DIFFERENTIAL EQUATIONS RELATED TO FAMILY $\mathcal{A}$

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Abstract. Let $h$ be a meromorphic function with few poles and zeros. By Nevanlinna’s value distribution theory we prove some new properties on the polynomials in $h$ with the coefficients being small functions of $h$. We prove that if $f$ is a meromorphic function and if $f^{(n)}$ is identically a polynomial in $h$ with the constant term not vanish identically, then $f$ is a polynomial in $h$. As an application, we are able to find the entire solutions of the differential equation of the type

$$f^{(n)} + P(f) = be^{cz} + Q(e^z),$$

where $P(f)$ is a differential polynomial in $f$ of degree at most $n - 1$, and $Q(e^z)$ is a polynomial in $e^z$ of degree $k \leq \max \{n - 1, s(n - 1)/n\}$ with small functions of $e^z$ as its coefficients.

1. Introduction and results

In this paper the term “meromorphic” will always mean meromorphic in the complex plane $\mathbb{C}$. Let $\mathcal{M}(\mathbb{C})$ be the set of all meromorphic functions. For $f(z) \in \mathcal{M}(\mathbb{C})$, we shall use Nevanlinna’s value distribution theory of meromorphic functions, and it is assumed that the reader is familiar with its basic notations and results (see [3]), such as $T(r, f)$, $N(r, f)$ and $m(r, f)$; they are called characteristic function, proximate function and counting function of $f$, respectively. The notation $S(r, f)$ is defined to be any quantity satisfying $S(r, f) = o(T(r, f))$ as $r \to \infty$ possibly outside a set of $r$ of finite linear measure. In the sequel, we call $a(z)$ a small function of $f$ provided that $T(r, a) = S(r, f)$. The notation $\mathfrak{A}$ is defined to be the family of all meromorphic functions which satisfy $N(r, 1/h) + N(r, h) = S(r, h)$. Note that all functions in the family $\mathfrak{A}$ are transcendental, and all functions of the form $ah$ are functions in family $\mathfrak{A}$, where $h \in \mathfrak{A}$ and $a \neq 0$ is a small function of $h$. Some properties related to the functions in $\mathfrak{A}$ have been studied in [1], [4], [5], [8] and [11].

Let $h$ be a function in $\mathfrak{A}$, and

$$S(h) = \{a \in M(\mathbb{C}) : T(r, a) = S(r, h)\}.$$
It is obvious that $S(h)$ is a field of functions, which is closed for product and differentiation. In the following, the notation $\mathcal{P}[h]$ is defined to be the ring of polynomials in $h$ with coefficients being the functions in $S(h)$. We call $p(h) \in \mathcal{P}[h]$ (deg $p(h) \geq 1$) is prime if $p(h)$ has no factors with degree $\leq 1$ except $ap(h)$, where $a$ is any nonzero small function of $h$. Obviously, any polynomials with degree one must be prime. We denote by $(p_1(h), p_2(h))$ the monic greatest common divisor of two polynomials $p_1(h), p_2(h) \in \mathcal{P}[h]$. We call $p_1(h)$ and $p_2(h)$ are relatively prime provided that $(p_1(h), p_2(h)) = 1$. By using the Eucliding algorithm for polynomials over small function field, we can derive the following result.

**Theorem A.** Suppose $p(h) \in \mathcal{P}[h]$ and deg $p(h) \geq 1$. Then we have the following decomposition

$$p(h) = a \prod_{i=1}^{s} p_i^{t_i}(h),$$

where $a \in S(h)$, $p_i(h) \in \mathcal{P}[h]$ ($i = 1, \ldots, s$) are prime and monic polynomials in $h$, $t_1, \ldots, t_s$ are positive integers. Moreover, any two of $p_i(h)$ ($i = 1, \ldots, s$) are relatively prime. And this decomposition is unique. We call such decomposition a standard decomposition.

If $h \in \mathcal{A}$ and $p(h) = ah + b$, where $a, b \in S(h)$ and $ab \neq 0$, then by Nevanlinna’s second fundamental theorem, we have $T(r, p(h)) = N \left( r, \frac{1}{p(h)} \right) + S(r, h)$, which implies that the zeros of $p(h)$ are mainly simple zeros. In this paper, we shall generalize this property by proving the following results.

**Theorem 1.** Let $h$ be a function in the family $\mathcal{A}$. If $p(h)$ is a polynomial in $h$ with the following standard decomposition

$$p(h) = a \prod_{i=1}^{s} p_i(h),$$

where

(1) $$p_i(h) = a_{i0}h^{n_i} + a_{i1}h^{n_i-1} + \cdots + a_{in_i} \in \mathcal{P}[h],$$

is prime for $i = 1, \ldots, s$, and all the “constant term” $a_{in_i} \neq 0$, then

$$T(r, p(h)) = N \left( r, \frac{1}{p(h)} \right) + S(r, h).$$

For example, suppose that $h \in \mathcal{A}$. It is easy to see that $p(h) = h^4 + z$ is a prime polynomial of degree 4 in $\mathcal{P}[h]$. Thus

$$T \left( r, \frac{1}{h^4 + z} \right) = N \left( r, \frac{1}{h^4 + z} \right) + S(r, h).$$
**Theorem 2.** Let \( h \in \mathcal{A} \), \( p(h) \) be a polynomial in \( h \) of degree \( n \) with the “constant term” not vanish identically. Suppose that \( f \) is a meromorphic function such that

\[
f^n = p(h)
\]

for a positive integers \( m \). Then \( f \in \mathcal{P}[h] \), and \( m \) is a factor of \( n \).

**Corollary 1.** Let \( h \in \mathcal{A} \), \( p(h) \) be a polynomial in \( h \) of degree \( n \), \( b_n \) be the “constant term” of \( p(h) \). Suppose that \( f \) is a meromorphic function satisfying

\[
a_0 f^2 + a_1 f + a_2 = p(h),
\]

where \( a_0, a_1, a_2 \) are small functions of \( f \) and \( a_0 \neq 0 \), \( \frac{4a_0a_2 - a_1^2}{a_0} - b_n \neq 0 \). Then \( f \in \mathcal{P}[h] \), and \( n \) is even.

The conclusion of Corollary 1 may not be true without the condition (4) \( \frac{4a_0a_2 - a_1^2}{a_0} - b_n \neq 0 \). For example, the functions \( f = e^{3z^2/2} + e^{z^2/2} \) and \( h = e^z \) satisfy \( f^2 = h^3 + 2h^2 + h \). However, if the degree of \( p(h) \) in Corollary 1 is 2, then we can still get \( f \in \mathcal{P}[h] \). In fact, we have the following result.

**Theorem 3.** Let \( h \) be a function in family \( \mathcal{A} \), \( Q(h) \in \mathcal{P}[h] \) be a polynomial in \( h \) with \( \deg Q(h) = n \). Suppose that \( f \) is a meromorphic function satisfying the following equation

\[
a_n f^n + a_{n-1} f^{n-1} + \cdots + a_1 f + a_0 = Q(h),
\]

where \( a_0, a_1, \ldots, a_n \) are small functions of \( f \) with \( a_n \neq 0 \). Then \( f = \omega h + \alpha \), where \( \omega \) and \( \alpha \) are small functions of \( h \).

This result is actually a corollary of the following more general result.

**Theorem 4.** Let \( h \) be a function in family \( \mathcal{A} \), \( Q_k(h) \in \mathcal{P}[h] \) be a polynomial in \( h \) with \( \deg Q_k(h) = k \). Let \( f \) be a nonconstant meromorphic function and \( P_{n-1}(f) \) is a differential polynomial in \( f \) of weight \( \leq n - 1 \) with small functions of \( f \) as its coefficients. If

\[
f^n + P_{n-1}(f) = b_n h^k + Q_k(h),
\]

where \( s (> k) \) is a positive integer, \( b_n \) is a nonconstant small function of \( h \), and \( k \leq \max\{n - 1, s(n - 1)/n\} \), then there exists a small function \( \alpha \) of \( h \) such that

\[
(f - \alpha)^n = b_n h^s.
\]

Note that the conclusion of Theorem 4 may not be true if the weight of \( P_{n-1}(f) \) exceed \( n - 1 \). For example, the function \( f(z) = e^z + \frac{1}{e^{z-1}} \) satisfies the following equation

\[
f^3 - \frac{1}{2} f'' - \frac{5}{2} f = e^{3z} - \frac{3}{2} e^z + \frac{3}{2}.
\]
However, if we restrict $f$ to be an entire function, then we can just assume that $P_{n-1}(f)$ is a differential polynomial in $f$ of degree $\leq n - 1$ without the restriction on the weight.

Theorem 4 enable us to solve some functional differential equations related to functions in the family $\mathcal{A}$.

**Corollary 2.** Let $a_0, a_3 \neq 0$ be constants, and $a_1, a_2$ be small functions of $e^z$. Then the following differential equation

\begin{equation}
 f'^3 + f' = a_3e^{3z} + a_2e^{2z} + a_1e^z + a_0
\end{equation}

has a meromorphic solution $f$ if and only if $a_1, a_2$ are constants and

\begin{equation}
 a_3^3 = 27a_0a_3^3, \quad (3a_3^2a_1 - a_2^3)^3 = 27a_3^7.
\end{equation}

If the above conditions holds, then the solution of (5) is

\[ f(z) = \left(a_1 - \frac{a_2^2}{3a_3^2}\right) \left( e^z + \frac{a_2}{3a_3} \right). \]

**Corollary 3.** Suppose that $b_0$ and $b_3$ are nonzero constants, and $b_1$ is a small function of $e^z$. Then the equation

\begin{equation}
 f'^2 + f' = b_3e^{3z} + b_1e^z + b_0
\end{equation}

has an entire solution $f$ if and only if $b_1 = 0$ and $b_0 = 9/16$. If this the conditions hold, then the entire solution of the above equation is $f(z) = \frac{1}{\sqrt{b_0}}e^{\frac{3}{2}z} - \frac{3}{4}$. 

**Corollary 4.** Let $n \geq 2$ be a positive integer. Then there exists no meromorphic function $f$ satisfy

\[ f^n + a_{n-1}f^{n-1} + \cdots + a_1f + a_0 = \sum_{k=0}^{n} b_k e^{kz}, \]

where $a_k (k = 0, 1, \ldots, n-1)$, $b_0$ and $b_n (\neq 0)$ are constants, $b_k (k = 1, \ldots, n-1)$ are rational functions, and one of them is not constant.

In [6], Li-Wang proved the following result.

**Theorem B.** Let $h \in \mathcal{A}$. Suppose that

\[ f = \frac{a_0(z)h^p(z) + a_1(z)h^{p-1}(z) + \cdots + a_p(z)}{b_0(z)h^q(z) + b_1(z)h^{q-1}(z) + \cdots + b_q(z)} \]

is an irreducible rational polynomial in $h$ with all coefficients being small functions of $h$ and $a_0(z)b_0(z) \neq 0$. If $N(r, f) = S(r, h)$, then $b_1 = \cdots = b_q = 0$.

We improve this result by proving the following theorem.

**Theorem 5.** Under the assumption of Theorem B. Suppose $\overline{N}(r, f) = S(r, h)$. Then we have $b_1 = \cdots = b_q = 0$. 
2. Some lemmas

The following lemmas will be needed in the proofs of our results.

**Lemma 1** ([9]). Suppose that \( h \) is a nonconstant meromorphic function. If

\[
R(h) = \frac{a_0(z)h^p(z) + a_1(z)h^{p-1}(z) + \cdots + a_p(z)}{b_0(z)h^q(z) + b_1(z)h^{q-1}(z) + \cdots + b_q(z)}
\]

is an irreducible rational polynomial in \( h \) with all coefficients being small functions of \( h \) and \( a_0(z)b_0(z) \neq 0 \), then

\[
T(r, R(h)) = \max\{p, q\}T(r, h) + S(r, h).
\]

**Lemma 2** ([6]). Suppose that \( h \) is a function in the family \( \mathcal{A} \). Let \( f = a_0h^p + a_1h^{p-1} + \cdots + a_p \) and \( g = b_0h^q + b_1h^{q-1} + \cdots + b_q \) be polynomials in \( h \) with all coefficients being small functions of \( h \) and \( a_0b_0a_p \neq 0 \). If \( q \leq p \), then \( m(r, g/f) = S(r, h) \).

**Lemma 3.** Suppose that \( h \) is a function in the family \( \mathcal{A} \). Let \( f = a_0h^p + a_1h^{p-1} + \cdots + a_p \) and \( g = b_0h^q + b_1h^{q-1} + \cdots + b_q \) be polynomials in \( h \) with all coefficients being small functions of \( h \) and \( a_0b_0 \neq 0 \). If \( f \) and \( g \) are relatively prime, then

\[
N(r, 0; f, g) = S(r, h),
\]

where \( N(r, 0; f, g) \) denotes the counting function of \( f \) and \( g \) related to the common zeros of \( f \) and \( g \). Moreover, any point \( z_0 \) which is a zero of \( f \) of multiplicity \( s \) and a zero of \( g \) of multiplicity \( t \) is counted \( \min\{s, t\} \) times in \( N(r, 0; f, g) \).

**Proof.** Since \( f \) and \( g \) are relatively prime, at least one of \( a_p, b_q \) is not zero. If one of \( a_p, b_q \) is zero, say \( a_p = 0 \), which implies that

\[
f = h^k(a_0h^{p-k} + a_1h^{p-k-1} + \cdots + a_k) = h^k f_1, \quad (a_k \neq 0),
\]

for some positive integer \( k \), and obviously \( N(r, 0; f, g) = S(r, h) + N(r, 0; f_1, g) \). Thus it is only needed to prove the conclusion holds provided \( a_pb_q \neq 0 \). Without loss of generality, we assume that \( p \geq q \). By Lemma 1 we have

\[
T\left(r, \frac{g}{f}\right) = pT(r, h) + S(r, h) = T(r, f) + S(r, h).
\]

By Lemma 2 we have

\[
T\left(r, \frac{g}{f}\right) = m\left(r, \frac{g}{f}\right) + N\left(r, \frac{g}{f}\right) = S(r, h) + N\left(r, \frac{g}{f}\right),
\]

and

\[
T(r, f) = N\left(r, \frac{1}{f}\right) + m\left(r, \frac{1}{f}\right) + O(1) = N\left(r, \frac{1}{f}\right) + S(r, h).
\]

From (8), (9) and (10) we have

\[
N\left(r, \frac{1}{f}\right) = N\left(r, \frac{g}{f}\right) + S(r, h).
\]
Since each pole of \( g/f \) can only be a pole of \( 1/f \) or a pole of the coefficients of \( g \), it follows from (11) that

\[
N(r,0;f,g) = N\left(r, \frac{1}{f}\right) - N\left(r, \frac{g}{f}\right) + S(r,h) = S(r,h),
\]

which completed the proof of Lemma 3. \( \square \)

**Lemma 4** ([7]). Let \( f_1, \ldots, f_n \) be nonconstant meromorphic functions such that \( f_1 + \cdots + f_n = 1 \). If \( f_1, \ldots, f_n \) are linearly independent, then the following inequality holds:

\[
T(r, f) < \sum_{i=1}^{n} N_{n-1}\left(r, \frac{1}{f_i}\right) + (n-1) \sum_{i=2}^{n} N(r, f_i) + o(T(r)), r \notin E.
\]

Here \( N_{n-1}(r,f) \) is the counting function of \( f \) which counts a pole of \( f \) according to its multiplicity if the multiplicity is less than or equal to \( n-1 \) and counts a pole \( n-1 \) times if the multiplicity is greater than \( n-1 \). Here \( T(r) = \sum_{i=1}^{n} T(r, f_i) \).

**Lemma 5.** Let \( h \) be a function in the family \( \mathcal{A} \) and

\[
f = a_0 h^n + a_1 h^{n-1} + \cdots + a_{n-1} h + a_n
\]

be a polynomial in \( h \) with all coefficients \( a_i \) (\( i = 0, 1, \ldots, n \)) being small functions of \( h \). If \( a_0 a_n \neq 0 \) and \( n \geq 1 \), then

\[
T(r, f) = N\left(r, \frac{1}{f}\right) + S(r,h),
\]

\[
T(r, f) \leq nN\left(r, \frac{1}{f}\right) + S(r,h).
\]

**Proof.** By Lemma 2 we have

\[
T(r, f) = N\left(r, \frac{1}{f}\right) + m\left(r, \frac{1}{f}\right) + O(1) = N\left(r, \frac{1}{f}\right) + S(r,h).
\]

Without loss of generality, we assume that none of \( a_0, a_1, \ldots, a_n \) is zero. Thus the functions \( f, a_0 h^n, \ldots, a_{n-1} h \) are linearly independent. Without loss of generality, we assume that \( a_n = 1 \). By Lemma 4 we can easily deduce that

\[
T(r, f) \leq nN\left(r, \frac{1}{f}\right) + S(r,h),
\]

which also completes the proof of Lemma 5. \( \square \)

**Lemma 6** ([2]). Suppose that \( f(z) \) is meromorphic and transcendental in the plane and that

\[
f^n(z)Q_1(f) = Q_2(f),
\]
where $Q_1(f)$ and $Q_2(f)$ are differential polynomials in $f$ with functions of small proximity related to $f$ as the coefficients. If the degree of $Q_2(f)$ is at most $n$, then

$$m(r, Q_1(f)) = S(r, f).$$

**Lemma 7** ([10]). Suppose that $f(z)$ is a transcendental meromorphic function and

$$F := a_n f^n + a_{n-1} f^{n-1} + \cdots + a_0, \quad a_n \neq 0,$$

where $a_k$ ($k = 0, 1, \ldots, n$) are small functions of $f$. Then either

$$F = a_n \left( f + \frac{a_{n-1}}{na_n} \right)^n$$

or

$$T(r, f) \leq \mathcal{N}(r, 1) + \mathcal{N}(r, f) + S(r, f).$$

**Lemma 8.** Let $h$ be a function in the family $\mathcal{A}$, $f$ a transcendental meromorphic function. Let $P(f)$ be a polynomial in $f$ of degree $n$ with first term $a_n f^n$, and $Q(h)$ a polynomial in $h$ of degree $s$ with first term $b_s h^s$. If

$$P(f) = Q(h)$$

and $f' = \omega_1 f + \omega_0$, where $\omega_1, \omega_0$ are small functions of $f$, then there exists a small function $\alpha$ of $f$ such that

$$a_n(f - \alpha)^n = b_s h^s.$$

**Proof.** It is obvious that $S(r, f) = S(r, h)$. Let $S(r) = S(r, f) = S(r, h)$. Without loss of generality, we assume $a_n = 1$. Let

$$P(f) = f^n + a_{n-1} f^{n-1} + \cdots + a_1 f + a_0.$$

We use mathematical induction on $s$ to prove the conclusion. When $s = 1$, let $Q(h) = b_1 h + b_0$. We have $P(f) - b_0 = b_1 h$. By Lemma 7, we get

$$\left( f + \frac{a_{n-1}}{a_n} \right)^n = b_1 h.$$

So, the conclusion is true in this case. Suppose that the conclusion is true for $s = 1, 2, \ldots, m - 1$. We now prove it is true for $s = m$. Let

$$Q(h) = b_m h^m + b_{m-1} h^{m-1} + \cdots + b_1 h + b_0.$$

If $b_{m-1} = b_{m-2} = \cdots = b_1 = 0$, then by Lemma 7, we see that the conclusion is obviously true. Suppose that $b_k$ is the first non-vanishing term in $\{b_{m-1}, b_{m-2}, \ldots, b_1\}$, i.e.,

$$Q(h) = b_m h^m + b_k h^k + \cdots + b_1 h + b_0.$$

By taking derivative in (12), we get

$$\sum_{j=0}^{n} a_j' f^j + \sum_{j=1}^{n} ja_j f^{j-1} f' = (b_m' + mb_m \beta) h^m + \sum_{j=0}^{k} (b_j' + jb_j \beta) h^j,$$
where $\beta := h'/h$ is a small function of $h$. By substituting $f' = \omega_1 f + \omega_0$ into the above equation, we get
\[
n\omega_1 f^n + \sum_{j=0}^{n-1} (a_j' + j\omega_1 a_j + (j + 1)\omega_0 a_{j+1}) f^j
\]
(16) \hspace{1cm} = (b_m' + mb_m\beta)h^m + \sum_{j=0}^{k} (b_j' + jb_j\beta)h^j.

Eliminating $h^m$ from (12) and (16), we get
\[
(nT(r; f) + kT(r; h) + S(r)).
\]
(17) \hspace{1cm} (b_m' + mb_m\beta - n\omega_1 b_m)f^n + \sum_{j=0}^{n-1} c_j f^j = \sum_{j=0}^{k} (b_j b_m' - b_m b_j' + (m - j)b_j b_m\beta)h^j,
\]
where
\[
c_j = a_j' (b_m' + mb_m\beta) - b_m (a_j' + j\omega_1 a_j + (j + 1)\omega_0 a_{j+1}).
\]
If the coefficient of $f^n$ in (17) not vanish identically, then we have $nT(r, f) \leq kT(r, h) + S(r)$. However, from (14) we have
\[
nT(r, f) = mT(r, h) + S(r).
\]
Therefore, we get $m \leq k$, a contradiction. Hence the coefficient of $f^n$ in (17) must be vanish identically, i.e., the left-hand side of (17) is a polynomial in $f$ of degree $\leq n - 1$. Since $b_k b_m \neq 0$, we can easily see that the coefficient of $h^k$ in (17) can not vanish identically. This means that the right-hand side of (17) is a polynomial in $h$ of degree $k$. By the hypothesis of mathematical induction, we see that there exist an integer $t$ and two small functions $\alpha, \varphi (\neq 0)$ of $h$ such that
\[
(f - \alpha)^t = \varphi h^k,
\]
which implies $tT(r, f) = kT(r, h) + S(r)$. This together with (19) implies $n/t = m/k$. It follows from the above equation that
\[
(f - \alpha)^n = c h^m,
\]
where $c$ is a small function of $h$. Compare the coefficient of $h^m$ in the above equation and (14), we get $c = b_m$. This also completes the proof of Lemma 8.

3. Proof of Theorem 1

It is sufficient to prove Theorem 1 for the simple case that
\[
p(h) = a_0 h^n + a_1 h^{n-1} + \cdots + a_n.
\]
Without loss of generality, we assume that $a_0 = 1$. Let $f = p(h)$. From (22), we have
\[
f' = \sum_{i=0}^{n} b_i h^{n-i},
\]
(23)
where $b_i = a'_i + a_i(n - i)h'/h$, $i = 0, 1, \ldots, n$. If $b_0 = a'_0 + a_0 nh'/h = 0$, then $a_0 h^n$ is a constant, which is impossible. Thus $b_0 \neq 0$. By the assumption of Theorem 1 and the lemma of logarithmic derivative, we see that $b_i \in S(h)$ for $i = 0, 1, \ldots, n$, which leads to $f' \in \mathcal{P}[h]$. Since $p(h)$ is prime, there are two possibilities: $(f, f') = f$, or $(f, f') = 1$.

If $(f, f') = f$, then $T(r, f'/f) = S(r, h)$, which implies $N(r, 1/f) = S(r, h)$. From this and by Lemma 5 we have $T(r, f) = S(r, h)$. On the other hand, by Lemma 1 we have $T(r, f) = nT(r, h) + S(r, h)$, which leads to $T(r, h) = S(r, h)$, a contradiction. Hence $(f, f') = 1$. Let $f' = q(h)$. Thus $p(h)$ and $q(h)$ are relatively prime and

\[(24) \quad \frac{f'}{f} = \frac{q(h)}{p(h)}.\]

By Lemma 5 we have

\[(25) \quad T(r, p(h)) = N\left(r, \frac{1}{p(h)}\right) + S(r, h).\]

Obviously

\[(26) \quad N\left(r, \frac{1}{p(h)}\right) = N\left(r, \frac{q(h)}{p(h)}\right) + N(r, 0; p(h), q(h)) + S(r, h).\]

Since $p(h)$ and $q(h)$ are relatively prime, by Lemma 3 we have

\[N(r, 0; p(h), q(h)) = S(r, h).\]

Combining this with (25) and (26) we have

\[(27) \quad T(r, p(h)) = N\left(r, \frac{q(h)}{p(h)}\right) + S(r, h).\]

From (24) we see that each pole of $q(h)/p(h)$ is simple and it can only be the pole of $1/p(h)$ or the pole of the coefficients of $q(h)$, which implies

\[N\left(r, \frac{q(h)}{p(h)}\right) = \mathcal{N}\left(r, \frac{q(h)}{p(h)}\right) \leq \mathcal{N}\left(r, \frac{1}{p(h)}\right) + S(r, h).\]

It follows from this and (27) that

\[T(r, p(h)) = \mathcal{N}\left(r, \frac{1}{p(h)}\right) + S(r, h),\]

which also completes the proof of Theorem 1.

4. Proof of Theorem 2

Obviously the conclusion is true for $m = 1$ or $n = 0$. Now we assume that $m \geq 2$ and $n \geq 1$. By Theorem A, there exist a function $a \in S(h)$, prime and
monic polynomials $p_i(h) \in \mathcal{P}[h]$ ($i = 1, \ldots, s$) and positive integers $t_1, \ldots, t_s$ such that

\[(28) \quad f^m = a \prod_{i=1}^{s} p_i^{t_i}(h),\]

with any two of $p_i(h)$ ($i = 1, \ldots, s$) being relatively prime. It is only needed to prove that $m$ is a factor of all positive integers $t_1, \ldots, t_s$. Since $p_1(h)$ is prime, and the “constant term” of $p(h)$ does not vanish identically, it follow that the “constant term” of $p_1(h)$ does not vanish identically, too. By Theorem 1 we have

\[T(r, p_1(h)) = N\left(r, \frac{1}{p_1(h)}\right) + S(r, h).\]

It follows that

\[N_{11}\left(r, \frac{1}{p_1(h)}\right) = T(r, p_1(h)) + S(r, h),\]

where $N_{11}(r, 1/p_1(h))$ denotes the counting function of $p_1(h)$ related to the simple zeros of $p_1(h)$, which counts such points only once. Since $p_1(h)$ and $p_i(h)$ are relatively prime for $i = 2, \ldots, s$, by Lemma 3 we have

\[N_{11}\left(r, \frac{1}{p_1(h)}\right) = \sum_{i=2}^{s} N(r, 0; p_1(h), p_i(h)) = T(r, p_1(h)) + S(r, h),\]

which means that there are many simple zeros of $p_1(h)$, and such zeros are not zeros of any $p_i(h)$ ($i = 2, \ldots, s$) or $a$. From (28) we see that these points must be zeros of $f^m$ with multiplicity $t_1$. This implies that $m$ must be a factor of $t_1$. By using a similar method we can deduce that $m$ is a factor of all positive integers $t_1, \ldots, t_s$. This also completes the proof of Theorem 2.

5. Proof of Theorem 4

Let

\[Q_k(h) = b_k h^k + b_{k-1} h^{k-1} + \cdots + b_1 h + b_0,\]

and $Q(h) = b_k h^k + Q_k(h)$. Since the weight of $P_{n-1}(f)$ is at most $n - 1$, from (3), we can easily get $N(r, f) = N(r, h) + S(r, h) = S(r, h)$, and

\[(29) \quad nT(r, f) = sT(r, h) + S(r, h).\]

Write $S(r) = S(r, f) = S(r, h)$. Let $a_0$ be the constant term of $P_{n-1}(f)$ (the term of degree zero). Without loss of generality, we assume $a_0 \neq b_0$, otherwise, we do a transformation $f = \tilde{f} + c$ for a suitable constant $c$. By Lemma 2, we have

\[(30) \quad m\left(r, \frac{1}{Q(h) - a_0}\right) = S(r).\]

It is obvious that $Q(h) - a_0 \neq 0$. Rewrite (3) as

\[(31) \quad \frac{1}{Q(h) - a_0} + \frac{1}{Q(h) - a_0} \frac{P_{n-1}(f) - a_0}{f^n} = \frac{1}{f^n}.\]
Note that the constant term of $P_{n-1}(f) - a_0$ is zero. We see that the left-hand side of the above equation is a polynomial in $1/f$ of degree at most $n - 1$ with functions of small proximity related to $f$ as the coefficients. Therefore, we have $m(r, 1/f) = S(r)$. Rewrite (3) as

$$\frac{h^s}{Q(h)} + \frac{h^s}{Q(h)} \frac{P_{n-1}(f)}{f^n} = \frac{h^s}{f^n}.$$  

By Lemma 2, we have $m(r, h^s/Q(h)) = S(r)$. It follows from the above equation that

$$m\left(r, \frac{h^s}{f^n}\right) = S(r).$$

Now we prove

$$m\left(r, \frac{h^j}{f^{n-1}}\right) = S(r)$$

for positive integers $j = 1, 2, \ldots, k$. For fixed $r > 0$, let

$$E_1 = \left\{ \theta \in [0, 2\pi) \mid \frac{|f(re^{i\theta})|}{|h(re^{i\theta})|^{s-j}} \leq 1 \right\},$$

$$E_2 = \left\{ \theta \in [0, 2\pi) \mid \frac{|f(re^{i\theta})|}{|h(re^{i\theta})|^{s-j}} > 1, \text{ and } |h(re^{i\theta})| \leq 1 \right\},$$

$$E_3 = \left\{ \theta \in [0, 2\pi) \mid \frac{|f(re^{i\theta})|}{|h(re^{i\theta})|^{s-j}} > 1, \text{ and } |h(re^{i\theta})| > 1 \right\}.$$

We see that the three sets $E_1, E_2, E_3$ are disjointed from each other and the union of them is $[0, 2\pi)$. By the definition of the proximity function, we have

$$m\left(r, \frac{h^j}{f^{n-1}}\right) = \frac{1}{2\pi} \int_{0}^{2\pi} \log^+ \left| \frac{h^j(re^{i\theta})}{f^{n-1}(re^{i\theta})} \right| d\theta = I_1 + I_2 + I_3,$$

where

$$I_l = \frac{1}{2\pi} \int_{E_l} \log^+ \left| \frac{h^j(re^{i\theta})}{f^{n-1}(re^{i\theta})} \right| d\theta, \quad l = 1, 2, 3.$$  

When $\theta$ belongs to $E_1$, we have $|f(re^{i\theta})| \leq |h(re^{i\theta})|^{s-j}$. From the expansion

$$\frac{h^j}{f^{n-1}} = \frac{h^s}{f^n} \cdot \frac{f}{h^{s-j}},$$

we get

$$I_1 \leq m\left(r, \frac{h^s}{f^n}\right) = S(r).$$

When $\theta$ belongs to $E_2$, we have $|h(re^{i\theta})| \leq 1$, and thus

$$\left| \frac{h^j(re^{i\theta})}{f^{n-1}(re^{i\theta})} \right| \leq \frac{1}{|f^{n-1}(re^{i\theta})|}.$$
which yields

\[ I_2 \leq m \left( r, \frac{1}{f^{n-1}} \right) = S(r) . \]

When \( \theta \) belongs to \( E_3 \), we can deduce that \( 1 < |h(re^{i\theta})| \leq |f(re^{i\theta})| \). If \( k \leq n-1 \), then

\[ \frac{|h^j(re^{i\theta})|}{|f^{n-1}(re^{i\theta})|} \leq \left( \frac{|h(re^{i\theta})|}{|f(re^{i\theta})|} \right)^{n-1} \leq 1 . \]

If \( k \leq (n-1)s/n \), then we still have

\[ \frac{|h^j(re^{i\theta})|}{|f^{n-1}(re^{i\theta})|} \leq \frac{1}{|f(re^{i\theta})|^{\frac{n-1-s}{s}}} \leq 1 . \]

Therefore, we have \( I_3 = 0 \). Hence the equation (34) holds.

Let \( \beta = h'/h \). Then \( \beta \) is a small function of \( h \). Taking derivative in (3) gives

\[ nf^{n-1}f' + \left( P_{n-1}(f) \right)' = (b'_s + s\beta b_s)h^s + \sum_{j=0}^{k} (b'_j + jb_j\beta)h^j . \]

Combining this with (3) we get

\[ f^{n-1} + R_{n-1}(f) = \sum_{j=0}^{k} \left( b_jb'_s - b_jb'_j + (s - j)b_jb_s\beta \right)h^j . \]

where

\[ \gamma = (b'_s + s\beta f - nb_s f') . \]

and

\[ R_{n-1}(f) = (b'_s + s\beta) P_{n-1}(f) - b_s (P_{n-1}(f))' . \]

which is a differential polynomial in \( f \) of degree at most \( n - 1 \), and weight at most \( n \). By (34) and Lemma 6, we get \( m(r, \gamma) = S(r) \). Since the weight of \( R_{n-1}(f) \) is at most \( n \), it follows from (36) that \( N(r, \gamma) \leq N(r, f) + S(r) = S(r) \). Hence \( T(r, \gamma) = S(r) \), i.e., \( \gamma \) is a small function of \( h \) and \( f \). Substituting

\[ f' = \frac{b'_s + sb_s\beta}{nb_s} f - \frac{\gamma}{nb_s} . \]

into \( P_{n-1}(f) \), we see that \( P_{n-1}(f) \) is a polynomial in \( f \) of degree at most \( n - 1 \). By Lemma 8, we have

\[ (f - \alpha)^n = b_sh^s , \]

where \( \alpha \) is a small function of \( f \). This also completes the proof of Theorem 4.
6. Proof of Theorem 5

Let
\[ P(h) = a_0(z)h^p(z) + a_1(z)h^{p-1}(z) + \cdots + a_p(z), \]
and
\[ Q(h) = b_0(z)h^q(z) + b_1(z)h^{q-1}(z) + \cdots + b_q(z). \]
Then \( P(h) \) and \( Q(h) \) are relatively prime and
\[ f = \frac{P(h)}{Q(h)}. \]
If \( b_r(z) \neq 0 \) for some \( r \in \{1, \ldots, q\} \), then \( Q(h) \) must have a prime factor \( Q_0(h) \) with the "constant term" not vanish identically. By Theorem 1 we have
\[
\mathcal{N}\left(r, \frac{1}{Q(h)}\right) \geq \mathcal{N}\left(r, \frac{1}{Q_0(h)}\right) + S(r, h) = T(r, Q_0(h)) + S(r, h) \\
\geq T(r, h) + S(r, h).
\]
Obviously
\[
\mathcal{N}\left(r, \frac{1}{Q(h)}\right) \leq \mathcal{N}(r, f) + N(r, 0; P(h), Q(h)) + S(r, h).
\]
By Lemma 3 we have
\[
N(r, 0; P(h), Q(h)) = S(r, h).
\]
Therefore, by the assumption we have \( \mathcal{N}\left(r, \frac{1}{Q(h)}\right) = S(r, h) \). Hence we get \( T(r, h) = S(r, h) \), a contradiction. And this also completes the proof of Theorem 5.

References


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