( \alpha \) be a monomorphism of \( R. \) Then \( R \) is \( \alpha \)-skew Armendariz if and only if \( R[x] \) is \( \bar{\alpha} \)-skew Armendariz.
Proof. It follows from [2, Theorem 1], [6, Proposition 1.7] and Theorem 13. □

Related to Corollary 14, notice that there exists a reduced $\alpha$-skew Armendariz ring which is not $\alpha$-Armendariz (Example 11).

Let $\alpha, \gamma$ be an endomorphism of a ring $R$, for each $\gamma \in \Gamma$. For the product $\prod_{\gamma \in \Gamma} R_{\gamma}$ of $R_{\gamma}$ and the endomorphism $\hat{\alpha} : \prod_{\gamma \in \Gamma} R_{\gamma} \rightarrow \prod_{\gamma \in \Gamma} R_{\gamma}$ defined by $\hat{\alpha}(a_{\gamma}) = (\alpha_{\gamma}(a_{\gamma}))$, it can be easily checked that $\prod_{\gamma \in \Gamma} R_{\gamma}$ is $\hat{\alpha}$-rigid if and only if each $R_{\gamma}$ is $\alpha_{\gamma}$-rigid.

Recall that for an endomorphism $\alpha$ and an ideal $I$ of a ring $R$, $I$ is called an $\alpha$-ideal if $\alpha(I) \subseteq I$, and if $I$ is an $\alpha$-ideal of $R$, then $\hat{\alpha} : R/I \rightarrow R/I$ defined by $\hat{\alpha}(a + I) = \alpha(a) + I$ for $a \in R$ is an endomorphism of the factor ring $R/I$. The homomorphic image of an $\alpha$-rigid ring is not $\alpha$-rigid, in general. The following example shows that there exists a ring $R$ with an automorphism $\alpha$ such that $R/I$ is $\hat{\alpha}$-rigid for a non-zero $\alpha$-ideal $I$ of $R$, but $R$ is not $\alpha$-rigid.

Example 15. Let $R = (\begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix})$ where $F$ is a field, and $\alpha$ be defined by $\alpha((\begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix})) = (\begin{pmatrix} 0 & a \\ 0 & 0 \end{pmatrix})$. Note that $R$ is not $\alpha$-Armendariz by [6, Example 1.12], and so it is not $\alpha$-rigid. However, for a nonzero proper ideal $I = (\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix})$ of $R$, it can be easily checked that $\alpha(I) \subseteq I$ and $R/I$ is $\hat{\alpha}$-rigid.

Let $\alpha$ be an automorphism of a ring $R$. Suppose that there exists the classical right quotient ring $Q(R)$ of $R$. Then for any $ab^{-1} \in Q(R)$ where $a, b \in R$ with $b$ regular, the induced map $\alpha : Q(R) \rightarrow Q(R)$ defined by $\alpha(ab^{-1}) = \alpha(a)\alpha(b)^{-1}$ is also an endomorphism. Note that $R$ is $\alpha$-rigid if and only if $Q(R)$ is $\alpha$-rigid.

Let $R$ be an algebra over a commutative ring $S$. Recall that the Dorroh extension of $R$ by $S$ is the ring $D = R \times S$ with operations $(r_1, s_1) + (r_2, s_2) = (r_1 + r_2, s_1 + s_2)$ and $(r_1, s_1)(r_2, s_2) = (r_1r_2 + s_1r_2 + s_2r_1, s_1s_2)$, where $r_1, r_2 \in R$ and $s_1, s_2 \in S$. For an endomorphism $\alpha$ of $R$ and the Dorroh extension $D$ of $R$ by $S$, $\hat{\alpha} : D \rightarrow D$ defined by $\hat{\alpha}(r, s) = (\alpha(r), s)$ is an $S$-algebra homomorphism.

In the following, we give some other example of $\alpha$-rigid rings. Observe that for an $\alpha$-rigid ring $R$, every subring $S$ of $R$ with $\alpha(S) \subseteq S$ is clearly $\alpha$-rigid, and $R$ is reduced with $\alpha(e) = e$ for $e^2 = e \in R$ by [4, Proposition 5].

Proposition 16. Let $\alpha$ be an endomorphism of a ring $R$.

(1) $R$ is an $\alpha$-rigid ring if and only if $eR$ and $(1 - e)R$ are $\alpha$-rigid for $e^2 = e \in R$.

(2) If $R$ is an $\alpha$-rigid ring and $S$ is a reduced ring, then the Dorroh extension $D$ of $R$ by $S$ is $\hat{\alpha}$-rigid.

Proof. (1) It is enough to show that $R$ is $\alpha$-rigid. Suppose that $eR$ and $(1 - e)R$ are $\alpha$-rigid. Let $ao(a) = 0$ for $a \in R$. Then $0 = eao(ea)$ and $0 = (1 - e)a(e(1 - e)a)$. By hypothesis, we get $ea = 0$ and $(1 - e)a = 0$, and so $a = 0$. Thus $R$ is $\alpha$-rigid.

(2) Let $(r, s) \in D$ with $(r, s)\hat{\alpha}(r, s) = 0$. Then $r\alpha(r) + s\alpha(r) + sr = 0$ and $s^2 = 0$. Since $S$ is reduced, we get $s = 0$. Thus $r\alpha(r) = 0$, and so $r = 0$ since $R$ is $\alpha$-rigid. Hence, $(r, s) = 0$, and therefore the Dorroh extension $D$ is $\hat{\alpha}$-reduced. □
For any endomorphism $\alpha$ of a ring $R$, $R$ is $\alpha$-rigid if and only if $R[x; \alpha]$ is reduced by [5, Proposition 3], but there exists a semiprime ring $R$ with an automorphism $\alpha$ such that the skew polynomial ring $R[x; \alpha]$ is not semiprime by the following example.

**Example 17.** Let $F$ be a field and $F_i = F$ for $i \in \mathbb{Z}$. Let $R$ be a $F$-subalgebra of $\prod_{i \in \mathbb{Z}} F_i$, generated by $\oplus_{i \in \mathbb{Z}} F_i$ and $1_{\prod_{i \in \mathbb{Z}} F_i}$. Let $\alpha$ be an automorphism of $R$ defined by $\alpha((a_i)) = (a_{i+1})$. Then

$$R = \{ (a_i) \in \prod_{i \in \mathbb{Z}} F_i \mid a_i \text{ is eventually constant} \}$$

is reduced and von Neumann regular, but $R[x; \alpha]$ is not semiprime by [7, Example 4.3].

**Lemma 18.** For a ring $R$, the following are equivalent:

1. $R$ is a semiprime ring.
2. For $a, b \in R$, $aRb = 0$ implies $aR \cap Rb = 0$.

**Proof.** Suppose that $R$ is semiprime and $aRb = 0$ for $a, b \in R$. Let $c \in aR \cap Rb$. Then $c = ar = sb$ for some $r, s \in R$. So $cRc = (ar)R(sb) \subseteq aRb = 0$, and thus $c = 0$. The converse is obvious. \qed

A ring $R$ is called *semicommutative* if $ab = 0$ implies $aRb = 0$ for $a, b \in R$, and so every reduced ring is semicommutative.

**Theorem 19.** Let $\alpha$ be an endomorphism of a ring $R$. The following are equivalent:

1. $R$ is an $\alpha$-rigid ring.
2. For $p(x), q(x) \in R[x; \alpha]$, $p(x)q(x) = 0$ implies $p(x)R[x; \alpha] \cap R[x; \alpha]q(x) = 0$.

**Proof.** Assume that $R$ is $\alpha$-rigid. By [5, Proposition 3], $R[x; \alpha]$ is reduced and so it is semiprime and semicommutative. If $p(x)q(x) = 0$ for $p(x), q(x) \in R[x; \alpha]$, then $p(x)R[x; \alpha]q(x) = 0$, and thus $p(x)R[x; \alpha] \cap R[x; \alpha]q(x) = 0$ by Lemma 18. Conversely, assume (2). Let $a \alpha(a) = 0$ for $a \in R$. For $p(x) = ax = q(x) \in R[x; \alpha]$, $p(x)q(x) = a \alpha(a)x^2 = 0$ and so $(ax)R[x; \alpha] \cap R[x; \alpha](ax) = 0$ by hypothesis. Then $ax = 0$, and hence $a = 0$. Therefore $R$ is $\alpha$-rigid. \qed

**Corollary 20.** For a ring $R$, the following are equivalent:

1. $R$ is a reduced ring.
2. $R[x] \cap R$ is a reduced ring.
3. For $a, b \in R$, $ab = 0$ implies $aR \cap Rb = 0$.
4. For $f(x), g(x) \in R[x]$, $f(x)g(x) = 0$ implies $f(x)R[x] \cap R[x]g(x) = 0$.

**Proof.** It follows from Theorem 13 and Theorem 19. \qed

Paralleled to Theorem 2 and Proposition 4 in this paper, Chen and Tong [3] proved the relationship between $\alpha$-rigid rings and $\alpha$-skew Armendariz rings. However, we note that Theorem 19 shows that the results in [3, Theorems 3.11
and 3.12(15)] is meaningless: In [3, Theorem 3.11], Chen and Tong claimed that the trivial extension \( T(R, R) \) of a ring \( R \) is \( \alpha \)-skew Armendariz (equivalently, \( R \) is an \( \alpha \)-rigid ring by [3, Theorem 3.4]) if and only if \( R \) is an \( \alpha \)-skew Armendariz ring and for \( p(x), q(x) \in R[x; \alpha], p(x)q(x) = 0 \) implies \( p(x)R[x; \alpha] \cap R[x; \alpha]q(x) = 0 \).

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**References**

Tai Keun Kwak  
Department of Mathematics  
Daejin University  
Pocheon 487-711, Korea  
E-mail address: tkkwak@daejin.ac.kr