NORMALITY OF FUZZY TOPOLOGICAL SPACES
IN KUBIAK-ŠOSTAK’S SENSE

M. Azab Abd-Allah

Abstract. The aim of this paper is to study the normality of fuzzy topological spaces in Kubiak-Šostak’s sense. Also, some characterizations and the effects of some types of functions on these types of normality are studied.

1. Introduction

The study of fuzzy sets was initiated with the famous paper of Zadeh [8], and thereafter Chang [1] paved the way for the subsequent tremendous growth of the numerous fuzzy topological concepts. In Chang’s definition, a fuzzy topology is a crisp subfamily of family of fuzzy subsets, and fuzziness in the openness of a fuzzy subset has not been considered. An essentially more general notion of fuzzy topology, in which each fuzzy subset has a certain degree of openness, was introduced by Kubiak [5] and Šostak [8].

In [4], Krsteska and Kim defined the concepts of fuzzy generalized α-closed sets and fuzzy generalized regular α-closed sets in Chang’s fuzzy topological space. By using the above mentioned classes of generalized fuzzy closed sets, they introduced and studied the concepts of fuzzy normal space, fuzzy almost normal space, and fuzzy mildly normal space.

In this paper, we introduce the concepts of fuzzy almost normal, fuzzy normal, fuzzy mildly normal spaces in fuzzy topological spaces in Kubiak-Šostak’s sense and then, we investigate some of their characteristic properties. Our results here represent a generalization of Krsteska and Kim’s study. We can simply obtain their results by taking the r-cut to min.

2. Preliminaries

Throughout this paper, let X be a nonempty set and I is the closed unit interval [0, 1], \( I_0 = (0, 1] \) and \( I_1 = [0, 1) \). The family of all fuzzy subsets on X denoted by \( I^X \). \( \emptyset \) and 1 denote the smallest and the greatest fuzzy

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For it satisfies the following conditions:

(O1) \( \tau(\emptyset) = \tau(\{\} ) = 1. \)

(O2) \( \tau(\lambda_1 \land \lambda_2) \geq \tau(\lambda_1) \land \tau(\lambda_2) \) for each \( \lambda_1, \lambda_2 \in I^X . \)

(O3) \( \tau(\land_{i \in I} \lambda_i) \geq \land_{i \in I} \tau(\lambda_i) \) for any \( \{\lambda_i\} \subset I^X . \)

The pair \((X, \tau)\) is called a fuzzy topological spaces (fts, for short). \( \tau(\lambda) \) may be interpreted as a gradation of openness for \( \lambda \). A function \( f : (X, \tau_1) \to (Y, \tau_2) \) is said to be a fuzzy continuous if \( \tau_1( f^{-1}(\nu) ) \geq \tau_2(\nu) \) for each \( \nu \in I^Y . \)

**Theorem 2.1** ([2]). Let \((X, \tau)\) be an fts. Then for each \( r \in I_0, \lambda \in I^X , \) one defines an operator \( C_\tau : I^X \times I_0 \to I^X \) as follows:

\[
C_\tau(\lambda, r) = \bigwedge \{\mu \in I^X | \lambda \leq \mu, \tau(\mu) \geq r \}.
\]

For \( \lambda, \mu \in I^X \) and \( r, s \in I_0 , \) the operator \( C_\tau \) satisfies the following statements:

(C1) \( C_\tau(\emptyset, r) = \emptyset . \)

(C2) \( \lambda \leq C_\tau(\lambda, r) . \)

(C3) \( C_\tau(\lambda, r) \lor C_\tau(\mu, r) = C_\tau(\lambda \lor \mu, r) . \)

(C4) \( C_\tau(\lambda, r) \leq C_\tau(\lambda, s) \) if \( r \leq s . \)

(C5) \( C_\tau(C_\tau(\lambda, r), r) = C_\tau(\lambda, r) . \)

**Theorem 2.2** ([3]). Let \((X, \tau)\) be an fts. Then for each \( r \in I_0 \) and \( \lambda \in I^X \), one defines an operator \( I_\tau : I^X \times I_1 \to I^X \) as follows:

\[
I_\tau(\lambda, r) = \bigvee \{\mu \in I^X | \lambda \leq \mu, \tau(\mu) \geq r \}.
\]

For \( \lambda, \mu \in I^X \) and \( r, s \in I_0 , \) the operator \( I_\tau \) satisfies the following statements:

(I1) \( I_\tau(\emptyset, r) = \emptyset . \)

(I2) \( I_\tau(\lambda, r) = \lambda . \)

(I3) \( I_\tau(\lambda, r) \leq \lambda . \)

(I4) \( I_\tau(\lambda, r) \land I_\tau(\mu, r) = I_\tau(\lambda \land \mu, r) . \)

(I5) \( I_\tau(\lambda, s) \leq I_\tau(\lambda, r) \) if \( r \leq s . \)

(I6) \( I_\tau(I_\tau(\lambda, r), s) = I_\tau(r, \lambda) . \)

(I7) If \( I_\tau(C_\tau(\lambda, r), r) = \lambda , \) then \( C_\tau(I_\tau(\lambda, r), r) = \lambda - \lambda . \)

**Definition** ([6]). Let \((X, \tau)\) be an fts, \( \lambda \in I^X \) and \( r \in I_0 . \)

(1) A fuzzy set \( \lambda \) is called \( r \)-regular fuzzy open (for short, \( r \)-rfo) if \( \lambda = I_\tau(C_\tau(\lambda, r), r) . \)
A fuzzy set $\lambda$ is called $r$-regular fuzzy closed (for short, $r$-rfc) if $\lambda = C_r(I_r(\lambda, r), r)$.

A fuzzy set $\lambda$ is called $r$-fuzzy $\alpha$-open (for short, $r$-fao) if

$$\lambda \leq I_\alpha(C_r(I_r(\lambda, r), r), r).$$

The $r$-fuzzy $\alpha$-closure and $r$-fuzzy $\alpha$-interior of $\lambda$ is defined by

$$\alpha C_r(\lambda, r) = \bigwedge \{ \mu \in I^X \mid \lambda \leq \mu, \mu \text{ is } r\text{-afc} \},$$

$$\alpha I_r(\lambda, r) = \bigvee \{ \mu \in I^X \mid \mu \leq \lambda, \mu \text{ is } r\text{-afo} \}.$$  

**Definition.** Let $(X, \tau)$ be an fts, $\lambda, \mu \in I^X$ and $r \in I_0$.

1. A fuzzy set $\lambda$ is called $r$-generalized fuzzy closed (for short, $r$-gfcc) if $C_r(\lambda, r) \leq \mu$ whenever $\lambda \leq \mu$ and $\tau(\mu) \geq r$. $\lambda$ is called $r$-generalized fuzzy open (for short, $r$-gfo) if $\lambda - \lambda$ is $r$-generalized fuzzy closed.

2. A fuzzy set $\lambda$ is called $r$-generalized regular fuzzy closed (for short, $r$-grfc) if $C_r(\lambda, r) \leq \mu$ whenever $\lambda \leq \mu$ and $\mu$ is $r$-rfo. $\lambda$ is called $r$-generalized regular fuzzy open (for short, $r$-grfo) if and only if $\lambda - \lambda$ is $r$-grfc.

3. A fuzzy set $\lambda$ is called $r$-generalized fuzzy $\alpha$-closed (for short, $r$-gfaco) if $\alpha C_r(\lambda, r) \leq \mu$ whenever $\lambda \leq \mu$ and $\tau(\mu) \geq r$. $\lambda$ is called $r$-generalized fuzzy $\alpha$-open (for short, $r$-gfao) if and only if $\lambda - \lambda$ is $r$-gfco.

4. A fuzzy set $\lambda$ is called $r$-generalized regular fuzzy $\alpha$-closed (for short, $r$-grfaco) if $\alpha C_r(\lambda, r) \leq \mu$ whenever $\lambda \leq \mu$ and $\mu$ is $r$-rfo. $\lambda$ is called $r$-generalized regular fuzzy $\alpha$-open (for short, $r$-grfao) if and only if $\lambda - \lambda$ is $r$-grfco.

**Remark 2.3.** From the above definitions it is not difficult to conclude that the following diagram of implications is true:

- $r$-gfcc set $\Rightarrow$ $r$-gfco set
- $\downarrow$ $\downarrow$
- $r$-gfaco set $\Rightarrow$ $r$-grfco set

**Counter example 2.1.** Let $X = \{a, b\}$ and let $\lambda_1, \mu_1$ and $\nu_1$ are fuzzy sets defined by

$$\lambda_1(a) = 0.2, \quad \lambda_1(b) = 0.4;$$

$$\mu_1(a) = 0.9, \quad \mu_1(b) = 0.4;$$

$$\nu_1(a) = 0.1, \quad \nu_1(b) = 0.4.$$  

Define $\tau_1$ and $\tau_2$ on $X$ as follows:

$$\tau_1(\lambda) = \begin{cases} 1, & \text{if } \lambda = \frac{0}{1}; \\ \frac{1}{4}, & \text{if } \lambda = \frac{1}{4}; \\ \frac{1}{2}, & \text{if } \lambda = \frac{1}{2}; \\ 0, & \text{otherwise}, \end{cases}$$

$$\tau_2(\lambda) = \begin{cases} 1, & \text{if } \lambda = \frac{0}{1}; \\ \frac{1}{2}, & \text{if } \lambda = \frac{1}{2}; \\ 0, & \text{otherwise}. \end{cases}$$
Definition. Let $f : (X, \tau_1) \to (Y, \tau_2)$ be a function between fts's $(X, \tau_1)$ and $(Y, \tau_2)$. Then the function $f$ is called:

1. fuzzy regular continuous if $f^{-1}(\nu) \text{ is } r\text{-rfc for each } \nu \in I^Y$ and $r \in I_0$ such that $\tau_2(\nu) \geq r$.
2. fuzzy regular irresolute if $f^{-1}(\nu) \text{ is } r\text{-rfc for each } \nu \in I^Y$, $r \in I_0$ such that $\nu \text{ is } r\text{-rfc}.$
3. fuzzy almost regular generalized continuous if $f^{-1}(\nu) \text{ is } r\text{-grfo for each } \nu \in I^X$ and $r \in I_0$ such that $\tau_2(\nu) \geq r$.
4. fuzzy almost continuous if $\tau_1(f^{-1}(\nu)) \geq r \text{ for each } \nu \in I^X$, $r \in I_0$ such that $\nu \text{ is } r\text{-rfc}.$

Definition. Let $f : (X, \tau_1) \to (Y, \tau_2)$ be a function between fts's $(X, \tau_1)$ and $(Y, \tau_2)$. Then the function $f$ is called:

1. fuzzy regular closed if $f(\lambda) \text{ is } r\text{-rfc for each } \lambda \in I^X$, $r \in I_0$ such that $\tau_1(\lambda) \geq r$.
2. fuzzy almost closed if $\tau_2(f(\lambda')) \geq r \text{ for each } \lambda \in I^X$, $r \in I_0$ such that $\lambda \text{ is } r\text{-rfc}$.
3. fuzzy almost generalized closed if $f(\lambda) \text{ is } r\text{-gfc for each } \lambda \in I^X$, $r \in I_0$ such that $\lambda \text{ is } r\text{-rfc}$.
4. fuzzy almost regular generalized closed if $f(\lambda) \text{ is } r\text{-gfc for each } \lambda \in I^X$, $r \in I_0$ such that $\lambda \text{ is } r\text{-rfc}$.

3. Fuzzy normal spaces

Definition. An fts $(X, \tau)$ is said to be:

1. fuzzy normal space if for each $\lambda_1, \lambda_2 \in I^X$ and $r \in I_0$ such that $\tau(\lambda_1') \geq r$, $\lambda_2$ is r-rfc set and $\lambda_1 \bar{\mu} \lambda_2$, there exist $\mu_1, \mu_2 \in I^X$ such that $\lambda_1 \leq \mu_1$, $\lambda_2 \leq \mu_2$ with $\mu_1 \bar{\mu} \mu_2$.
2. fuzzy normal space if for each $\lambda_1, \lambda_2 \in I^X$ and $r \in I_0$ such that $\tau(\lambda_1') \geq r$, $\tau(\lambda_2') \geq r$ and $\lambda_1 \bar{\mu} \lambda_2$, there exists $\mu_1, \mu_2 \in I^X$ such that $\tau(\mu_1) \geq r$, $\tau(\mu_2) \geq r$, $\lambda_1 \leq \mu_1$, $\lambda_2 \leq \mu_2$ and $\mu_1 \bar{\mu} \mu_2$.
3. fuzzy mildly normal space if for each $\lambda_1, \lambda_2 \in I^X$ and $r \in I_0$ such that $\lambda_1$ and $\lambda_2$ are r-rfc sets and $\lambda_1 \bar{\mu} \lambda_2$, there exists $\mu_1, \mu_2 \in I^X$ such that $\tau(\mu_1) \geq r$, $\tau(\mu_2) \geq r$, $\lambda_1 \leq \mu_1$, $\lambda_2 \leq \mu_2$ and $\mu_1 \bar{\mu} \mu_2$.

Clearly, every fuzzy normal space is almost fuzzy normal space and every fuzzy almost normal space is fuzzy mildly normal space, but the converse is not true in general as the following example shows:
Example 3.1. Let $X = \{x\}$ and define the fuzzy sets $\mu$, $\nu$, $\rho$, $\gamma$ and $\omega$ as follows:

$$
\mu(x) = 0.4, \quad \nu(x) = 0.7, \quad \rho(x) = 0.8, \quad \gamma(x) = 0.6, \quad \omega(x) = 0.2.
$$

Define the fuzzy topologies $\tau_1, \tau_2$ as follows:

$$
\tau_1(\lambda) = \begin{cases} 
1, & \text{if } \lambda = 0, 1; \\
\frac{1}{2}, & \text{if } \lambda = \mu; \\
\frac{1}{4}, & \text{if } \lambda = \nu; \\
0, & \text{otherwise.}
\end{cases}
$$

$$
\tau_2(\lambda) = \begin{cases} 
1, & \text{if } \lambda = 0, 1; \\
\frac{1}{2}, & \text{if } \lambda = \nu; \\
\frac{1}{4}, & \text{if } \lambda = \rho; \\
\frac{1}{4}, & \text{if } \lambda = \gamma; \\
0, & \text{otherwise.}
\end{cases}
$$

(1) $(X, \tau_1)$ is fuzzy mildly normal space but not fuzzy almost normal space.
(2) $(X, \tau_2)$ is fuzzy almost normal space but not fuzzy normal space.

Theorem 3.1. Let $(X, \tau)$ be an fts. Then the following statements are equivalent:

1. $(X, \tau)$ is a fuzzy mildly normal space;
2. For each pair of $r$-rfc sets $\lambda_1, \lambda_2 \in I^X$ and $r \in I_0$ such that $\lambda_1 \tilde{\lambda} \lambda_2$, there exist $r$-rfc sets $\mu_1, \mu_2$ such that $\lambda_1 \leq \mu_1, \lambda_2 \leq \mu_2$ and $\mu_1 \tilde{\mu} \mu_2$;
3. For each pair of $r$-rfc sets $\lambda_1, \lambda_2 \in I^X$ and $r \in I_0$ such that $\lambda_1 \tilde{\lambda} \lambda_2$, there exist $r$-gfo sets $\mu_1, \mu_2$ such that $\lambda_1 \leq \mu_1, \lambda_2 \leq \mu_2$ and $\mu_1 \tilde{\mu} \mu_2$;
4. For each pair of $r$-rfc sets $\lambda_1, \lambda_2 \in I^X$ and $r \in I_0$ such that $\lambda_1 \tilde{\lambda} \lambda_2$, there exist $r$-rfc sets $\mu_1$ and $\mu_2$, such that $\lambda_1 \leq \mu_1, \lambda_2 \leq \mu_2$ and $\mu_1 \tilde{\mu} \mu_2$;
5. For each $r$-rfc set $\lambda \in I^X$ and each $r$-rfc set $\mu \in I^X$ such that $\lambda \leq \mu$, there exists $\rho \in I^X$ such that $\tau(\rho) \geq r$ and

$$
\lambda \leq \rho \leq C_\tau(\rho, r) \leq \mu;
$$
6. For each $r$-rfc set $\lambda \in I^X$ and each $r$-rfc set $\mu \in I^X$ such that $\lambda \leq \mu$, there exists an $r$-rfc set $\rho \in I^X$ such that

$$
\lambda \leq \rho \leq C_\tau(\rho, r) \leq \mu;
$$
7. For each $r$-rfc set $\lambda \in I^X$ and $r$-rfc set $\mu \in I^X$ such that $\lambda \leq \mu$, there exists an $r$-gfo set $\rho \in I^X$ such that

$$
\lambda \leq \rho \leq C_\tau(\rho, r) \leq \mu;
$$
8. For each $r$-rfc set $\lambda \in I^X$ and each $r$-rfc set $\mu \in I^X$ such that $\lambda \leq \mu$, there exists an $r$-gfo set $\rho \in I^X$ such that

$$
\lambda \leq \rho \leq \alpha C_\tau(\rho, r) \leq \mu;\]
(9) For each r-rfc set \( \lambda \in I^X \) and each r-rfo set \( \mu \in I^X \) such that \( \lambda \leq \mu \), there exists an r-fao set \( \rho \in I^X \) such that
\[
\lambda \leq \rho \leq \alpha C_r(\rho, r) \leq \mu;
\]
(10) For each r-rfc sets \( \lambda_1, \lambda_2 \in I^X \) and \( r \in I_0 \) such that \( \lambda_1 \leq \lambda_2 \), there exist r-fao sets \( \mu_1 \) and \( \mu_2 \), such that \( \lambda_1 \leq \mu_1, \lambda_2 \leq \mu_2 \) and \( \mu_1 \leq \mu_2 \).

Proof. (1) \( \Rightarrow \) (5) For each \( r \in I_0 \), let \( \lambda \) be r-rfc set and let \( \mu \) be an r-rfo set such that \( \lambda \leq \mu \). Then \( \lambda \leq \mu \). By (1), there exist an r-rfo sets \( \lambda_1 \) and \( \mu_1 \), such that \( \lambda \leq \lambda_1, \lambda_1 = 1 - \mu \leq \lambda, \) such that \( \lambda \leq \lambda_1, \lambda_1 \leq \lambda_2 \) such that \( \lambda_1 \leq \lambda_2 \). Thus
\[
\lambda \leq \lambda_1 \leq C_r(\lambda_1, r) \leq 1 - \mu \leq \mu.
\]
(5) \( \Rightarrow \) (2) Let \( \lambda_1 \) and \( \lambda_2 \) be an r-rfc set such that \( \lambda_1 \leq \lambda_2 \). Then \( \lambda_1 \leq 1 - \lambda_2 \). By (5), there exists \( \rho \in I^X \) such that \( \tau(\rho) \geq r \) and
\[
\lambda_1 \leq \rho \leq C_r(\rho, r) \leq 1 - \lambda_2.
\]
It follows that
\[
\lambda_1 \leq I_r(C_r(\rho, r), r) \leq 1 - \lambda_2
\]
and
\[
\lambda_2 \leq 1 - C_r(\rho, r) = I_r(1 - \rho, r).
\]
Then \( \mu_1 = I_r(C_r(\rho, r), r) \) and \( \mu_2 = I_r(1 - \rho, r) \) are r-rfo sets such that \( \lambda_1 \leq \mu_1, \lambda_2 \leq \mu_2 \) and \( \mu_1 \leq \mu_2 \).
(2) \( \Rightarrow \) (6) Let \( \lambda_1 \) be any r-rfc set and let \( \lambda_2 \) be r-rfo set such that \( \lambda_1 \leq \lambda_2 \). Then \( \lambda_1 = 1 - \lambda_2 \). According to (2), there exist r-rfo sets \( \mu_1 \) and \( \mu_2 \) such that \( \lambda_1 \leq \mu_1, \lambda_2 \leq \mu_2, \mu_1 \leq \mu_2 \). Let \( \rho = I_r(C_r(\mu_1, r), r) \). Then \( \rho \) is an r-rfo set such that
\[
\lambda_1 \leq \rho \leq C_r(\rho, r) \leq 1 - \mu_2 \leq \lambda_2.
\]
(6) \( \Rightarrow \) (2) Let \( \lambda_1 \) and \( \lambda_2 \) be r-rfc sets such that \( \lambda_1 \leq \lambda_2 \). Then \( \lambda_1 \leq 1 - \lambda_2 \). By (6), there exists r-rfo set \( \rho \) such that
\[
\lambda_1 \leq \rho \leq C_r(\rho, r). \leq 1 - \lambda_2.
\]
For \( C_r(\rho, r) \leq 1 - \lambda_2 \), there exists r-rfo set \( \mu \) such that
\[
C_r(\rho, r) \leq \mu \leq C_r(\rho, r) \leq 1 - \lambda_2.
\]
Then \( \lambda_2 \leq 1 - C_r(\mu, r) = I_r(1 - \mu, r), I_r(1 - \mu, r) \) is r-rfo set and \( \rho \leq I_r(1 - \mu, r) \).
(4) \( \Rightarrow \) (8) Let \( \lambda_1 \) be an r-rfc set and let \( \lambda_2 \) be r-rfo set such that \( \lambda_1 \leq \lambda_2 \). Then \( \lambda_1 \leq \lambda_2 \). By (4), there exist r-gfo set \( \mu_1 \) and \( \mu_2 \) such that \( \lambda_1 \leq \mu_1, \mu_2 \leq \lambda_2 \). Since \( \mu_2 \) is r-gfo set and \( 1 - \lambda_2 \) is r-rfo set, from \( \lambda_2 \leq \mu_2 \) follows that \( 1 - \lambda_2 \leq \alpha I_r(\mu_2, r) \). Thus
\[
1 - \lambda_2 \leq \alpha I_r(\mu_2, r) \leq \mu_2 \leq 1 - \mu_1.
\]
Since \( 1 - I_r(\mu_2) \) is a r-gfo set, from \( \mu_1 \leq 1 - \alpha I_r(\mu_2, r) \), we obtain that
\[
\alpha C_r(\mu_1, r) \leq 1 - \alpha I_r(\mu_2, r).
\]
Hence
\[
\lambda_1 \leq \mu_1 \leq \alpha C_r(\mu_1, r) \leq \lambda_2.
\]
(8) ⇒ (9) Let \( \lambda_1 \) be an r-rfc set and let \( \lambda_2 \) be an r-rfo set such that \( \lambda_1 \leq \lambda_2 \). Then \( \lambda_1 \leq \mu \). By (8), there exists an r-gf set \( \mu_1 \) such that
\[
\lambda_1 \leq \mu_1 \leq \alpha C_{\tau}(\mu_1, r) \leq \lambda_2.
\]
Since \( \mu_1 \) is an r-gf set, from \( \lambda_1 \leq \mu_1 \) follows that \( \lambda_1 \leq \alpha I_{\tau}(\mu_1, r) \). Then, \( \mu_2 = \alpha I_{\tau}(\mu_1, r) \) is an r-f set and
\[
\lambda_1 \leq \mu_2 \leq \alpha C_{\tau}(\mu_2, r) \leq \alpha C_{\tau}(\mu_1, r) \leq \lambda_2.
\]
(9) ⇒ (10) Let \( \lambda_1 \) and \( \lambda_2 \) be r-rfc sets such that \( \lambda_1 \leq \lambda_2 \). Then \( \lambda_1 \leq 1 - \lambda_2 \).
By (9), there exists r-f set \( \mu_1 \) such that
\[
\lambda_1 \leq \mu_1 \leq \alpha C_{\tau}(\mu_1, r) \leq 1 - \lambda_2.
\]
Then \( \mu_2 = 1 - \alpha C_{\tau}(\mu_1, r) \) is an r-f set and \( \mu_1 \leq \mu_2 \).

(10) ⇒ (1) Let \( \lambda_1 \) and \( \lambda_2 \) be r-rfc sets such that \( \lambda_1 \leq \lambda_2 \). Then \( \lambda_1 \leq 1 - \lambda_2 \).
By (10), there exist r-f sets \( \mu_1 \) and \( \mu_2 \) such that \( \lambda_1 \leq \mu_1 \), \( \lambda_2 \leq \mu_2 \) and \( \mu_1 \leq \mu_2 \). Let \( \mu_1 = I_{\tau}(C_{\tau}(\mu_1, r), r) \) and \( \mu_2 = I_{\tau}(C_{\tau}(\mu_2, r), r) \). Then \( \tau(\mu_1) \geq r \) and \( \tau(\mu_2) \geq r \) and \( \mu_1 \leq \mu_2 \). Hence \( (X, \tau) \) is a fuzzy normal space.

The implications (1) ⇒ (3) ⇒ (4), (3) ⇒ (7) ⇒ (8) and (2) ⇒ (1) are trivial. □

**Theorem 3.2.** Let \( (X, \tau) \) be an fts. Then the following statements are equivalent:

1. \((X, \tau)\) is a fuzzy normal space (resp. fuzzy almost normal space);
2. For each \( \lambda_1, \lambda_2 \in I^X \) and \( r \in I_0 \) such that \( \tau(\lambda_1) \geq r, \tau(\lambda_2) \geq r, \lambda_1 \leq \lambda_2 \), there exist r-rfo sets \( \mu_1, \mu_2 \) such that \( \lambda_1 \leq \mu_1, \lambda_2 \leq \mu_2 \); and \( \lambda_1 \leq \mu_1 \leq \mu_2 \);
3. For each \( \lambda_1, \lambda_2 \in I^X \) and \( r \in I_0 \) such that \( \tau(\lambda_1) \geq r, \tau(\lambda_2) \geq r, \lambda_1 \leq \lambda_2 \), there exist r-gf sets \( \mu_1, \mu_2 \) such that \( \lambda_1 \leq \mu_1, \lambda_2 \leq \mu_2 \);
4. For each \( \lambda_1, \lambda_2 \in I^X \) and \( r \in I_0 \) such that \( \tau(\lambda_1) \geq r, \tau(\lambda_2) \geq r, \lambda_1 \leq \lambda_2 \), there exist r-gf sets \( \mu_1, \mu_2 \) such that \( \lambda_1 \leq \mu_1, \lambda_2 \leq \mu_2 \);
5. For each \( \lambda_1, \lambda_2 \in I^X \) and \( r \in I_0 \) such that \( \tau(\lambda_1) \geq r, \tau(\lambda_2) \geq r, \lambda_1 \leq \lambda_2 \), there exist \( \rho \in I^X \) such that \( \tau(\rho) \geq r \) and \( \lambda_1 \leq \rho \leq C_{\tau}(\rho, r) \leq \lambda_2 \);
6. For each \( \lambda_1, \lambda_2 \in I^X \) and \( r \in I_0 \) such that \( \tau(\lambda_1) \geq r, \tau(\lambda_2) \geq r, \lambda_1 \leq \lambda_2 \), there exists an r-f set \( \rho \in I^X \) such that \( \lambda_1 \leq \rho \leq C_{\tau}(\rho, r) \leq \lambda_2 \);
7. For each \( \lambda_1, \lambda_2 \in I^X \) and \( r \in I_0 \) such that \( \tau(\lambda_1) \geq r, \tau(\lambda_2) \geq r, \lambda_1 \leq \lambda_2 \), there exists an r-gf set \( \rho \in I^X \) such that \( \lambda_1 \leq \rho \leq C_{\tau}(\rho, r) \leq \lambda_2 \).
(8) For each \( \lambda_1, \lambda_2 \in I^X \) and \( r \in I_0 \) such that \( \tau(\lambda_1^r) \geq r \), \( \tau(\lambda_2^r) \geq r \) and \( \lambda_1 \leq \lambda_2 \), there exists an \( r \)-fuzzy set \( \rho \in I^X \) such that
\[
\lambda_1 \leq \rho \leq \alpha C_r(\rho, r) \leq \lambda_2;
\]
(9) For each \( \lambda_1, \lambda_2 \in I^X \) and \( r \in I_0 \) such that \( \tau(\lambda_1^r) \geq r \), \( \tau(\lambda_2^r) \geq r \) and \( \lambda_1 \leq \lambda_2 \), there exists an \( r \)-fuzzy set \( \rho \in I^X \) such that
\[
\lambda_1 \leq \rho \leq \alpha C_r(\rho, r) \leq \lambda_2;
\]
(10) For each \( \lambda_1, \lambda_2 \in I^X \) and \( r \in I_0 \) such that \( \tau(\lambda_1^r) \geq r \), \( \tau(\lambda_2^r) \geq r \) and \( \lambda_1 \leq \lambda_2 \), there exist \( r \)-fuzzy sets \( \mu_1 \) and \( \mu_2 \), such that \( \lambda_1 \leq \mu_1 \), \( \lambda_2 \leq \mu_2 \) and \( \mu_1 \leq \mu_2 \).

Proof. It is clear from Theorem 3.1. \( \square \)

**Theorem 3.3.** Let \((X, \tau_1)\) and \((Y, \tau_2)\) be fts’s such that \((X, \tau_1)\) is a fuzzy normal space. If \( f : (X, \tau_1) \to (Y, \tau_2) \) is a fuzzy almost continuous, fuzzy almost closed and surjective function, then \((Y, \tau_2)\) is a fuzzy mildly normal space.

Proof. For each \( r \in I_0 \), let \( \nu_1, \nu_2 \in I^X \) be \( r \)-fuzzy sets such that \( \nu_1 \leq \nu_2 \). Since \( f \) is a fuzzy almost continuous function, \( \tau_1(f^{-1}(\nu_1)^r) \geq r \), \( \tau_1(f^{-1}(\nu_2)^r) \geq r \), and \( f^{-1}(\nu_1 \cap \nu_2) = f^{-1}(\nu_1) \cap f^{-1}(\nu_2) \). Since \((X, \tau_1)\) is a fuzzy normal space, there exist \( \mu_1, \mu_2 \in I^X \) such that \( \tau_1(\mu_1) \geq r \), \( \tau_1(\mu_2) \geq r \), and \( f^{-1}(\nu_1) \leq \mu_1 \), \( f^{-1}(\nu_2) \leq \mu_2 \) with \( \mu_1 \leq \mu_2 \).

Since \( I_{\tau_1}(C_{\tau_1}(\mu_1, r), r) \) and \( I_{\tau_1}(C_{\tau_1}(\mu_2, r), r) \) are \( r \)-fuzzy sets and
\[
I_{\tau_1}(C_{\tau_1}(\mu_1, r), r) \leq \alpha I_{\tau_1}(C_{\tau_1}(\mu_2, r), r),
\]
Furthermore,
\[
f^{-1}(\nu_i) \leq \mu_i \leq \alpha I_{\tau_1}(C_{\tau_1}(\mu_i, r), r)
\]
for each \( i \in \{1, 2\} \).

Since \( f \) is fuzzy almost closed, there exist \( \gamma_1, \gamma_2 \in I^Y \) such that \( \tau_2(\gamma_1) \geq r \), \( \tau_2(\gamma_2) \geq r \), and \( \nu_i \leq \gamma_1, \nu_i \leq \gamma_2 \) with
\[
f^{-1}(\gamma_i) \leq I_{\tau_1}(C_{\tau_1}(\mu_i, r), r)
\]
for each \( i \in \{1, 2\} \).

Moreover \( \gamma_1 \leq \gamma_2 \). Hence \((Y, \tau_2)\) is a fuzzy mildly normal space. \( \square \)

**Corollary 3.4.** Let \((X, \tau_1)\) and \((Y, \tau_2)\) be fuzzy topological spaces and let \((X, \tau_1)\) be a fuzzy normal space. If \( f : (X, \tau_1) \to (Y, \tau_2) \) is a fuzzy almost continuous and fuzzy closed function, then \((Y, \tau_2)\) is a fuzzy mildly normal space.

**Corollary 3.5.** Let \((X, \tau_1)\) and \((Y, \tau_2)\) be fuzzy topological spaces and let \((X, \tau_1)\) be fuzzy mildly normal space. If \( f : (X, \tau_1) \to (Y, \tau_2) \) is a fuzzy almost continuous, fuzzy almost closed and fuzzy open (resp. fuzzy continuous, fuzzy closed, fuzzy open) function, then \((Y, \tau_2)\) is a fuzzy mildly normal space.

**Theorem 3.6.** Let \((X, \tau_1)\) and \((Y, \tau_2)\) be fuzzy topological spaces and let \((Y, \tau_2)\) be a fuzzy mildly normal space (resp. fuzzy normal space). If \( f : (X, \tau_1) \to (Y, \tau_2) \) is a fuzzy almost regular generalized continuous, fuzzy regular closed (resp. fuzzy almost closed) injective function, then \((X, \tau_1)\) is a fuzzy mildly normal space.
Proof. For each \( r \in I_0 \), let \( \lambda_1, \lambda_2 \in I^X \) be \( r \)-rfc sets such that \( \lambda_1 \triangleleft \lambda_2 \). Since \( f \) is a fuzzy regular closed (resp. fuzzy normal) function, \( f(\lambda_1) \) and \( f(\lambda_2) \) are \( r \)-rfc sets (resp. \( \tau_2(f(\lambda_1)) \geq r \) and \( \tau_2(f(\lambda_2)) \geq r \)). Since \((Y, \tau_2)\) is a fuzzy mildly normal (resp. fuzzy normal) space, there exist \( \nu_1, \nu_2 \in I^Y \) such that \( \tau_2(\nu_1) \geq r, \tau_2(\nu_2) \geq r \) and \( f(\lambda_1) \leq \nu_1, f(\lambda_2) \leq \nu_2 \) with \( \nu_1 \triangleleft \nu_2 \).

Now let \( \gamma_1 = I_{\tau_2}(C_{\tau_2}(\nu_1, r), r) \) and let \( \gamma_2 = I_{\tau_2}(C_{\tau_2}(\nu_2, r), r) \). Then \( \gamma_1, \gamma_2 \) are \( r \)-rfc sets such that \( f(\lambda_1) \leq \gamma_1, f(\lambda_2) \leq \gamma_2 \) and \( \gamma_1 \triangleleft \gamma_2 \). Since \( f \) is a fuzzy almost regular generalized continuous function, then \( f^{-1}(\gamma_1) \) and \( f^{-1}(\gamma_2) \) are \( r \)-rgfo sets. Furthermore, \( \lambda_1 \leq f^{-1}(\gamma_1), \lambda_2 \leq f^{-1}(\gamma_2) \) and \( f^{-1}(\gamma_1) \triangleleft f^{-1}(\gamma_2) \).

Hence by Theorem 3.1, \((X, \tau_1)\) is a fuzzy mildly normal space. \( \square \)

**Theorem 3.7.** Let \( f : (X, \tau_1) \to (Y, \tau_2) \) be a fuzzy regular continuous, fuzzy normal space, then \((Y, \tau_2)\) is a fuzzy mildly normal space.

**Proof.** For each \( r \in I_0 \), let \( \nu_1, \nu_2 \in I^Y \) such that \( \tau_2(\nu_1') \geq r, \tau_2(\nu_2') \geq r \) and \( \nu_1 \triangleleft \nu_2 \). Since \( f \) is fuzzy regular continuous, \( f^{-1}(\nu_1) \) and \( f^{-1}(\nu_2) \) are \( r \)-rfc sets with \( f^{-1}(\nu_1) \triangleleft f^{-1}(\nu_2) \). Since \((X, \tau_1)\) is fuzzy mildly normal, there exist \( \lambda_1, \lambda_2 \in I^X \) such that \( \tau_1(\lambda_1) \geq r, \tau_1(\lambda_2) \geq r \) and \( f^{-1}(\nu_1) \leq \lambda_1, f^{-1}(\nu_2) \leq \lambda_2 \) with \( \lambda_1 \triangleleft \lambda_2 \).

Let \( \mu_1 = I_{\tau_1}(C_{\tau_1}(\lambda_1, r), r) \) and let \( \mu_2 = I_{\tau_1}(C_{\tau_1}(\lambda_2, r), r) \). Then clearly \( \mu_1 \) and \( \mu_2 \) are \( r \)-rfc sets such that \( f^{-1}(\nu_1) \leq \mu_1, f^{-1}(\nu_2) \leq \mu_2 \) and \( \mu_1 \triangleleft \mu_2 \). Since \( f \) is fuzzy almost regular generalized closed, there exist \( \gamma_1, \gamma_2 \in I^Y \) such that \( \gamma_1, \gamma_2 \) are \( r \)-rgfo sets and \( \nu_1 \leq \gamma_1, \nu_2 \leq \gamma_2, f^{-1}(\gamma_1) \leq \mu_1 \) and \( f^{-1}(\gamma_2) \leq \mu_2 \). Since \( \mu_1 \triangleleft \mu_2 \), then \( \gamma_1 \triangleleft \gamma_2 \). But \( \gamma_1, \gamma_2 \) are \( r \)-rgfo, \( \nu_1 \leq I_{\tau_2}(\gamma_1, r) \) and \( \nu_2 \leq I_{\tau_2}(\gamma_2, r) \). Furthermore \( I_{\tau_2}(\gamma_1, r) \triangleleft I_{\tau_2}(\gamma_2, r) \). Hence \((Y, \tau_2)\) is a fuzzy normal space. \( \square \)

**Corollary 3.8.** Let \( f : (X, \tau_1) \to (Y, \tau_2) \) be a fuzzy regular continuous, fuzzy normal space, then \((Y, \tau_2)\) is a fuzzy normal space.

**Theorem 3.9.** Let \( f : (X, \tau_1) \to (Y, \tau_2) \) be a fuzzy regular irresolute (resp. fuzzy almost continuous), fuzzy normal space, then \((Y, \tau_2)\) is a fuzzy normal space.

**Proof.** For each \( r \in I_0 \), let \( \nu_1, \nu_2 \in I^Y \) be \( r \)-rfc sets such that \( \nu_1 \triangleleft \nu_2 \). Since \( f \) is a fuzzy regular irresolute (resp. fuzzy almost continuous) function, \( f^{-1}(\nu_1) \) and \( f^{-1}(\nu_2) \) are \( r \)-rfc sets (resp. \( \tau_1(f^{-1}(\nu_1)) \geq r, \tau_1(f^{-1}(\nu_2)) \geq r \) ) such that \( f^{-1}(\nu_1) \triangleleft f^{-1}(\nu_2) \). Since \((X, \tau_1)\) is a fuzzy mildly normal space (resp. fuzzy normal space), there exist \( \lambda_1, \lambda_2 \in I^X \) such that \( f^{-1}(\nu_1) \leq \lambda_1, f^{-1}(\nu_2) \leq \lambda_2 \) and \( \lambda_1 \triangleleft \lambda_2 \).

Let \( \mu_1 = I_{\tau_1}(C_{\tau_1}(\lambda_1, r), r) \) and let \( \mu_2 = I_{\tau_1}(C_{\tau_1}(\lambda_2, r), r) \). Then clearly \( \mu_1 \) and \( \mu_2 \) are \( r \)-rfc sets such that \( f^{-1}(\nu_1) \leq \mu_1, f^{-1}(\nu_2) \leq \mu_2 \) and \( \mu_1 \triangleleft \mu_2 \). Since \( f \) is fuzzy almost regular generalized closed function, there exist \( \gamma_1, \gamma_2 \in I^Y \) such that \( \gamma_1, \gamma_2 \) are \( r \)-rgfo sets, \( \nu_1 \leq \gamma_1, \nu_2 \leq \gamma_2, f^{-1}(\gamma_1) \leq \mu_1 \) and
f^{-1}(\gamma_2) \leq \mu_2 \). Since \( \mu_1 \tilde{\gamma}_2 \), then \( \gamma_1 \tilde{\gamma}_2 \). Hence \((Y, \tau_2)\) is a fuzzy mildly normal space. \(\square\)

**Corollary 3.10.** Let \( f : (X, \tau_1) \to (Y, \tau_2) \) be a fuzzy almost continuous, fuzzy almost closed and surjective function. If \((X, \tau_1)\) is a fuzzy normal space, then \((Y, \tau_2)\) is a fuzzy mildly normal space.

**Proof.** The proof is determined straightforward. \(\square\)

**References**


Department of Mathematics
Faculty of Science
Assuit University
Assuit, Egypt

E-mail address: mazab57@yahoo.com