SEMI-DIVISORIALITY OF HOM-MODULES AND INJECTIVE COGENERATOR OF A QUOTIENT CATEGORY

Hwankoo Kim

Abstract. In this paper, we study \( w \)-nullity and (co-)semi-divisoriality of Hom-modules and the semi-divisorial envelope of \( \text{Hom}_R(M, N) \) under suitable conditions on \( R, M, \) and \( N \). We also investigate an injective cogenerator of a quotient category.

1. Introduction

Let \( R \) be an integral domain. In [17] Wang and McCasland defined semi-divisorial closure, or \( w \)-closure for torsion-free \( R \)-modules. In [7], H. Kim extended this notion to any \( R \)-module and introduced and studied the related notions of co-semi-divisoriality and \( w \)-nullity. In [7, 8, 9] these concepts were then used to give new module-theoretic characterizations of \( t \)-linkative domains, generalized GCD domains, and strong Mori domains, classes of domains widely considered in multiplicative ideal theory.

Earlier, in [1, 12, 13], Beck, Nishi and Shinagawa investigated injective modules over a Krull domain in terms of co-divisorial modules, pseudo-null modules, and divisorial modules and investigated pseudo-nullity and (co-)divisoriality of Home-modules. In particular, it was shown that in the case of a Krull domain \( R \) with quotient field \( K \), the injective envelope \( E(K/R) \) of \( K/R \) is a cogenerator of the quotient category \( \text{Mod}(R)/\mathbb{M}_0 \), where \( \text{Mod}(R) \) is the category of all unitary \( R \)-modules and \( \mathbb{M}_0 \) is the thick subcategory of the modules with trivial maps into the codivisorial modules. Recently, in [11] Moucouf characterized the rings of Krull type \( R \) with quotient field \( K \) such that the (canonical) functorial image of \( E(K/R) \) is an injective cogenerator of the quotient category \( \text{Mod}(R)/\mathbb{M}_0 \). Also in [16], Wang investigated the case when Hom-modules are semi-divisorial in torsion-free.

Received July 31, 2009.

2010 Mathematics Subject Classification. Primary 13A15; Secondary 13D30.

Key words and phrases. (co-)semi-divisorial, \( w \)-null, cogenerator, Hom-module, H-domain, Krull domain, torsion theory.

This research was supported by Basic Science Research Program through the National Research Foundation of Korea(NRF) funded by the Ministry of Education, Science and Technology(2010-0011996).
In this paper, we study an injective cogenerator of a quotient category and \( w \)-nullity and (co-)semi-divisoriality of Hom-modules using methods developed in [1, 11, 12, 13]. As a corollary, for the class of completely integrally closed domains, we characterize Krull domains in terms of an injective cogenerator of a quotient category. We also investigate the semi-divisorial envelope of \( \text{Hom}_R(M, N) \) under suitable conditions on \( R, M, \) and \( N \).

Throughout this paper, \( R \) denotes an integral domain with quotient field \( K \). Let \( \mathcal{F}(R) \) denote the set of nonzero fractional ideals of \( R \). Recall that the function on \( \mathcal{F}(R) \) defined by \( A \mapsto (A^{-1})^{-1} = A_w \) is a star operation called the \( w \)-operation, where \( A^{-1} = R :_K A = \{ x \in K \mid xA \subseteq R \} \). An ideal \( J \) of \( R \) is called a Glaz-Vasconcelos ideal if \( J \) is a finitely generated ideal of \( R \) with \( J^{-1} = R \). We abbreviate this as \( \text{GV-ideal} \), denoted by \( J \in \text{GV}(R) \). Following [17], a torsion-free \( R \) module \( M \) is called a \( w \)-module if \( Jx \subseteq M \) for \( J \in \text{GV}(R) \) and \( x \in M \otimes K \) implies that \( x \in M \), which is said to be semi-divisorial in [4]. For a torsion-free \( R \)-module \( M \), Wang and McCasland defined the \( w \)-envelope of \( M \) in [17] as \( M_w = \{ x \in M \otimes K \mid Jx \subseteq M \text{ for some } J \in \text{GV}(R) \} \). In particular, if \( I \) is a nonzero fractional ideal, then \( I_w = \{ x \in K \mid Jx \subseteq I \text{ for some } J \in \text{GV}(R) \} \).

The canonical map \( I \mapsto I_w \) on \( \mathcal{F}(R) \) is a star-operation, denoted \( w \). It was shown in [17] that a prime ideal \( P \) of \( R \) is a \( w \)-ideal if and only if \( P_w \neq R \). Therefore, all prime ideals contained in a proper \( w \)-ideal of \( R \) are also \( w \)-ideals. We denote by \( w \)-Max\((R) \) the set of \( w \)-maximal ideals of \( R \). It is also worth noting that \( w \) distributes over (finite) intersections [17, Proposition 2.5]. For unexplained terminology and notation, we refer to [2, 3, 14].

2. \( w \)-null and (co-)semi-divisorial Hom-modules

In [7], H. Kim introduced the notions of “co-semi-divisoriality” and “\( w \)-nullity” of a module as follows. Let \( M \) be a module over an integral domain \( R \) and let \( \tau(M) := \{ x \in M \mid (\mathcal{O}(x))_w = R \} \), where \( \mathcal{O}(x) := (0 :_R x) = \text{ann}_R(x) \) is the order ideal of \( x \). Then \( \tau(M) \) is a submodule of \( M \). \( M \) is said to be co-semi-divisorial (resp., \( w \)-null) if \( \tau(M) = 0 \) (resp., \( \tau(M) = M \)). Note that the notions of co-semi-divisoriality and \( w \)-nullity can be interpreted in terms of a suitable torsion theory [2, Proposition IX.6.2 and Proposition IX.6.4] (with \( \mathcal{P} = \text{w-Max}(R) \)).

Let \( R \) be an integral domain, let \( \mathcal{T}_c(R) \) denote the full subcategory of \( \text{Mod}(R) \) consisting of all modules \( M \) such that \( M_P = 0 \) for all \( P \in \text{w-Max}(R) \), and let \( \mathcal{T}_c(R) \) denote the full subcategory of all \( R \)-modules \( M \) that have no sub-object other than zero belonging to \( \mathcal{T}_c(R) \). Finally let \( \mathcal{E}_c(R) \) be the full subcategory of \( \text{Mod}(R) \) consisting of all co-semi-divisorial and semi-divisorial \( R \)-modules.

In an abelian category \( \mathcal{A} \), we have the following definitions:

(a) An injective object \( E \) is called an injective cogenerator if \( \text{Hom}_{\mathcal{A}}(M, E) \neq 0 \) for every \( M \in \mathcal{A} \) that is not a zero object.
(b) A nonempty full subcategory \( C \) of \( \mathcal{A} \) is said to be \textit{thick} if, for each short exact sequence \( 0 \to L \to M \to N \to 0 \) in \( \mathcal{A} \), \( M \) is an object of \( C \) if and only if \( L \) and \( N \) are objects of \( C \). It is also called a \textit{Serre subcategory} of \( \mathcal{A} \).

It is clear that \( \mathcal{T}(R) \) is a thick subcategory of \( \text{Mod}(R) \). Then we can now consider the quotient category \( \text{Mod}(R) = \mathcal{T}(R) \) and the canonical functor \( T : \text{Mod}(R) \to \text{Mod}(R)/\mathcal{T}(R) \).

As usual, we denote by \( E(M) \) the injective envelope of an \( R \)-module \( M \).

The following result will be useful later on.

\textbf{Proposition 2.1.} The following statements are equivalent for an \( R \)-module \( M \).

1. \( M \) is co-semi-divisorial, i.e., \( M \in \mathcal{T}(R) \).
2. \( \mathcal{O}(x) \) is a \( w \)-ideal for every element \( x \in M \).
3. \( (\mathcal{O}(x))_w \neq R \) for every nonzero element \( x \in M \).
4. \( \text{Hom}_R(N, M) = 0 \) for every \( w \)-null \( R \)-module \( N \).
5. \( \text{Hom}_R(N, E(M)) = 0 \) for every \( w \)-null \( R \)-module \( N \).

\textbf{Proof.} The equivalences of (1), (2), (3), and (4) are given in [7, Proposition 2.6], while the equivalence of (1) and (5) follows from [6, Proposition 1.2]. \( \square \)

Note from [17, Proposition 1.4] that the annihilator ideal of any submodule of a co-semi-divisorial module is a \( w \)-ideal. Recall from [1] that a module \( M \) is said to be \textit{cdivisorial} if the annihilator of every nonzero element of \( M \) is a divisorial ideal. Thus in a Krull domain, the notions of co-semi-divisoriality and cdivisoriality are the same.

Recall from [16, Definition 4.5] that an \( R \)-module \( M \) is said to be \textit{\( w \)-vanishing} if \( M_P = 0 \) for any maximal \( w \)-ideal \( P \) of \( R \).

\textbf{Proposition 2.2.} Let \( N \) be an \( R \)-module. Then the following statements are equivalent.

1. \( N \) is \( w \)-null, i.e., \( M \in \mathcal{T}(R) \).
2. For each \( x \in N \), \( \mathcal{O}(x) \) is not contained in any maximal \( w \)-ideal.
3. \( N \) is \( w \)-vanishing.
4. There is a torsion-free \( R \)-module \( F \) with \( N \cong F_w/F \).
5. \( \text{Hom}_R(N, E(M)) = 0 \) for every co-semi-divisorial \( R \)-module \( M \).

\textbf{Proof.} The equivalences of (1), (2), (3), and (4) are given in [7, Proposition 9.3], while the equivalence of (1) and (5) follows from [6, Proposition 1.2]. \( \square \)

Now we study \( w \)-nullity and (co-)semi-divisoriality of \( \text{Hom} \)-modules. It was shown in [7, Proposition 3.1] that an \( R \)-module \( M \) is co-semi-divisorial if and only if \( \text{Hom}_R(\mathcal{Z}(R), M) = 0 \), where \( \mathcal{Z}(R) := \bigoplus_{I \leq R \mid I_w = R} R/I \).

\textbf{Proposition 2.3.} Let \( R \) be an integral domain and let \( M \) and \( N \) be \( R \)-modules. If \( M \) is co-semi-divisorial, then so is \( \text{Hom}_R(N, M) \).
Proof. By [7, Proposition 2.6], it suffices to show that $\text{Hom}_R(L, \text{Hom}_R(N, M)) = 0$ for every $w$-null $R$-module $L$. But this follows from $\text{Hom}_R(L, \text{Hom}_R(N, M)) \cong \text{Hom}_R(N, \text{Hom}_R(L, M)) = 0$ since $M$ is co-semi-divisorial.

Proposition 2.4. Let $R$ be an integral domain and let $M$ and $N$ be any $R$-module. If $M$ is $w$-null, then so is $\text{Tor}_n^R(N, M)$ for all $n \geq 0$.

Proof. First we consider the case $n = 0$. For every co-semi-divisorial $R$-module $L$ we have $\text{Hom}_R(N \otimes_R M, E(L)) = \text{Hom}_R(N, \text{Hom}_R(M, E(L))) = 0$ since $M$ is $w$-null; therefore $N \otimes_R M$ is $w$-null by Proposition 2.2. For the case when $n \geq 1$, we consider a projective resolution of $N$:

$$\cdots \to P_n \to P_{n-1} \to \cdots \to P_2 \to P_1 \to P_0 \to N \to 0.$$ 

Then, since each $P_i \otimes M$ is $w$-null, we can see that $\text{Tor}_n^R(N, M)$ is $w$-null for every $n \geq 0$ by noting that the submodules and homomorphic images of $w$-null modules are also $w$-null.

Now we recall some definitions from [7]: Let $M$ be an $R$-module. Then $W(M) := \pi^{-1}(\tau(E(M)/M))$ is called the semi-divisorial envelope of $M$, where $\pi : E(M) \to E(M)/M$ is the canonical projection, $M$ is said to be semi-divisorial if $W(M) = M$, and $M$ is said to be weakly $w$-flat if $\text{Tor}_1^R(\mathbb{Z}(R), M) = 0$. It is clear from the definition that every injective $R$-module is semi-divisorial.

Let $N$ be an $R$-module. Then we denote $U_w(N) := \{ L \mid L \text{ is a submodule of } N \text{ such that } (L :_R x)_w = R \text{ for every } x \in N \}$.

Proposition 2.5. The following statements are equivalent for an $R$-module $M$.

1. $M$ is weakly $w$-flat.
2. $M^\flat := \text{Hom}_\mathbb{Z}(M, \mathbb{Q}/\mathbb{Z})$ is semi-divisorial.
3. $I \otimes_R M \to M$ is a monomorphism for all $I \in U_w(R)$.
4. $L \otimes_R M \to N \otimes_R M$ is a monomorphism for all $L \in U_w(N)$.

Proof. The equivalence of (1) and (2) is given in [7, Proposition 4.3], while the equivalences of (2), (3), and (4) are given in [14, IX, Exercise 25].

Let $M$ be a semi-divisorial $R$-module and $N$ be an $R$-module. Then it was shown in [7, Corollary 3.4] that if $\text{Hom}_R(\text{Tor}_1^R(\mathbb{Z}(R), N), M) = 0$, then $\text{Hom}_R(N, M)$ is semi-divisorial.

Theorem 2.6. Let $R$ be an integral domain, $M$ be a semi-divisorial $R$-module, and $N$ be an $R$-module. Then $\text{Hom}_R(N, M)$ is semi-divisorial if one of the following conditions is satisfied:

1. $M$ is co-semi-divisorial;
2. $N$ is weakly $w$-flat.

Proof. It suffices to show that $\text{Hom}_R(\text{Tor}_1^R(\mathbb{Z}(R), N), M) = 0$ by [7, Corollary 3.4].
(i) Note that $R/I$ is $w$-null for every $I \in \mathcal{U}_w(R)$ ([7, Proposition 2.5]). Thus we have that $\text{Tor}_1^R(R/I, N)$ is $w$-null for every $I \in \mathcal{U}_w(R)$. Now since $\text{Tor}$ commutes with direct sums and $w$-nullity is closed under direct sums, we have $\text{Tor}_1^R(\mathcal{Z}(R), N)$ is $w$-null. Therefore $\text{Hom}_R(\text{Tor}_1^R(\mathcal{Z}(R), N), M) = 0$ by the co-semi-divisoriality of $M$ ([7, Proposition 2.6]).

(ii) This follows from the definition of “weakly $w$-flat”. □

It was shown in [5, Proposition 2.2] that for a rank one $\nat$ ideal $I$ in $K$, the endomorphism $\text{End}_R(I) = I : I$ of $I$ is semi-divisorial. We extend this result to any $\nat$ module in the following corollary. Note that $\nat$ $R$-modules are torsion-free (and so co-semi-divisorial) for every integral domain $R$.

Corollary 2.7. Let $R$ be an integral domain.

(1) If $M$ is a flat $R$-module, then $\text{End}_R(M)$ is a semi-divisorial $R$-module.

(2) If $M$ is a co-semi-divisorial and semi-divisorial $R$-module, then so is $\text{End}_R(M)$.

(3) If $M$ is co-semi-divisorial, then $M^* = \text{Hom}_R(M, R)$ is semi-divisorial.

3. Semi-divisorial equivalence

In this section, we investigate the semi-divisorial envelope of $\text{Hom}_R(M, N)$ under suitable conditions on $R, M,$ and $N$. To do so, we need some definitions and results.

Lemma 3.1 ([15, Proposition 1.1]). Let $R$ be an integral domain and let $L \to M \to N$ be an exact sequence of $R$-modules. If $L$ and $N$ are $w$-null, then so is $M$.

Let $M$ and $N$ be $R$-modules and let $f : M \to N$ be an $R$-homomorphism. Then $f$ is said to be $w$-injective (resp., $w$-surjective) if $\ker(f)$ (resp., $\text{coker}(f)$) is $w$-null. And $f$ is said to be $w$-isomorphic if $f$ is both $w$-injective and $w$-surjective.

Lemma 3.2 ([15, Lemma 1.2]). Let $R$ be an integral domain and let $f : L \to M$ and $g : M \to N$ be homomorphisms of $R$-modules. If $f$ and $g$ are $w$-injective (resp., $w$-surjective or $w$-isomorphic), then so is $gf$.

Theorem 3.3 ([7, Theorem 8.1]). The following statements are equivalent for an integral domain $R$.

(1) If an $R$-module $M$ is injective, then so is $\tau(M)$.

(2) $E(\tau(M)) = \tau(E(M))$ for any $R$-module $M$.

(3) Let $N$ be an essential extension of $M$. If $M$ is $w$-null, then so is $N$.

(4) Let $I \leq R$ such that $I_w \neq R$. Then $I : R \alpha$ is a $w$-ideal for some $\alpha \in R \setminus I_w$.

(5) If $M$ is not $w$-null, then $M$ has a nonzero co-semi-divisorial submodule.

(6) If $I \leq R$, then there exists an ideal $J$ of $R$ such that $J_w = R$ and $I = I_w \cap J$. 

Recall that an integral domain $R$ is said to be pseudo-$t$-linkative if $R$ satisfies one of the equivalent conditions of Theorem 3.3.

**Proposition 3.4.** Let $R$ be a pseudo-$t$-linkative domain with quotient field $K(\neq R)$. Let $f : M \to N$ be a homomorphism of $R$-modules and $p : M \to M/\tau(M)$, $q : N \to N/\tau(N)$ be the canonical projections.

1. There is a unique homomorphism $f_* : M/\tau(M) \to N/\tau(N)$ such that $f_* p = q f$.
2. If $f$ is $w$-injective, then $f_*$ is injective, and if $f$ is $w$-isomorphic, then so is $f_*$. 
3. If $f$ is $w$-isomorphic and $M$ is semi-divisorial, then $f_*$ is an isomorphism.

**Proof.** 
\begin{itemize}
    \item (1) The existence of $f_*$ follows from [7, Proposition 2.8] and its uniqueness is clear.
    \item (2) Suppose first that $f$ is $w$-injective. Since $\tau(M) \subseteq f^{-1}(\tau(N))$, we have the following exact sequence 
        \[0 \to \ker(f) \to f^{-1}(\tau(N)) \to \tau(N).\] 
        This implies, by Lemma 3.1, that $f^{-1}(\tau(N))$ is $w$-null; therefore $\tau(M) = f^{-1}(\tau(N))$. Thus $f_*$ must be injective. If, moreover, $f$ is $w$-surjective, then $\ker(f)$ is $w$-null. Since the induced homomorphism of $\ker(f)$ to $\ker(f_*)$ is surjective, $\ker(f_*)$ must be $w$-null.
    \item (3) Suppose that $M$ is semi-divisorial. Then $M \cong \tau(M) \oplus M/\tau(M)$ by [7, Corollary 8.9], and hence $M/\tau(M)$ is also semi-divisorial. Now the assertion follows from [7, Corollary 5.3].
\end{itemize} 

It was shown in [16, Proposition 2.1] that $\text{Hom}_R(M, N) = \text{Hom}_R(M_w, N)$ for a torsion-free $R$-module $M$ and a $w$-module $N$. It follows from this result that $w$, as a functor from the category of all torsion-free $R$-modules to the category of all $w$-modules, is a reflector. The following result shows that the functor $W$ is a reflector from the category $\mathcal{F}_r(R)$ to the category $\mathcal{C}_r(R)$. By the $R$-dual of an $R$-module $M$ is meant the $R$-module $M^* = \text{Hom}_R(R, M)$.

**Proposition 3.5.** Let $R$ be an integral domain and let $M, N$ be $R$-modules. Let $i$ be the canonical injection of $M$ to $W(M)$. If $N$ is co-semi-divisorial, then 
\[\text{Hom}_R(i, W(N)) : \text{Hom}_R(W(M), W(N)) \to \text{Hom}_R(M, W(N))\] 
is an isomorphism. In particular, we have $M^* = (W(M))^*$.

**Proof.** Since $N$ is co-semi-divisorial, so is $W(N)$ by [7, Proposition 2.9]. On the other hand, $W(M)/M$ is $w$-null by the definition of a semi-divisorial envelope $W$. Therefore $\text{Hom}(W(M)/M, W(N)) = 0$, which implies that $\text{Hom}_R(i, W(N))$ is an injection. By [7, Proposition 3.2], we can see that $\text{Hom}_R(i, W(N))$ is a surjection.
Corollary 3.6. Let $R$ be a pseudo-$t$-linkative domain with quotient field $K(\neq R)$. Let $f : M \to N$ be a homomorphism of $R$-modules. Then there exists a unique homomorphism $T(f) : T(M) \to T(N)$ such that $T(f)i = jf$, where $i$ (resp., $j$) is the canonical homomorphism of $M$ (resp., $N$) into $T(M)$ (resp., $T(N)$). Moreover, if $f$ is a $w$-isomorphism, then $T(f)$ is an isomorphism.

Proof. The homomorphism $f$ induces the homomorphism $f^*_w$ of $M/\tau(M)$ to $N/\tau(N)$ by Proposition 3.4. Applying Proposition 3.5 to $f^*_w$, we can obtain a homomorphism $T(f) : T(M) \to T(N)$ such that $T(f)i = jf$.

It is easy to show that, similarly to the proof of Proposition 3.5, $\text{Hom}(i; T(N))$ is an injection. This shows the uniqueness of $T(f)$.

Suppose now that $f$ is a $w$-isomorphism. Then by Proposition 3.4, $f^*_w$ is a $w$-isomorphism ($f^*_w$ is necessarily injective). Since the canonical injection of $M/\tau(M)$ to $T(M)$ is an essential extension, $T(f)$ must be an injection. Since both $f^*_w$ and the canonical injection of $N/\tau(N)$ to $T(N)$ are $w$-surjective, so is the composition of them by Lemma 3.2. We can conclude from this fact that $T(f)$ is a $w$-surjection. Since a $w$-isomorphism of co-semi-divisorial and semi-divisorial modules is an isomorphism by [7, Corollary 5.3], $T(f)$ must be an isomorphism.

It was shown in [16, Proposition 2.3] that $(\text{Hom}_R(M, N))_w = \text{Hom}_R(M, N_w)$ for a torsion-free finitely generated $R$-module $M$ and a torsion-free $R$-module $N$. As a corollary, Wang obtained that $(\text{End}_R(M))_w = \text{End}_R(M_w)$ for a torsion-free finitely generated $R$-module $M$ ([16, Corollary 2.4]).

Theorem 3.7. Let $R$ be a pseudo-$t$-linkative domain. Let $M$ and $N$ be co-semi-divisorial $R$-modules. If $M$ is a submodule of a finitely generated $R$-module $L$, then we have

$$W(\text{Hom}_R(M, N)) \cong \text{Hom}_R(W(M), W(N)).$$

Proof. By Proposition 3.5, we have only to prove

$$W(\text{Hom}_R(M, N)) \cong \text{Hom}_R(M, W(N)).$$

Consider the following exact sequence

$$0 \to \text{Hom}_R(M, N) \to \text{Hom}_R(M, W(N)) \to \text{Hom}_R(M, W(N)/N).$$

Since $N$ is co-semi-divisorial, so is $W(N)$; thus, by Proposition 2.3, $\text{Hom}_R(M, N)$ and $\text{Hom}_R(M, W(N))$ are co-semi-divisorial. Also we have that $\text{Hom}_R(M, W(N))$ is semi-divisorial by Theorem 2.6. Since a $w$-isomorphism of co-semi-divisorial modules is an essential extension, it suffices to show that $\text{Hom}_R(M, W(N)/N)$ is $w$-null.

In general, for a submodule $M_1$ of a finitely generated $R$-module $M_2$ and a $w$-null $R$-module $N_1$, we will show that $\text{Hom}_R(M_1, N_1)$ is $w$-null. Set $N_2 := E(N_1)$. Then $N_2$ is $w$-null by [7, Theorem 8.1], since $R$ is pseudo-$t$-linkative. Let $\{x_1, \ldots, x_n\}$ be a system of generators of $M_2$ and let $f \in \text{Hom}_R(M_2, N_2)$. Then $O(f) = O(f(x_1)) \cap \cdots \cap O(f(x_n))$. Since each $(O(f(x_i)))_w = R$, we
have $(\mathcal{O}(f))_w = R$ by the distributivity of the star-operation $w$ over finite intersection. Hence $\text{Hom}_R(M_2, N_2)$ is $w$-null. Therefore, $\text{Hom}_R(M_1, N_2)$ is $w$-null, since it is a homomorphic image of $\text{Hom}_R(M_2, N_2)$. Thus $\text{Hom}_R(M_1, N_1)$ is $w$-null since it is isomorphic to a submodule of $\text{Hom}_R(M_1, N_2)$.

\textbf{Corollary 3.8.} Let $R$ be a pseudo-t-linkative domain with quotient field $K(\neq R)$ and let $M$ and $N$ be co-semi-divisorial and semi-divisorial $R$-modules. If $M$ is a submodule of a finitely generated $R$-module, then $\text{Hom}_R(M, N)$ is semi-divisorial.

Let $M$ and $N$ be an $R$-modules. We say that $M$ is semi-divisorially equivalent to $N$ if there exists a $w$-isomorphism of $W(M)$ to $W(N)$.

\textbf{Proposition 3.9.} Let $R$ be a pseudo-t-linkative domain with quotient field $K(\neq R)$. Let $M$ and $N$ be $R$-modules.

(1) $M$ is semi-divisorially equivalent to $N$ if and only if $W(M/\tau(M))$ is isomorphic to $W(N/\tau(N))$. In particular, the “semi-divisorial equivalence” is an equivalence relation.

(2) If $M$ is $w$-isomorphic to $N$, then $M$ is semi-divisorially equivalent to $N$.

\textbf{Proof.} (1) The necessity follows from the facts that $W(M) \cong W(\tau(M)) \oplus W(M/\tau(M))$ and $W(N) \cong W(\tau(N)) \oplus W(N/\tau(N))$ by [7, Corollary 8.9] and $W(\tau(M))$ and $W(\tau(N))$ are $w$-null by [7, Theorem 8.1] since $R$ is pseudo-t-linkative. The sufficiency follows from Proposition 3.4.

(2) The assertion follows immediately from Corollary 3.6.

\section{4. Injective cogenerator of a quotient category}

In this section, we generalize some results of [1, 11] related to an injective cogenerator in a quotient category. We recall from [4] that a domain $R$ is said to be an $H$-domain if every ideal $I$ of $R$ with $I^{-1} = R$ is quasi-finite (i.e. $I^{-1} = J^{-1}$ for some finitely generated subideal $J$ of $I$).

\textbf{Theorem 4.1.} Let $R$ be an $H$-domain with quotient field $K(\neq R)$, and let $M$ be any $R$-module. Then $M$ is $w$-null if and only if $\text{Hom}_R(M, E(K/R)) = 0$.

\textbf{Proof.} ($\Rightarrow$): This follows from Proposition 2.1 since $E(K/R)$ is co-semi-divisorial by [7, Corollary 2.11].

($\Leftarrow$): Suppose that $M$ is not $w$-null and let $N = M/\tau(M)$. By Proposition 2.1 and [7, Proposition 2.8], there is a non-zero element of $x \in N$ such that $\mathcal{O}(x)$ is a proper $w$-ideal and hence $R : \mathcal{O}(x) \not\subseteq R$ (since $R$ is an $H$-domain). Let $a \in R : \mathcal{O}(x) \setminus R$. Then $R : R a \supset \mathcal{O}(x)$. Let $f : R \to K/R$ be the homomorphism defined by $f(b) = ab$, where $ab$ is the class of $ab$ in $K/R$. Since $\ker(f) = R : R a \supset \mathcal{O}(x)$, there is a non-zero homomorphism $g : R/\mathcal{O}(x) \to K/R$ such that $f = gp$, where $p$ is the canonical projection of $R$ to $R/\mathcal{O}(x)$. Let $i$ be the canonical injection of $R/\mathcal{O}(x)(\cong Rx)$ to $N$. Then there is a non-zero homomorphism $h$ of $N$ to $E(K/R)$ such that $ih = hj$, and hence $hq$ is a
non-zero homomorphism of $M$ to $E(K/R)$, where $q$ is the canonical projection of $M$ to $N$.

Since $K/R \in \mathcal{F}(R)$, i.e., $K/R$ has no subobject other than zero belonging to $\mathcal{F}(R)$, then $T(E(K/R))$ is the injective envelope of the object $T(K/R)$ of $\text{Mod}(R)/\mathcal{I}(R)$.

**Corollary 4.2.** If $R$ is an $H$-domain, then $T(E(K/R))$ is an injective cogenerator in the quotient category $\text{Mod}(R)/\mathcal{I}(R)$. Hence every co-semi-divisorial and semi-divisorial module over an $H$-domain can be embedded in an injective module.

**Proof.** Let $T(N) \in \text{Mod}(R)/\mathcal{I}(R)$ with $\text{Hom}_{\text{Mod}(R)/\mathcal{I}(R)}(T(N), T(E(K/R))) = 0$. Then by [11, Lemma 2.6] we have $\text{Hom}_{\text{Mod}(R)}(N, E(K/R)) = 0$, and by Theorem 4.1 we have $N \in \mathcal{I}(R)$, and then $T(N) = 0$. It is clearly seen that $T(E(K/R))$ is a cogenerator object of $\text{Mod}(R)/\mathcal{I}(R)$. The last assertion follows from [14, Proposition I.6.6].

**Lemma 4.3.** Let $R$ be an integral domain, let $P \in w\text{-Max}(R)$, let $M$ be a co-semi-divisorial $R$-module and let $f : R/P \to M$ a homomorphism. Then either $f \equiv 0$ or $f$ is injective.

**Proof.** Suppose that $f \neq 0$ and let $f(1) = x$. Then we have $x \in M$. Since $M$ is co-semi-divisorial, then $O(x)$ is a $w$-ideal, and since $x \neq 0$, there exists $Q \in w\text{-Max}(R)$ such that $O(x) \subseteq Q$, but since $P \subseteq O(x)$, we have $P \subseteq Q$ and hence $P = Q$, so $O(x) = P$ and $f$ is injective.

We recall from [10, III.1.4] two facts related to $\mathcal{C}(R)$, $\text{Mod}(R)/\mathcal{I}(R)$, and $T$.

(a) The subcategory $\mathcal{C}(R)$ of $\text{Mod}(R)$ may be identified with $\text{Mod}(R)/\mathcal{I}(R)$.

(b) Let $M$ be an $R$-module. Then $T(M) = W(M/\tau(M))$.

Therefore, we have that $T(E(K/R)) = W(E(K/R)/\tau(E(K/R))) \cong E(K/R)$.

**Theorem 4.4.** Let $R$ be an integral domain with quotient field $K$ satisfying $(R :_R x)_v = (R :_R x)$ for every $x \in K$. If $T(E(K/R))$ is an injective cogenerator in the quotient category $\text{Mod}(R)/\mathcal{I}(R)$, then $R$ is an $H$-domain.

**Proof.** Note that if $R$ satisfies that $(R :_R x)_v = (R :_R x)$ for every $x \in K$, then $K/R$ is co-divisorial. Suppose that $R$ is not an $H$-domain. Then by [17, Proposition 5.7] there exists a prime ideal $P$ which is $w$-maximal but not a $v$-ideal. First we show that the module $R/P$ cannot be injected in $E(K/R)$. If this were not so, then the kernel of the composition $R \xrightarrow{\Pi} R/P \to E(R/K)$ is $P$, where $\Pi$ is the canonical projection. Then by [1, Corollary 1.7] $P$ is a $v$-ideal, which is a contradiction. Thus by Lemma 4.3, $\text{Hom}_{\text{Mod}(R)}(R/P, E(K/R)) \neq 0$. So $\text{Hom}_{\text{Mod}(R)/\mathcal{I}(R)}(T(R/P), T(E(K/R))) \cong \text{Hom}_{\text{Mod}(R)}(W(R/P), E(K/R)) \cong \text{Hom}_{\text{Mod}(R)}(R/P, E(K/R)) = 0$ (note that the last isomorphism follows from Proposition 3.5). Since $T(E(K/R))$ is a cogenerator object in $\text{Mod}(R)/\mathcal{I}(R)$,
$T(R/P) = 0$, and thus $R/P \in \mathcal{T}_\mathcal{R}(R)$, i.e., $R/P$ is $w$-null. Hence $P_w = R$, which is a contradiction. Therefore $R$ is an $H$-domain.

It is well known that if $R$ is a completely integrally closed domain, then $R$ satisfies the hypothesis of Theorem 4.4. Now the following result follows from Corollary 4.2, Theorem 4.4, and the fact that an integral domain $R$ is a Krull domain if and only if $R$ is a completely integrally closed $H$-domain ([4, 3.2(d)]).

**Corollary 4.5.** Let $R$ be a completely integrally closed domain. Then $R$ is a Krull domain if and only if $E(K/R)$ is an injective cogenerator in the quotient category $\text{Mod}(R)/\mathcal{T}_\mathcal{R}(R)$.

Let $M$ be any $R$-module. We have a canonical mapping:

$$\lambda_M : M \to \text{Hom}_R(\text{Hom}_R(M, E(K/R)), E(K/R)).$$

Let $f \in \text{Hom}_R(M, E(K/R))$. Then define $\lambda_M(m)$ by the equation $\lambda_M(m)(f) = f(m)$ for all $m \in M$.

**Theorem 4.6.** Let $R$ be an $H$-domain with quotient field $K(\neq R)$, and let $M$ be any $R$-module. Then $M$ is co-semi-divisorial if and only if $\lambda_M$ is injective.

**Proof.** $(\Leftarrow)$: This follows from the facts that $E(K/R)$ is co-semi-divisorial and $\text{Hom}_R(L, N)$ is co-semi-divisorial whenever $N$ is co-divisorial.

$(\Rightarrow)$: Let $x \in M \setminus \{0\}$. Since $Rx$ is not $w$-null, we can find a homomorphism $f : Rx \to E(K/R)$ such that $f(x) \neq 0$ by Theorem 4.1. Since $E(K/R)$ is injective, we can lift $f$ to a mapping $\bar{f} : M \to E(K/R)$. This shows that $\lambda_M$ is injective, since $\lambda_M(x)(\bar{f}) = f(x) = f(x) \neq 0$ and hence $\lambda_M(x) \neq 0$.

**References**


Department of Information Security
Hoseo University
Asan 336-795, Korea
E-mail address: hkkim@hoseo.edu