ON HYPONORMALITY OF TOEPLITZ OPERATORS WITH POLYNOMIAL AND SYMMETRIC TYPE SYMBOLS

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ABSTRACT. In [6], it was shown that hyponormality for Toeplitz operators with polynomial symbols can be reduced to classical Schur’s algorithm in function theory. In [6], Zhu has also given the explicit values of the Schur’s functions \( \Phi_0, \Phi_1, \) and \( \Phi_2. \) Here we explicitly evaluate the Schur’s function \( \Phi_3. \) Using this value we find necessary and sufficient conditions under which the Toeplitz operator \( T_\varphi \) is hyponormal, where \( \varphi \) is a trigonometric polynomial given by \( \varphi(z) = \sum_{n=-N}^{N} a_n z^n \) \((N \geq 4)\) and satisfies the condition \( \begin{pmatrix} a_{-1} \\ a_{-4} \\ \vdots \\ a_{-N} \end{pmatrix} = a_{-N} \begin{pmatrix} a_1 \\ a_4 \\ \vdots \\ a_N \end{pmatrix}. \) Finally we illustrate the easy applicability of the derived results with a few examples.

1. Introduction

A bounded linear operator \( T \) on a Hilbert space is said to be hyponormal if its self commutator \([T^*, T] := T^*T - TT^*\) is positive semi definite. Given \( \varphi \in L^\infty(T), \) the Toeplitz operator with symbol \( \varphi \) is the operator \( T_\varphi \) on the Hardy space \( H^2(T) \) of the unit circle \( T = \partial \mathbb{D} \) defined by \( T_\varphi f := P(\varphi f), \) where \( f \in H^2(T) \) and \( P \) denotes the orthogonal projection that maps \( L^2(T) \) onto \( H^2(T). \)

We are interested in the following question: “if \( \varphi \) is a trigonometric polynomial, then when is the trigonometric Toeplitz operator \( T_\varphi \) hyponormal?” If \( \varphi \) is a trigonometric polynomial of the form \( \varphi(z) = \sum_{n=-m}^{N} a_n z^n \) where \( a_{-m} \) and \( a_N \) are non zero, then it was shown in [4] and [2] that hyponormality of \( T_\varphi \) implies \( m \leq N \) and \( |a_{-m}| \leq |a_N|. \) In [2] it was shown that for \( \varphi(z) = \sum_{n=-m}^{N} a_n z^n, \) if \( |a_{-m}| = |a_N| \neq 0, \) then \( T_\varphi \) is hyponormal if and only...
if the coefficients of $\varphi$ satisfy the following ‘symmetry’ condition:

$$
\begin{pmatrix}
  a_{-1} \\
  a_{-2} \\
  \vdots \\
  a_{-m}
\end{pmatrix}
= a_{-m} 
\begin{pmatrix}
  \bar{a}_{N-m+1} \\
  \bar{a}_{N-m+2} \\
  \vdots \\
  \bar{a}_N
\end{pmatrix}.
$$

But if $|a_{-m}| \neq |a_N|$, then the case of arbitrary polynomial $\varphi$, though solved in principle by Cowen’s Theorem [1] or Zhu’s Theorem [6], is in practice very complicated. In [5], the hyponormality of $T_\varphi$ was studied when $\varphi(z) = \sum_{n=-N}^N a_n z^n$ satisfies a ‘partial’ symmetry condition:

$$
\begin{pmatrix}
  a_{-m} \\
  a_{-(m+1)} \\
  \vdots \\
  a_{-N}
\end{pmatrix}
= a_{-N} 
\begin{pmatrix}
  \bar{a}_m \\
  \bar{a}_{m+1} \\
  \vdots \\
  \bar{a}_N
\end{pmatrix}
$$

for $m \leq N/2$.

The authors gave in the paper a necessary and sufficient condition for the hyponormality of $T_\varphi$ when $\varphi$ satisfies (1) and has the property $|\bar{a}_N a_{-(m-1)} - a_{-N} \bar{a}_{m-1}| = |a_N|^2 - |a_{-N}|^2$. Further, in [3] we find a complete criterion for hyponormality of $T_\varphi$ when $\varphi$ satisfies (1).

In this paper, we consider the trigonometric polynomial $\varphi(z) = \sum_{n=-N}^N a_n z^n$ with $|a_{-N}| < |a_N|$ such that

$$
\begin{pmatrix}
  a_{-1} \\
  a_{-2} \\
  \vdots \\
  a_{-N}
\end{pmatrix}
= a_{-N} 
\begin{pmatrix}
  \bar{a}_1 \\
  \bar{a}_2 \\
  \vdots \\
  \bar{a}_N
\end{pmatrix}
$$

and and we give necessary and sufficient conditions for hyponormality of $T_\varphi$.

2. Evaluation of Schur’s function $\Phi_3$

In [6], Kehe Zhu applied Schur’s algorithm to the Schur’s function $\Phi_N$ in order to determine hyponormality of Toeplitz operators with polynomial symbols. We begin the section with the brief description of Zhu’s idea.

Suppose that $f(z) = \sum_{j=0}^\infty c_j z^j$ is in the closed unit ball of $H^\infty(\mathbb{T})$. If $f_0 = f$, define by induction a sequence $\{f_n\}$ of functions in the closed unit ball of $H^\infty(\mathbb{T})$ as follows:

$$
f_{n+1}(z) = \frac{f_n(z) - f_n(0)}{z(1 - f_n(0)f_n(z))}, \quad |z| < 1, \quad n = 0, 1, 2, \ldots
$$

We write $f_n(0) = \Phi_n(c_0, \ldots, c_n)$, $n = 0, 1, 2, \ldots$, where $\Phi_n$ is a function of $n + 1$ complex variables. We call the $\Phi_n$’s Schur’s functions. Zhu’s theorem can now be written as follows:
Theorem 2.1 ([6]). If \( \varphi(z) = \sum_{n=-N}^{N} a_n z^n \), where \( a_N \neq 0 \) and if
\[
\begin{pmatrix}
c_0 \\
c_1 \\
\vdots \\
c_{N-1}
\end{pmatrix} = \begin{pmatrix}
a_1 & a_2 & \cdots & a_{N-1} & a_N \\
a_2 & a_3 & \cdots & a_N & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
a_N & 0 & \cdots & 0 & 0
\end{pmatrix}^{-1} \begin{pmatrix}
\bar{a}_{-1} \\
\bar{a}_{-2} \\
\vdots \\
\bar{a}_{-N}
\end{pmatrix},
\]
then \( T_\varphi \) is hyponormal if and only if \( |\Phi_n(c_0, \ldots, c_n)| \leq 1 \) for each \( n = 0, 1, \ldots, N - 1 \).

Note that each \( \Phi_n(c_0, \ldots, c_n) \) is a rational function of the form
\[
\Phi_n(c_0, \ldots, c_n) = \frac{F_n(c_0, \ldots, c_n)}{G_n(c_0, \ldots, c_n)},
\]
where \( F_n \) and \( G_n \) are polynomials. Thus the inequalities \( |\Phi_n(c_0, \ldots, c_n)| \leq 1 \) should be understood as \( |F_n(c_0, \ldots, c_n)| \leq |G_n(c_0, \ldots, c_n)| \).

In [6] Zhu also listed the first three Schur’s functions as follows:
\[
\begin{align*}
\Phi_0(c_0) &= c_0, \\
\Phi_1(c_0, c_1) &= \frac{c_1}{1 - |c_0|^2}, \\
\Phi_2(c_0, c_1, c_2) &= \frac{c_2(1 - |c_0|^2) + c_0 c_1^2}{(1 - |c_0|^2)^2 - |c_1|^2}.
\end{align*}
\]

Here in this section we evaluate \( \Phi_3(c_0, c_1, c_2, c_3) \) using the Schur’s algorithm, and we get,
\[
\Phi_3(c_0, c_1, c_2, c_3) = \frac{(1 - |c_0|^2)^2 - |c_1|^2)((1 - |c_0|^2)c_2 + \bar{c}_0 c_1 c_2 + c_2(1 - |c_0|^2) + \bar{c}_0 c_1^2)(\bar{c}_0(1 - |c_0|^2)c_1 + c_1 c_3)}{(1 - |c_0|^2)^2 - |c_1|^2)^2 - |c_2|^2((1 - |c_0|^2)^2 - |c_1|^2)^2 - |c_2|^2(1 - |c_0|^2) + c_0 c_1^2)^2}.
\]

We briefly outline the steps leading to this result. For the function \( f(z) = \sum_{j=0}^{\infty} c_j z^j \) in the closed unit ball of \( H^\infty(\mathbb{T}) \):

Step(1): \( f_0(z) = f(z) \) and so \( \Phi_0(c_0) = f_0(0) = f(0) = c_0 \).

Step(2): \( f_1(z) = \frac{f_0(z) - f_0(0)}{z(1 - f_0(0)f_0(z))} = \frac{\sum_{j=1}^{\infty} c_j z^{j-1}}{1 - |c_0|^2 - c_0 \sum_{j=1}^{\infty} c_j z^j} \) and so \( \Phi_1(c_0, c_1) = f_1(0) = \frac{c_1}{1 - |c_0|^2} \).

Step(3): \( f_2(z) = \frac{f_1(z) - f_1(0)}{z(1 - f_1(0)f_1(z))} = \frac{N}{zD} \), where
\[
N = \frac{(1 - |c_0|^2)^2 - |c_1|^2 - \bar{c}_0 c_1^2 \sum_{k=0}^{\infty} c_k z^k + c_0 c_1 \sum_{k=0}^{\infty} c_k z^k}{(1 - |c_0|^2)^2 - |c_1|^2 - \bar{c}_0 c_1^2 \sum_{k=0}^{\infty} c_k z^k + c_0 c_1 \sum_{k=0}^{\infty} c_k z^k},
\]
\[
D = \frac{(1 - |c_0|^2)^2 - |c_1|^2 - c_0(1 - |c_0|^2) \sum_{k=1}^{\infty} c_k z^k - \bar{c}_0 c_1 \sum_{k=2}^{\infty} c_k z^{k-1}}{((1 - |c_0|^2)^2 - |c_1|^2 - c_0(1 - |c_0|^2) \sum_{k=0}^{\infty} c_k z^k - \bar{c}_0 c_1 \sum_{k=0}^{\infty} c_k z^k)^2}.
\]

Therefore,
\[
f_2(z) = \frac{(1 - |c_0|^2)^2 \sum_{k=2}^{\infty} c_k z^{k-2} + \bar{c}_0 c_1 \sum_{k=1}^{\infty} c_k z^{k-1}}{(1 - |c_0|^2)^2 - |c_1|^2 - c_0(1 - |c_0|^2) \sum_{k=1}^{\infty} c_k z^k - \bar{c}_0 c_1 \sum_{k=2}^{\infty} c_k z^{k-1}}.
\]
and

$$\Phi_2(c_0, c_1, c_2) = f_2(0) = \frac{c_2(1 - |c_0|^2) + c_0 c_1^2}{(1 - |c_0|^2)^2 - |c_1|^2}.$$  

Step (4):

$$f_3(z) = \frac{f_2(z) - f_2(0)}{z(1 - f_2(0)f_2(z))} = \frac{N}{D},$$

where

$$N = ((1 - |c_0|^2)^2 - |c_1|^2) \left( (1 - |c_0|^2) \sum_{k=2}^{\infty} c_k z^{k-2} + c_0 c_1 \sum_{k=1}^{\infty} c_k z^{k-1} \right)$$

$$- (c_2(1 - |c_0|^2) + c_0 c_1^2) \left( (1 - |c_0|^2)^2 - |c_1|^2 \right) - c_0(1 - |c_0|^2) \sum_{k=1}^{\infty} c_k z^k - c_1 \sum_{k=2}^{\infty} c_k z^{k-1}$$

and

$$D = ((1 - |c_0|^2)^2 - |c_1|^2) \left( (1 - |c_0|^2)^2 - |c_1|^2 \right) - c_0(1 - |c_0|^2) \sum_{k=1}^{\infty} c_k z^k - c_1 \sum_{k=2}^{\infty} c_k z^{k-1}.$$  

Thus,

$$\Phi_3(c_0, c_1, c_2, c_3) = f_3(0) = \frac{((1 - |c_0|^2)^2 - |c_1|^2)((1 - |c_0|^2)c_3 + c_0 c_1 c_2) + (c_2(1 - |c_0|^2) + c_0 c_1^2)(c_0(1 - |c_0|^2)c_1 + c_1 c_2)}{((1 - |c_0|^2)^2 - |c_1|^2)^2 - |c_2(1 - |c_0|^2) + c_0 c_1|^2}.$$  

### 3. Hyponormality condition

**Theorem 3.1.** Suppose \( \varphi(z) = \sum_{n=-N}^{N} a_n z^n \) is such that

$$\bar{a}_N \begin{pmatrix} a_{-1} \\ a_{-2} \\ \vdots \\ a_{-N} \end{pmatrix} = a_{-N} \begin{pmatrix} \bar{a}_1 \\ \bar{a}_2 \\ \vdots \\ \bar{a}_N \end{pmatrix} \quad \text{and} \quad \alpha := \frac{\det \begin{pmatrix} a_{-3} & a_{-N} \\ \bar{a}_3 & \bar{a}_N \end{pmatrix}}{|a_N|^2 - |a_{-N}|^2}.$$  

(1) If \( N = 4 \), then \( T_\varphi \) is hyponormal if and only if

(a) \( |\alpha| \leq 1 \),

(b) \( |\bar{a}_{-4} - a_3| \leq |a_4| \left( \frac{1}{|\alpha|} - |\alpha| \right) \),

(c) \( |(1 - |\alpha|^2)(\bar{a}_3^2 - 2a_4 - \bar{a}_3\bar{a}_{-4}) + \alpha(\bar{a}_{-4}\alpha - \bar{a}_3)(\bar{a}_{-4} - \bar{a}_3\alpha) - \alpha(\bar{a}_{-4}\alpha - \bar{a}_3)(\bar{a}_{-4} - \bar{a}_3\alpha)| \leq |\alpha| \left( |a_4|^2 \left( \frac{1}{|\alpha|} - |\alpha| \right)^2 - |a_{-4}\alpha - a_3|^2 \right) \).

(2) If \( N \geq 5 \), then \( T_\varphi \) is hyponormal if and only if

(a) \( |\alpha| \leq 1 \),

(b) \( |\frac{a_{N-1}}{a_N}| \leq \frac{1}{|\alpha|} - |\alpha| \),
Using expressions (6) to (10) in (11), (12), (13) and (14) and simplifying, we get

\[(c) \quad \left| \frac{1}{\alpha} \left( \frac{a_{N-1}}{a_N} \right)^2 - \left( \frac{1}{\alpha} - \bar{\alpha} \right) \frac{a_{N-2}}{a_N} \right| \leq \left( \frac{1}{|\alpha|} - |\alpha| \right)^2 - \left( \frac{a_{N-1}}{a_N} \right)^2.\]

**Remark 3.2.** Here we assume that \( \det \left( \begin{array}{cc} a_3 & a_{N-1} \\ \bar{a}_3 & \bar{a}_{N-1} \end{array} \right) \) and \( |a_N|^2 - |a_{-N}|^2 \) are non-zero, because otherwise the results follow from [2] and [5]. Thus, \( a_{-3} \bar{a}_N \neq a_{-N} \bar{a}_3 \) and \( |a_{-N}| < |a_N| \).

**Proof.** (1) When \( N = 4 \).

Using Proposition 1 [5], \( c_0, c_1, c_2, c_3 \) are determined as follows:

\[(2) \quad c_0 = \frac{a_{-4}}{\bar{a}_4},\]
\[(3) \quad c_1 = (\bar{a}_4)^{-1}(a_{-3} - c_0 \bar{a}_3),\]
\[(4) \quad c_2 = (\bar{a}_4)^{-1}(a_{-2} - c_0 \bar{a}_2 - c_1 \bar{a}_3),\]
\[(5) \quad c_3 = (\bar{a}_4)^{-1}(a_{-1} - c_0 \bar{a}_1 - c_1 \bar{a}_2 - c_2 \bar{a}_3).\]

Calculating and simplifying we get

\[(6) \quad c_1 = (\bar{a}_4)^{-2} \left( |a_4|^2 - |a_{-4}|^2 \right) \alpha,\]
\[(7) \quad c_2 = (\bar{a}_4)^{-1} \left( a_{-2} - \frac{a_{-4}}{\bar{a}_4} \bar{a}_2 - c_1 \bar{a}_3 \right) = -\frac{c_1 \bar{a}_3}{\bar{a}_4},\]
\[(8) \quad c_3 = \left( \frac{\bar{a}_3^2 - \bar{a}_3 \bar{a}_4}{\bar{a}_4^2} \right) c_1,\]
\[(9) \quad 1 - |c_0|^2 = 1 - \frac{|a_{-4}|^2}{|a_4|^2} = |a_4|^2 \left( |a_4|^2 - |a_{-4}|^2 \right),\]
\[(10) \quad \frac{|c_1|}{1 - |c_0|^2} = |\alpha|.\]

We also have,

\[(11) \quad \Phi_0 = c_0,\]
\[(12) \quad \Phi_1 = \frac{c_1}{1 - |c_0|^2},\]
\[(13) \quad \Phi_2 = \frac{c_2(1 - |c_0|^2) + c_0 c_1^2}{(1 - |c_0|^2)^2 - |c_1|^2},\]
\[(14) \quad \Phi_3 = \frac{(1-|c_0|^2)^2 - |c_1|^2}{(1-|c_0|^2)^2 - |c_2|^2} \left( (1-|c_0|^2)^2 + c_0 c_2 \right).\]

Using expressions (6) to (10) in (11), (12), (13) and (14) and simplifying, we get

\[|\Phi_0| \leq 1 \text{ if and only if } |a_{-4}| \leq |a_4|,\]
\[|\Phi_1| \leq 1 \text{ if and only if } |\alpha| \leq 1,\]
\[|\Phi_2| \leq 1 \text{ if and only if } |\bar{a}_{-4} \alpha - \bar{a}_3| \leq |a_4| \left( \frac{1}{|\alpha|} - |\alpha| \right),\]
\[|\Phi_3| \leq 1,\]
Thus, \(|\Phi_3| \leq 1\) if and only if

\[
|(1 - |a|^2)(\bar{a}_2^2 - \bar{a}_2a_4 - a_3\bar{a}_{-4}) + \alpha(\bar{a}_{-4}\alpha - \bar{a}_3)(\bar{a}_{-4} - \bar{a}_3\bar{a}_{-4})|
\leq |\alpha| \left( |a|^2 \left( \frac{1}{|\alpha|} - |\alpha| \right)^2 - |\bar{a}_{-4}\alpha - \bar{a}_3\bar{a}_{-4}|^2 \right).
\]

The result follows from Theorem 2.1.

(2) When \(N > 4\).

Let \(c_0, c_1, \ldots, c_{N-1}\) be determined by Proposition 1 [5]. Then,

\[
c_0 = \frac{a_{-N}}{\bar{a}_N},
\]

\[
c_1 = (\bar{a}_N)^{-1}(a_{-(N-1)} - c_0\bar{a}_{N-1}) = (\bar{a}_N)^{-2}(\bar{a}_N a_{-(N-1)} - a_{-N}a_{N-1}) = 0,
\]

\[
\vdots
\]

\[
c_{N-4} = (\bar{a}_N)^{-1}(a_{-4} - c_0\bar{a}_4 - c_1\bar{a}_5 - \ldots - c_{N-5}\bar{a}_{N-1}) = 0,
\]

\[
c_{N-3} = (\bar{a}_N)^{-1}(a_{-3} - c_0\bar{a}_3 - c_1\bar{a}_4 - \ldots - c_{N-4}\bar{a}_{N-1})
\]

\[
= (\bar{a}_N)^{-2}(a_{-3}\bar{a}_N - a_{-3}a_{N-1}),
\]

\[
c_{N-2} = (\bar{a}_N)^{-1}(a_{-2} - c_0\bar{a}_2 - c_1\bar{a}_3 - c_2\bar{a}_4 - \ldots - c_{N-3}\bar{a}_{N-1}),
\]

\[
= - \left( \frac{\bar{a}_{N-1}}{\bar{a}_N} \right) c_{N-3},
\]

\[
c_{N-1} = (\bar{a}_N)^{-1}(a_{-1} - c_0\bar{a}_1 - c_1\bar{a}_2 - \ldots - c_{N-2}\bar{a}_{N-1} - c_{N-2}a_{N-1})
\]

\[
= (\bar{a}_N)^{-1}(-c_{N-1}\bar{a}_{N-2} + \frac{\bar{a}_{N-1}}{\bar{a}_N}c_{N-3}\bar{a}_{N-1})
\]

\[
= \left( \frac{\bar{a}_{N-1}}{\bar{a}_N} \right)^2 - \frac{\bar{a}_{N-2}}{\bar{a}_N} c_{N-3}.
\]

Thus, \(b_\varphi(z) = c_0 + c_{N-3}z^{N-3} + c_{N-2}z^{N-2} + c_{N-1}z^{N-1}\) is the unique analytic polynomial of degree less than \(N\) satisfying \(\varphi - b_\varphi \varphi \in H^\infty\). By our assumption we have \(c_{N-3} \neq 0\). Thus, by Proposition 3 [5],

\[
\Phi_0 = c_0,
\]

\[
\Phi_1 = \Phi_2 = \ldots = \Phi_{N-4} = 0,
\]

\[
\Phi_{N-3} = \frac{c_{N-3}}{1 - |c_0|^2},
\]

\[
\Phi_{N-2} = \frac{c_{N-2}}{(1 - |c_0|^2)(1 - |\Phi_{N-3}|^2)},
\]

\[
\Phi_{N-1} = \frac{(1 - |\Phi_{N-3}|^2)c_{N-1}c_{N-3} + |\Phi_{N-3}|^2c_{N-2}^2}{c_{N-3}(1 - |c_0|^2)(1 - |\Phi_{N-3}|^2)(1 - |\Phi_{N-2}|^2)}.
\]

Computing and simplifying from these relations, we get,
\[ |\Phi| \leq 1 \] if and only if \[ |a_{-N}| \leq |a_N|, \] which is always true since we are taking \[ |a_{-N}| < |a_N|, \]
\[ |\Phi_{N-3}| \leq 1 \] if and only if \[ |\alpha| \leq 1, \]
\[ |\Phi_{N-2}| \leq 1 \] if and only if \[ \left| \frac{a_{N-1}}{a_N} \right| \leq \frac{1}{|\alpha|} - |\alpha|, \]
\[ |\Phi_{N-1}| \leq 1 \] if and only if \[ \left| \frac{a_{N-1}}{a_N} \right|^2 - \left( \frac{1}{|\alpha|} - |\alpha| \right) \left| \frac{a_{N-2}}{a_N} \right| \leq \left( \frac{1}{|\alpha|} - |\alpha| \right)^2 - \frac{|a_{N-1}|}{a_N} \]
\[ \left( \frac{a_{N-1}}{a_N} \right)^2. \]

The result follows from Theorem 2.1.

\[ \square \]

4. Examples

Example 1. For \( \varphi(z) = z^{-4} + \lambda z^{-3} + 2z^{-1} + 4z^3 + 2z^6 \) we want to determine the values of \( \lambda \) for which the Toeplitz operator \( T_{\varphi} \) is hyponormal.

By Proposition 2 [5], \( T_{\varphi} \) is hyponormal if and only if \( T_{\psi} \) is hyponormal where \( \psi(z) = z^{-4} + \lambda z^{-3} + 2z^{-1} + 4z + 2z^4 \). Expressing \( \psi(z) \) as \( \sum_{n=-N}^{N} a_n z^n \) and comparing with the given expression, we have, \( N = 4 \) and also \( a_4 \left( \frac{a_{-1}}{a_4} \right) = a_{-4} \left( \frac{a_1}{a_4} \right) \). Hence, we can apply Case 1 of Theorem 3.1 to determine the values of \( \lambda \) for which \( T_{\psi} \) will be hyponormal.

It may be noted that if \( \lambda = 0 \), then \( T_{\psi} \) is hyponormal by Theorem 1.4 [2].

As \( \alpha = |a_3 a_{-4}| / (|a_4|^2 - |a_{-4}|^2) = \frac{2a_3}{3} \), thus referring to the notations used in Theorem 3.1 we have,

(i) \( |\alpha| \leq 1 \) if and only if \( |\lambda| \leq \frac{2}{3} = 1.5, \)

(ii) \( |\alpha a_{-4} - a_3| \leq |a_4| \left( \frac{1}{|\alpha|} - |\alpha| \right) \) if and only if \( |\lambda| \leq \sqrt{\frac{2}{3}} = 1.22 \), correct upto two decimal places,

(iii) \[ |(1 - |\alpha|^2)(a_3^2 - \bar{a}_2 a_4 - \bar{a}_3 a_{-4} \alpha) + \alpha(\bar{a}_{-4} \alpha - \bar{a}_3)(\alpha a_{-4} - a_3 \bar{\alpha})| \]
\[ \leq |\alpha|^2 \left( \frac{1}{|\alpha|} - |\alpha| \right)^2 - |a_{-4} \alpha - a_3|^2 \]

if and only if \( 12|\lambda|^4 - 6|\lambda|^3 - 72|\lambda|^2 + 81 \geq 0 \)

if and only if \( |\lambda| \leq 1.13. \)

This can be seen from the graph of \( 12|\lambda|^4 - 6|\lambda|^3 - 72|\lambda|^2 + 81 = 0 \) in Figure 1:

Thus, \( T_{\varphi} \) is hyponormal if and only if \( |\lambda| \leq 1.13 \) (correct upto two decimal places).

Example 2. If \( \varphi(z) = z^{-4} + \lambda z^{-3} + z^{-2} + 2z^{-1} + 1 + 4z + 2z^2 + 2z^4 \), then using Theorem 3.1 (Case 1) as in Example 1, we get that the Toeplitz operator \( T_{\varphi} \) is hyponormal if and only if \( |\lambda| \leq 0.95 \) (correct upto two decimal places).
Example 3. If $\phi(z) = 2z^{-5} + 2z^{-4} + \lambda z^{-3} + 4z^{-2} + 2z^{-1} + 3z + 6z^2 + 3z^3 + 3z^4 + 3z^5$, then using Theorem 3.1 (Case 2) we get that the Toeplitz operator $T_{\phi}$ is hyponormal if and only if $0.34 \leq \lambda \leq 0.58$ (correct up to two decimal places).

Example 4. If $\phi(z) = z^{-6} + 3z^{-5} + z^{-4} + 2z^{-2} + z^{-1} + 2z + 4z^2 + \lambda z^3 + 2z^4 + 6z^5 + 2z^6$, then using Theorem 3.1 (Case 2) we get that the Toeplitz operator $T_{\phi}$ is hyponormal if and only if $|\lambda| \leq 0.32$ (correct up to two decimal places).

5. Conclusion

If for $N \geq 4$ and $\phi(z) = \sum_{n=-N}^{N} a_n z^n$ with $|a_N| > |a_{-N}|$, we have

$$\bar{\alpha}_N \begin{pmatrix} a_{-1} \\ a_{-2} \\ a_{-4} \\ \vdots \\ a_{-N} \end{pmatrix} = a_{-N} \begin{pmatrix} \bar{\alpha}_1 \\ \bar{\alpha}_2 \\ \bar{\alpha}_4 \\ \vdots \\ \bar{\alpha}_N \end{pmatrix}$$

and $\bar{\alpha}_N a_{-3} \neq a_{-N} \bar{\alpha}_3$.

Then for $N \geq 6$ we can also apply Theorem 1 [3]. Particularly, for very large $N$, application of Theorem 1 [3] would significantly reduce the workload in determining the hyponormality of $T_{\phi}$. However, this theorem does not give a ready set of conditions to finally determine hyponormality. In view of this, Theorem 3.1 offers an alternative method to determine hyponormality of $T_{\phi}$ under the given restrictions on $\phi$.

For $N \geq 3$, if

$$\bar{\alpha}_N \begin{pmatrix} a_{-1} \\ a_{-3} \\ \vdots \\ a_{-N} \end{pmatrix} = a_{-N} \begin{pmatrix} \bar{\alpha}_1 \\ \bar{\alpha}_3 \\ \vdots \\ \bar{\alpha}_N \end{pmatrix}$$

and $\bar{\alpha}_N a_{-2} \neq a_{-N} \bar{\alpha}_2$,

then Theorem 8 [5] gives a similar alternative method to determine hyponormality of $T_{\phi}$.
References


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