A NOTE ON HYPONORMAL TOEPLITZ OPERATORS

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Abstract. In this note we are concerned with the hyponormality of Toeplitz operators $T_\phi$ with polynomial symbols $\phi = g + f$ ($f, g \in H^\infty(\mathbb{T})$) when $g$ divides $f$.

1. Introduction

A bounded linear operator $A$ on a complex Hilbert space is said to be hyponormal if its selfcommutator $[A^*, A] = A^*A - AA^*$ is positive semidefinite. Recall that the Hilbert space $L^2(\mathbb{T})$ has a canonical orthonormal basis given by the trigonometric functions $e_n(z) = z^n$ for all $n \in \mathbb{Z}$, and that the Hardy space $H^2(\mathbb{T})$ is the closed linear span of $\{e_n : n = 0, 1, \ldots\}$. Recall that given $\phi \in L^\infty(\mathbb{T})$, the Toeplitz operator with symbol $\phi$ is the operator $T_\phi$ on $H^2(\mathbb{T})$ defined by

$$T_\phi f = P(\phi \cdot f) \quad \text{for} \quad f \in H^2(\mathbb{T})$$

and $P$ denotes the projection that maps $L^2(\mathbb{T})$ onto $H^2(\mathbb{T})$. Let $H^\infty(\mathbb{T}) := L^\infty(\mathbb{T}) \cap H^2(\mathbb{T})$. The hyponormality of Toeplitz operators has been studied by many authors (cf. [1-7, 9-16, 18]). In 1988, C. Cowen [3] characterized the hyponormality of a Toeplitz operator $T_\phi$ on $H^2(\mathbb{T})$ by properties of the symbol $\phi \in L^\infty(\mathbb{T})$. K. Zhu [18] reformulated Cowen’s criterion and then showed that the hyponormality of $T_\phi$ with polynomial symbols $\phi$ can be decided by a method based on the classical interpolation theorem of I. Schur [17]. Also D. Farenick and W. Y. Lee [6] characterized the hyponormality of $T_\phi$ in terms of the Fourier coefficients of the trigonometric polynomial $\phi$ in the cases that the outer coefficients of $\phi$ have the same modulus. In [12], it was shown that the hyponormality of $T_\phi$ with polynomial symbols of the form $\phi(z) = \sum_{n=-m}^{N} a_n z^n$ can be determined by the zeros of the analytic polynomial $z^m \phi$. In this note we consider the hyponormality of Toeplitz operators $T_\phi$ with polynomial symbols $\phi = g + f$ ($f, g \in H^\infty(\mathbb{T})$) when $g$ divides $f$. 

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2. Main results

We begin with:

**Lemma 1** (Cowen’s Theorem, [3, 16]). Let $\phi \in L^\infty(\mathbb{T})$. If $\mathcal{E}(\phi) := \{k \in H^\infty(\mathbb{T}) : ||k||_{\infty} \leq 1 \text{ and } \phi - k \overline{\phi} \in H^\infty(\mathbb{T})\}$, then $T_\phi$ is hyponormal if and only if $\mathcal{E}(\phi) \neq \emptyset$.

On the other hand, in 1993, T. Nakazi and K. Takahashi characterized the hyponormality of a Toeplitz operator in the cases that its self-commutator is of finite rank.

**Lemma 2** (Nakazi-Takahashi Theorem, [16]). A Toeplitz operator $T_\phi$ is hyponormal and the rank of the selfcommutator $[T_\phi^*, T_\phi]$ is finite (e.g., $\phi$ is a trigonometric polynomial) if and only if there exists a finite Blaschke product $k \in E(\phi)$ such that $\deg(k) = \text{rank } [T_\phi^*, T_\phi]$.

We record here results on the hyponormality of Toeplitz operators with polynomial symbols, which have been recently developed in the literature. The statement (vii) appears to be new.

**Lemma 3.** Suppose that $\phi$ is a trigonometric polynomial of the form $\phi(z) = \sum_{n=-m}^{N} a_n z^n$, where $a_{-m}$ and $a_N$ are nonzero.

(i) If $T_\phi$ is a hyponormal operator, then $m \leq N$ and $|a_{-m}| \leq |a_N|$.

(ii) If $T_\phi$ is a hyponormal operator, then $N - m \leq \text{rank } [T_\phi^*, T_\phi] \leq N$.

(iii) The hyponormality of $T_\phi$ is independent of the particular values of $a_0, a_1, \ldots, a_{N-m}$ of $\phi$. Moreover the rank of the selfcommutator $[T_\phi^*, T_\phi]$ is also independent of those coefficients.

(iv) Write $\phi = \overline{\phi} + f$ ($f, g \in H^\infty$) and put $\tilde{\phi} = \overline{\phi} + T_\phi f$ ($r \leq N - m$). Then $T_\phi$ is hyponormal if and only if $T_{\tilde{\phi}}$ is.

(v) If $|a_{-m}| = |a_N| \neq 0$, then $T_\phi$ is hyponormal if and only if the following symmetric condition holds:

\[
\sigma_N a_{-j} = a_{-m} \overline{a_{N-m+j}} \quad (1 \leq j \leq m).
\]

In this case, the rank of $[T_\phi^*, T_\phi]$ is $N - m$ and

\[
\mathcal{E}(\phi) = \{a_{-m} (\overline{\sigma_N})^{-1} z^{N-m}\}.
\]

(vi) $T_\phi$ is normal if and only if $m = N$, $|a_{-m}| = |a_N|$, and (3.1) holds with $m = N$.

(vii) Write $\phi := \tilde{\phi} + f$, where $f$ and $g$ are in $H^\infty(\mathbb{T})$ and put $\tilde{\phi} := \alpha \tilde{\phi} + f$ ($|\alpha| \leq 1$). If $T_{\tilde{\phi}}$ is hyponormal, then so is $T_{\tilde{\phi}}$.

**Proof.** The assertions (i) – (vi) were shown from [4, 6, 7, 10, 11, 12, 13, 16]. For the assertion (vii), suppose that there exists a function $k \in H^\infty(\mathbb{T})$ such that $\phi - k \overline{\phi} \in H^\infty(\mathbb{T})$ and $||k||_{\infty} \leq 1$. Thus $\tilde{\phi} - k \tilde{f} \in H^\infty(\mathbb{T})$. Since $|\alpha| \leq 1$ it follows that if we let $\tilde{k} = \alpha k$, then $\alpha \tilde{\phi} - \tilde{k} \tilde{f} = \alpha (\tilde{\phi} - k \tilde{f}) \in H^\infty(\mathbb{T})$ and $||\tilde{k}||_{\infty} = |\alpha| ||k||_{\infty} \leq 1$. Therefore by Lemma 1, $T_{\tilde{\phi}}$ is hyponormal. □
Suppose $\phi = g + f$, where $f = \sum_{n=1}^{N} a_n z^n$ and $g = \sum_{n=1}^{N} b_n z^n$. If $T_\phi$ is normal, then $g$ divides $f$: indeed, by Lemma 3 (v),(vi), $g = e^{i\theta} \sum_{n=1}^{N} a_n z^n$ for some $\theta \in [0, 2\pi)$, so that $g$ divides $f$. But if $T_\phi$ is hyponormal, then $g$ need not divide $f$. For example, consider $g(z) = (z + \frac{1}{2})^2$ and $f(z) = 3(z + 1)^2$.

Using an argument of P. Fan [5, Theorem 1] – for every trigonometric polynomial $\phi$ of the form $\phi(z) = \sum_{n=-2}^{2} a_n z^n$,

$$T_\phi \text{ is hyponormal} \iff |\det \left( \begin{array}{cc} a_{-1} & a_{-2} \\ \pi & \pi^2 \end{array} \right)| \leq |a_2|^2 - |a_{-2}|^2,$$

a straightforward calculation shows that $T_\phi$ is hyponormal. How is the converse? That is, if $g$ divides $f$, does it follow that $T_\phi$ is hyponormal? However we cannot also expect the hyponormality of $T_\phi$ when $g$ divides $f$: for example, if $\phi = (z + 1)^2 + (z + 1)^3$, then by Lemma 3 (v), $T_\phi$ is not hyponormal.

We now consider the hyponormality of $T_\phi$ with $\phi = g + f$ ($f$ and $g$ are analytic polynomials) when $g$ divides $f$. If $\psi$ is in $H_\infty(\mathbb{T})$, write $Z(\psi)$ for the set of all zeros of $\psi$.

**Theorem 4.** Suppose $\phi = g + f$ with $f, g \in H_\infty(\mathbb{T})$ and $\psi := \frac{f}{g}$ has a factorization $\psi = up$, where $u$ is an inner function and $p$ is an analytic polynomial. If

(i) $Z(\psi) \subseteq \mathbb{D}$;
(ii) $\text{ess inf } |\psi| \geq 1$,

then $T_\phi$ is hyponormal.

**Proof.** By the condition (i), $p$ is of the form

$$p(z) = c \prod_{j=1}^{n} (z - \zeta_j) \quad \text{with } |\zeta_j| < 1 \quad (j = 1, \ldots, n).$$

Then we have

$$\frac{1}{p} = \frac{1}{c \prod_{j=1}^{n} (z - \zeta_j)} = \frac{z^n}{c \prod_{j=1}^{n} (1 - \zeta_j z)},$$

which is in $H_\infty(\mathbb{T})$. Put $k := \frac{u}{p}$. Then evidently, $k \in H_\infty(\mathbb{T})$. By the condition (ii),

$$||k||_\infty = \text{ess sup } \left| \frac{u}{p} \right| = \text{ess sup } \left| \frac{1}{p} \right| = \frac{1}{\text{ess inf } |p|} = \frac{1}{\text{ess inf } |\psi|} \leq 1.$$

Since $\tilde{f} = \tilde{g}u\tilde{p}$, it follows

$$\phi - k \tilde{\phi} = (\tilde{g} + \tilde{f}) - \frac{u}{p} (g + \tilde{g}u\tilde{p}) = f - \frac{u}{p} g = f - kg \in H_\infty(\mathbb{T}).$$

Therefore by Cowen’s theorem $T_\phi$ is hyponormal. \qed
The conditions (i) and (ii) in Theorem 4 need not be necessary for \( T_\phi \) to be hyponormal. To see this consider the trigonometric polynomial \( \phi = \tilde{g} + f \), where
\[
g(z) = (z - 1)^2 \quad \text{and} \quad f(z) = 2(z - 1)^2(z - \frac{3}{2}).
\]
Then \( \phi(z) = z^{-2} - 2z^{-1} - 2 + 8z - 7z^2 + 2z^3 \). Put \( \tilde{\phi}(z) := z^{-2} - 2z^{-1} - 7z + 2z^2 \). By Lemma 3 (iv), \( T_\phi \) is hyponormal if and only if \( T_{\tilde{\phi}} \) is. A straightforward calculation with (3.2) shows that \( T_{\tilde{\phi}} \) is hyponormal and hence so is \( T_\phi \). But note that \( \mathcal{Z}(\frac{\tilde{\phi}}{f}) \subseteq \mathbb{C} \setminus \mathbb{D} \). Also if \( \phi(z) = z^{-1} + z(z - \frac{1}{2}) \), then \( T_\phi \) is hyponormal, whereas \( \text{ess inf} |\psi| = \frac{1}{2} \).

**Corollary 5** ([1]). Suppose \( \phi = \tilde{g} + f \) with \( f \) and \( g \) inner. If \( g \) divides \( f \), then \( T_\phi \) is hyponormal.

**Proof.** Apply Theorem 4 with \( p = 1 \). \( \square \)

**Corollary 6.** Let \( \phi = \tilde{g} + f \) with \( f, g \in H^\infty(\mathbb{T}) \) and suppose
\[
f(z) = cg(z) \prod_{j=1}^{\infty} (z - \zeta_j)
\]
with \( |\zeta_j| < 1 \) (\( j = 1, \ldots, n \)). If \( |c| \geq \frac{1}{\prod_{j=1}^{n} (1 - |\zeta_j|)} \), then \( T_\phi \) is hyponormal.

**Proof.** If \( \psi := \frac{f}{g} \), then \( \mathcal{Z}(\psi) \subseteq \mathbb{D} \). Further by assumption,
\[
\text{ess inf} |\psi| = \text{ess inf} |c| \prod_{j=1}^{n} |z - \zeta_j| \geq \text{ess inf} \prod_{j=1}^{n} \frac{|z - \zeta_j|}{1 - |\zeta_j|} \geq 1.
\]
Therefore by Theorem 4, \( T_\phi \) is hyponormal. \( \square \)

For example if \( \phi = \tilde{g} + f \), where
\[
g(z) = \prod_{j=1}^{n} (z - \zeta_j) \quad \text{and} \quad f(z) = \left( \frac{1}{1 - \alpha} \right)^m (z - \alpha)^m \prod_{j=1}^{n} (z - \zeta_j) \quad (|\alpha| < 1),
\]
then by Corollary 6, \( T_\phi \) is hyponormal.

If \( \phi = \tilde{g} + f \) (\( f \) and \( g \) are analytic polynomials), if \( g \) divides \( f \), and if the modulus of the leading coefficient of \( \psi := \frac{f}{g} \) is 1, then we can easily check the hyponormality of \( T_\phi \).

**Theorem 7.** Let \( \phi = \tilde{g} + f \), where \( f \) and \( g \) are analytic polynomials of degrees \( N \) and \( m \) (\( m \geq 2 \)), respectively. Suppose that \( g \) divides \( f \) and the modulus of the leading coefficient of \( \psi := \frac{f}{g} \) is 1. Then \( T_\phi \) is hyponormal if and only if \( \hat{\psi}(n) = 0 \) for \( N - 2m + 1 \leq n \leq N - m - 1 \), where \( \hat{\psi}(n) \) is the \( n \)-th Fourier coefficient of \( \psi \). Hence, in particular, if \( N < 2m \), then \( T_\phi \) is hyponormal if and only if \( \psi(z) = e^{i\omega} z^{N-m} \) for some \( \omega \in [0, 2\pi) \).
Proof. By assumption, we may write $$\psi(z) = e^{i\omega} \prod_{j=1}^{N-m} (z - \zeta_j)$$ for some $$\omega \in [0, 2\pi)$$. If $$T_\phi$$ is hyponormal, then by Lemma 3(v), the finite Blaschke product $$k \in \mathcal{E}(\phi)$$ should be of the form $$k(z) = e^{i\theta} z^{N-m}$$ for some $$\theta \in [0, 2\pi)$$. Thus we have

$$T_\phi$$ hyponormal $$\iff$$ $$\phi - k \bar{\phi} \in H^\infty$$ with $$k(z) = e^{i\theta} z^{N-m}$$

$$\iff$$ $$\bar{g} - e^{i\theta} z^{N-m} \bar{\bar{f}} \in H^\infty$$

$$\iff$$ $$\bar{g} - e^{i\theta} z^{N-m} \cdot \bar{\bar{g}} e^{-i\omega} \prod_{j=1}^{N-m} (\bar{z} - \bar{\zeta}_j) \in H^\infty$$

$$\iff$$ $$\bar{g} \left( 1 - e^{i(\theta - \omega)} \prod_{j=1}^{N-m} (1 - \zeta_j) \right) \in H^\infty$$

$$\iff$$ $$1 - e^{i(\theta - \omega)} \prod_{j=1}^{N-m} (1 - \zeta_j) \in z^m H^\infty.$$ 

Therefore if $$T_\phi$$ is hyponormal and $$N < 2m$$ then $$e^{i(\theta - \omega)} \prod_{j=1}^{N-m} (1 - \zeta_j) = 1$$, which implies $$\zeta_j = 0$$ for $$1 \leq j \leq N - m$$. Thus we have that $$\psi(z) = e^{i\omega} z^{N-m}$$. The converse immediately follows from applying Theorem 4 with $$p = 1$$. If instead $$N \geq 2m$$, write $$\eta(z) := e^{i(\theta - \omega)} \prod_{j=1}^{N-m} (1 - \zeta_j)$$. Then we have

$$T_\phi$$ hyponormal $$\iff$$ $$\eta(z) = 1 + \sum_{j=m}^{N-m} a_j z^j$$ for some $$a_j (j = m, \ldots, N - m)$$

$$\iff$$ $$z^{N-m} \eta(z) = \sum_{j=0}^{N-2m} a_{N-m-j} z^j + z^{N-m}$$ for some $$a_j (j = m, \ldots, N - m)$$

$$\iff$$ $$e^{i(\omega - \theta)} \prod_{j=1}^{N-m} (z - \zeta_j) = \sum_{j=0}^{N-2m} a_{N-m-j} z^j + z^{N-m}$$

for some $$a_j (j = m, \ldots, N - m)$$

$$\iff$$ $$\hat{\psi}(n) = 0$$ for $$N - 2m + 1 \leq n \leq N - m - 1.$$ 

This completes the proof. \hfill \Box

Since the hyponormality is translation-invariant, it follows from Lemma 3(ii) and Theorem 4 that the conclusion of Theorem 7 via its proof can be rewritten as: $$z^m T_{e^{i\omega}} f = \frac{a_N}{a_m} z^m g$$, or equivalently, $$T_{e^{i\omega}} f = \frac{a_N}{a_m} g$$. Therefore we can recapture Lemma 3(v): if $$\phi(z) = \sum_{n=-m}^{N-m} a_n z^n$$ with $$|a_{-m}| = |a_N| \neq 0$$, then
$T_\phi$ is hyponormal if and only if
\[
\begin{pmatrix}
\alpha_{m-1} \\
\alpha_{m-2} \\
\vdots \\
\alpha_1 \\
\alpha_0 \\
0
\end{pmatrix} = \begin{pmatrix}
\alpha_{N-m+1} \\
\alpha_{N-m+2} \\
\vdots \\
\alpha_N \\
\alpha_N \\
\alpha_N
\end{pmatrix}.
\]

Example 8. If
\[
\phi(z) = \prod_{j=1}^{m} (z - \alpha_j) + \prod_{j=1}^{N} (z - \alpha_j) \quad (m < N < 2m, \ \alpha_N \neq 0),
\]
then $T_\phi$ is not hyponormal.

Proof. This follows immediately from Theorem 7.

Theorem 7 is not true in general if the leading coefficient of $f/g$ does not have modulus 1. Hyponormality for such a case is very complicated.

Corollary 9. Let $\phi = g + f$, where $f$ and $g$ are analytic polynomials of degrees $N$ and $m$, respectively ($m \geq 2$). Suppose that $g$ divides $f$ and the leading coefficient of $\psi \equiv f/g$ has modulus $\geq 1$. If $Z(\psi) \subseteq T$ and $\psi(n) = 0$ for $n = 1, \ldots, m - 1$, then $T_\phi$ is hyponormal.

Proof. Without loss of generality we may write $g(z) = \prod_{j=1}^{m} (z - \gamma_j)$. Define
\[
k(z) := \frac{\psi(z)}{\psi(0)\overline{\psi(z)}}.
\]
Since $Z(\psi) \subseteq T$, it follows that $|\psi(0)| = 1$. But since $\overline{\psi}/\psi$ is unimodular it follows that $k \in H^\infty$ and $||k||_\infty \leq 1$. Thus
\[
g - k f = \prod_{j=1}^{m} (z - \gamma_j) \left(1 - \frac{\psi(z)}{\psi(0)\overline{\psi(z)}}\overline{\psi(z)}\right)
\]
\[
= \frac{1}{z_m} \prod_{j=1}^{m} (1 - \gamma_j z) \left(1 - \frac{\psi(z)}{\psi(0)}\right)
\]
\[
\in H^\infty \quad \text{(because } \psi(n) = 0 \text{ for } n = 1, \ldots, m - 1),
\]
which implies that $\phi - k \overline{\phi} \in H^\infty$, and hence $T_\phi$ is hyponormal.

References


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