CONJUGACY CLASSES OF AUTOMORPHISMS $p$-GROUPS

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Abstract. In this paper we provide examples of pairs of conformally non-equivalent, but topologically equivalent, $p$-groups $H_1, H_2 < \text{Aut}(S)$, where $S$ is a closed Riemann surface of genus $g \geq 2$, so that $S/H_j$ has genus zero and all its cone points are of order equal to $p$.

1. Introduction

We denote by $\text{Aut}(S)$ the group of conformal automorphisms of a Riemann surface $S$. If $S_1$ and $S_2$ are Riemann surfaces, then we say that $H_1 < \text{Aut}(S_1)$ and $H_2 < \text{Aut}(S_2)$ are topologically equivalent (respectively, conformally equivalent) if there is an orientation preserving homeomorphism (respectively, a conformal homeomorphism) $f : S_1 \to S_2$ so that $H_2 = fH_1f^{-1}$. In this paper, we assume $S_1 = S_2$. Sources for the matter of characterization of topological conjugacy for surface automorphisms by certain purely algebraic data are J. Nielsen [14], W. J. Harvey [12], and J. Gilmann [9]. In general, it is not hard to construct an example of a closed Riemann surface $S$, of genus $g \geq 2$, and pairs $H_1, H_2 < \text{Aut}(S)$, $H_1 \cong H_2$, so that $H_1$ and $H_2$ are topologically non-equivalent.

If $H < \text{Aut}(S)$, where $S$ is a closed Riemann surface of genus $g \geq 2$, and $R = S/H$, then denote by $\mathcal{M}_g(H)$ the locus in $\mathcal{M}_g$ (the moduli space of genus $g$) consisting of points parametrizing Riemann surfaces $S'$ which admit a group $H'$ of conformal automorphisms topologically equivalent to $H$. One has, of course, $\mathcal{M}_g(H) = \mathcal{M}_g(H')$. It is well known that $\mathcal{M}_g(H)$ is an irreducible subvariety of $\mathcal{M}_g$ of dimension $3g' - 3 + r$; where $g'$ is the genus of $R$ and $r$ is the number of points over which the natural projection $S \to R$ is ramified ([5], [6]). Moreover, $\mathcal{M}_g(H)$ fails to be normal if and only if there is a Riemann surface $S'$ of genus $g$ admitting two groups $H_1, H_2 < \text{Aut}(S')$ which are topologically equivalent to $H$, but not conformally equivalent to each other ([5], [6]).
From now on, let $S$ be a closed Riemann surface of genus $g \geq 2$, let $p$ be a prime, let $G$ be a finite $p$-group, and let $H_1, H_2 < \text{Aut}(S)$ be so that $H_1 \cong H_2 \cong G$.

If $G \cong \mathbb{Z}_p$ and $S/H_j$ (for $j = 1, 2$) has genus zero, then $H_1$ and $H_2$ are conformally equivalent. This fact is consequence of a classical Castelnuovo-Severi theorem [3] in the case that $g > (p-1)^2$ and, for the general case, this was proved by González-Diez in [4]. An alternative proof of this result was also obtained by Gromadzki [10]. If we drop the condition for $S/H_j$ to be of genus zero, then this may be false [6]; but if $p \geq 2\gamma + 1$, where $\gamma$ is the genus of $S/H_j$, then $H_1$ and $H_2$ are conformally equivalent [11] (in fact, the condition $p \geq 2\gamma + 1$ asserts that $H_j$ is a $p$-Sylow subgroup of $\text{Aut}(S)$).

If $p = 2$ and $G \cong \mathbb{Z}_8$, then in [7] there is a construction of (a 1-dimensional family) a closed Riemann surface $S$ of genus $g = 9$ and $H_1, H_2 < \text{Aut}(S)$ so that $H_j \cong \mathbb{Z}_8$ are topologically equivalent but not conformally equivalent. In this example, the quotient orbifolds $S/H_j$ has signature $(0; 4, 4, 8, 8)$. A generalization has been provided in [2], where $G \cong \mathbb{Z}_{2n+1}$ and $n \geq 2$, so that $S/H_j$ (for $j = 1, 2$) has signature $(0; 2^n, 2^n, 2^{n-1}, 2^{n+1})$. Note that in these examples, there are cone points of order different from the prime $p = 2$.

If $G \cong \mathbb{Z}_p^n$, where $n \geq 2$ is an integer, then in [8] it is proved that if $S/H_j$ (for $j = 1, 2$) has signature $(0; p, \ldots, 1, p)$, then $H_1$ and $H_2$ are conformally equivalent.

This makes us to wonder if the above is true without this relation between the exponent in the order of the group $G$ and the number of ramification points in $S/H_j$ which, in the above, are respectively $n$ and $n+1$. The following provides counter-examples to such an expectation.

**Theorem 1.** (1) Let $n \geq 3$ be an integer. Then there exists a prime $p_n$ so that, for every prime $p \geq p_n$, there exists a closed Riemann surface $S$, of genus $g \geq 2$, and subgroups $H_1, H_2 < \text{Aut}(S)$, $H_1 \cong H_2 \cong \mathbb{Z}_p^n$, with $S/H_j$ of signature of the form $(0; p, \ldots, p)$, which are topologically equivalent but not conformally equivalent.

(2) There exists a closed Riemann surface $S$, of genus $g = 5$, and subgroups $H_1, H_2 < \text{Aut}(S)$, $H_1 \cong H_2 \cong \mathbb{Z}_2^n$, with $S/H_j$ of signature of the form $(0; 2, 2, 2, 2, 2)$, which are topologically equivalent but not conformally equivalent.

**2. Proof of Theorem 1**

Let us consider an integer $n \geq 3$ and $p$ a prime (if $p = 2$, then we assume that $n \geq 4$). Let $\lambda_1, \ldots, \lambda_{n-2} \in \mathbb{C} - \{0, 1\}$ be so that $\lambda_i \neq \lambda_j$, for $i \neq j$.

We assume that these values are so that the group of Möbius transformations keeping invariant the set $\{\infty, 0, 1, \lambda_1, \ldots, \lambda_{n-2}\}$ is the trivial group.
Let us consider the closed Riemann surface $S$ defined by the following equations

$$S = \left\{ \begin{array}{l}
  x_1^p + x_2^p + x_3^p = 0 \\
  \lambda_1 x_1^p + x_2^p + x_3^p = 0 \\
  \vdots \\
  \lambda_n - 2 x_1^p + x_2^p + x_{n+1}^p = 0
\end{array} \right\} \subset \mathbb{P}_C^3.$$

Let $H = \langle a_1, \ldots, a_n \rangle \cong \mathbb{Z}_p^n$, where $a_j$ is defined by multiplication of the coordinate $x_j$ by $e^{2\pi i/p}$. Set $a_{n+1} = (a_1 a_2 \cdots a_n)^{-1}$. Then $a_{n+1}$ is multiplication of the coordinate $x_{n+1}$ by $e^{2\pi i/p}$.

It is well known [8] that $H < \text{Aut}(S)$ with $S/H$ of signature $(0; p, n+1, p)$ and that $H$ is the unique subgroup $K$, up to conjugation in $\text{Aut}(S)$, satisfying that $K \cong \mathbb{Z}_p^n$ and that $S/K$ has such a signature. Moreover, we may identify $S/H$ with the orbifold whose Riemann surface structure is the Riemann sphere and whose cone points (all of them of order $p$) are given by $\infty$, 0, 1, $\lambda_1$, $\ldots$, $\lambda_{n-2}$. The natural regular branched cover, with $H$ as Deck group, is given by

$$\pi([x_1 : \cdots : x_{n+1}]) = -\left(\frac{x_2}{x_1}\right)^p.$$ 

The map $\pi$ sends the set of fixed points of $a_1$ to $\infty$; the set of fixed points of $a_2$ to 0; the set of fixed points of $a_3$ to 1; and, for $j \in \{4, \ldots, n+1\}$, it sends the set of fixed points of $a_j$ to $\lambda_{j-3}$.

It is also known that $\pi : S \to S/H$ is a homology branched cover, that is, if $\Gamma$ is a Fuchsian group so that $\mathbb{H}^2/\Gamma = S/H$, then $S = \mathbb{H}^2/\Gamma'$, where $\Gamma'$ denotes the derived subgroup of $\Gamma$, and $H = \Gamma/\Gamma'$.

Set $H_1 = \langle a_1, a_2, \ldots, a_{n-1} \rangle \cong \mathbb{Z}_p^{n-1}$ and $H_2 = \langle a_2, a_3, \ldots, a_n \rangle \cong \mathbb{Z}_p^{n-1}$. It is clear that, for $j = 1, 2$, $S/H_j$ has signature $(0; p, (n-1)p, p)$.

Let us consider an orientation preserving homeomorphism $f : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$, of order two, so that $f(\infty) = \lambda_{n-3}$, $f(0) = 0$, $f(1) = 1$, $f(\lambda_1) = \lambda_1$, $\ldots$, $f(\lambda_{n-4}) = \lambda_{n-4}$, $f(\lambda_{n-2}) = \lambda_{n-2}$.

As $\pi : S \to S/H$ is a homology branched cover, the homeomorphism $f$ lifts to an orientation preserving homeomorphism $\tilde{f} : S \to S$ so that $\tilde{f} H \tilde{f}^{-1} = H$ and, by the property of $f$ at the cone point of $S/H$, that $f H_1 f^{-1} = H_2$, that is, these two groups are topologically equivalent.

2.1. Part (1)

As a consequence of the results in [13], it is possible to find a prime number $p_0$ so that, if $p \geq p_0$, then $H$ is a normal subgroup of $\text{Aut}(S)$. It follows that, in this case, $\text{Aut}(S)/H$ acts as a group of conformal automorphisms of the orbifold $S/H$. As we have supposed that no M"{o}bius transformation, different from the identity, may keep invariant the set $\{\infty, 0, 1, \lambda_1, \ldots, \lambda_{n-2}\}$, we have that $\text{Aut}(S) = H$. It follows that, for $p \geq p_0$, the groups $H_1$ and $H_2$ cannot be conformally equivalent.
2.2. Part (2)

If \( p = 2 \) and \( n = 4 \), under the above conditions, we have (see [1]) that \( S \) is a closed Riemann surface of genus 5 for which \( \text{Aut}(S) = H \). If we set \( H_1 = \langle a_1, a_2 \rangle \cong \mathbb{Z}_2^2 \) and \( H_2 = \langle a_1, a_3 \rangle \cong \mathbb{Z}_2^2 \), then these two groups are topologically equivalent but cannot be conformally equivalent.

3. A final remark

In [13] is proved that, if we fix \( \gamma, r \geq 0 \) and \( s \geq 1 \) integers so that \( 2\gamma - 2 + r > 0 \), then there exists a prime \( q = q(\gamma, r, s) \) with the following property: “if \( p \geq q \) is a prime, \( S \) is a closed Riemann surface of genus \( g \geq 2 \), \( H_1, H_2 < \text{Aut}(S) \), \( |H_1| = |H_2| = p^s \), \( S/H_1 \) has genus \( \gamma \) and exactly \( r \) cone points, then \( H_1 = H_2 \).” This property is not in contradiction with Theorem 1. In fact, in our family of examples, the quotient \( S/H \) has signature \( (0; p, \cdots, p) \) and the quotients \( S/H_j \) have signature \( (0; p, 2+(n-1)p, p) \). We are considering primes \( p \geq p_n = q(0, n+1, n) \). On the other hand, the prime \( q(0, 2+(n-1)p, n-1) \) is necessarily bigger than \( p \).

References


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