MULTIPLICATIVE SET OF IDEMPOTENTS
IN A SEMIPERFECT RING

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Abstract. Let $R$ be a ring with identity 1, $I(R)$ be the set of all idempotents in $R$ and $G$ be the group of all units of $R$. In this paper, we show that for any semiperfect ring $R$ in which $2 = 1+1$ is a unit, $I(R)$ is closed under multiplication if and only if $R$ is a direct sum of local rings if and only if the set of all minimal idempotents in $R$ is closed under multiplication and $eGe$ is contained in the group of units of $eRe$. In particular, for a left Artinian ring in which $2$ is a unit, $R$ is a direct sum of local rings if and only if the set of all minimal idempotents in $R$ is closed under multiplication.

1. Introduction and basic definitions

Let $R$ be a ring with identity 1, $G$ the group of all units of $R$, $J$ the Jacobson radical of $R$ and $I(R)$ the set of all idempotents of $R$. In this case, $I(R)$ is called commuting if $ef = fe$ for all $e, f \in I(R)$. Observe that if $I(R)$ is commuting, then $I(R)$ is closed under multiplication, that is, $I(R) = I(R)^2$ where $I(R)^2 = \{ ef : e, f \in I(R) \}$.

By the following example, there is a ring $R$ without identity such that the converse is not true.

Example 1. Let $R = \begin{pmatrix} Z_2 & Z_2 \\ 0 & 0 \end{pmatrix}$ be a ring without identity where $Z_2$ is the ring of integers modulo 2. Then $I(R) = \{ (\begin{smallmatrix} 0 & 0 \\ 0 & 0 \end{smallmatrix}), (\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}), (\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}) \}$.

Note that $I(R)$ is closed under multiplication, but $(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix})(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}) \neq (\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix})(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix})$, and hence $I(R)$ is not commuting.

On the other hand, in Section 2, we will prove that if $R$ is a ring with identity, then $I(R)$ is commuting if and only if $I(R)$ is closed under multiplication if and only if every idempotent of $R$ is central.

Let $\preceq$ denote the usual relation on $I(R)$, that is, $e \preceq f$ means that $ef = fe = e$ (refer [2]). A nonzero idempotent $e$ is called minimal if there is no idempotent strictly between 0 and $e$ according to the partial ordering $\preceq$. Note that the

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1033
minimal idempotents in this sense are precisely the primitive idempotents of $R$. In Section 2, we will show that if $R$ is a ring with identity such that $I(R)$ is closed, then (1) $|I(R)| = |I(R/J)|$ where $|\cdot |$ is the cardinality of a set; (2) $I_m(R)$ is closed under multiplication and $|I_m(R)| = |I_m(R/J)|$ where $I_m(R)$ (resp. $I_m(R/J)$) is the set of all minimal idempotents of $R$ (resp. $R/J$).

We also define a relation $e \preceq_1 f$ by $efe = e$ (refer [2]). In [2], by considering the relation $\preceq_1$ Dolzan has shown that for a finite ring $R$ with identity, if the set of all minimal idempotents of $R$ is closed under multiplication, then for all minimal idempotents $e$ in $R$, $eGe$ is contained in the group of units of $eRe$. As a corollary he also has shown that the set of all minimal idempotents of $R$ is closed under multiplication if and only if $R$ is a direct product of local rings. It is clear that if $I(R)$ is closed under the multiplication, then the above two relations $\preceq$ and $\preceq_1$ are equal.

In Section 3, we will show that for any semiperfect ring $R$ with identity 1 in which $2 = 1 + 1$ is a unit, $I(R)$ is closed under multiplication if and only if $R$ is a direct sum of local rings if and only if the set of all minimal idempotents of $R$ is closed under multiplication and $eGe$ is contained in the group of units of $eRe$. We also will show that for a left Artinian ring in which 2 is a unit, the fact that the set of all minimal idempotents in $R$ is closed under multiplication implies that $eGe$ is contained in the group of units of $eRe$. Hence as a corollary we have that for a left Artinian ring $R$ with identity 1 in which 2 is a unit, $I(R)$ is closed under multiplication if and only if $R$ is a direct sum of local rings if and only if the set of all minimal idempotents of $R$ is closed under multiplication.

Throughout this paper, let $R$ be a ring with identity 1, $G$ be the group of units of $R$, let $J$ denote the Jacobson radical of $R$ and let $I(R)$ (resp. $I_m(R)$) be the set of all idempotents of $R$ (resp. the set of all minimal idempotents of $R$).

2. Some properties of a ring with commuting idempotents

In this section, we will find some properties of a ring $R$ such that $I(R)$ is closed under multiplication.

Lemma 2.1. Let $R$ be a ring. Then the following are equivalent:

1. $I(R)$ is commuting;
2. $I(R)$ is closed under multiplication;
3. every idempotent of $R$ is central.

Proof. (1) $\Rightarrow$ (2). Clear.

(2) $\Rightarrow$ (3). Let $e \in I(R)$ be arbitrary, and let $f = 1 - e \in I(R)$. Then for all $a \in R$, $e + eaf \in I(R)$. Since $I(R)$ is closed under multiplication, $(e + eaf)f = ef + eaf = eaf \in I(R)$, and so $eaf = (eaf)(eaf) = 0$, which implies that $ae = eae$ for all $a \in R$. Similar argument yields $ae = eae$ for all $a \in R$. Thus $e$ is central.

(3) $\Rightarrow$ (1). Clear.
Lemma 2.2. Let $R$ be a ring such that $I(R)$ is commuting. If $e - f \in J$ for some $e, f \in I(R)$, then $e = f$.

Proof. Since $I(R)$ is commuting, $ef = fe$. Note that $(e - f)^2 = e - 2ef + f = (e - f)^2$, and so $(e - f)^2 \in I(R)$. Thus $(e - f)^2 \in I(R) \cap J$. Since $I(R) \cap J = \{0\}$, $(e - f)^2 = e - 2ef + f = 0$. Hence

\[ e + f = 2ef. \]

By multiplying with $e$ (resp. $f$) from the both sides of $(*)$, we have $e = ef$ (resp. $f = ef$). Hence $e - f = ef - ef = 0$. $\square$

Corollary 2.3. Let $R$ be a ring. If $I(R)$ is commuting, then $|I(R)| = |I(R)/J|$.

Proof. Clearly, $|I(R)| \geq |I(R)/J|$. Assume that there exist two idempotents $e, f$ of $R$ ($e \neq f$) such that $e + J = f + J$. Then $e - f \in J$, and so $e = f$ by Lemma 2.2, a contradiction. Hence $|I(R)| = |I(R)/J|$. $\square$

Proposition 2.4. Let $R$ be a ring such that $I(R)$ is commuting. Then

1. If $e$ is an idempotent in $R$ such that $\bar{e} = e + J \in I_m(R/J)$, then $e \in I_m(R)$.
2. $|I_m(R)| = |I_m(R/J)|$.
3. $I_m(R)$ is closed under multiplication.

Proof. (1) Suppose that there exists an idempotent $e_1 \in R$ such that $0 \neq e_1 \preceq e$. Then clearly $\bar{e}_1 \preceq \bar{e}$. If $\bar{e}_1 = 0$, then $e_1 \in I(R) \cap J = \{0\}$, and so $e_1 = 0$, a contradiction. Hence $\bar{e}_1 = \bar{e}$, and then $e_1 - e \in J$. Since $I(R)$ is commuting and $e_1 - e \in J$, $e_1 = e$ by Lemma 2.2. Hence $e \in I_m(R)$.

(2) It follows from (1).

(3) Let $e, f \in I_m(R)$ be arbitrary. Since $I(R)$ is commuting, we have $e(ef) = ef = (ef)e$, which implies that $ef \preceq e$. Since $e$ is a minimal idempotent, $ef = 0$ or $ef = e$. Hence $ef$ is a minimal idempotent in $R$. $\square$

Remark 1. Let $R$ be a ring and $N$ be a nil ideal of $R$. By the similar argument as the one given in Lemma 2.2 and Proposition 2.4, we have that if $I(R)$ is commuting (in particular, if $R$ is a commutative ring), then $|I(R)| = |I(R/N)|$ and $|I_m(R)| = |I_m(R/N)|$.

Proposition 2.5. Let $R$ be a ring in which 2 is a unit such that $G$ is an abelian group. If every idempotent of $R/N$ is central for some nil ideal $N$ of $R$, then every idempotent of $R$ is central.

Proof. Let $e \in I(R), a \in R$ be arbitrary. Since $e + N \in I(R/N)$ is central, $(e + N)(a + N) = (a + N)(e + N)$, and so $ea - ae \in N$. Note that since $G$ is abelian and $2 \in G$, $eg = ge$ for all $g \in G$. Since $1 + (ea - ae) \in G$, $e(1 + (ea - ae)) = (1 + (ea - ae))e$, and so $ea = cae = ae$, which implies that $e$ is central. $\square$
Proposition 2.6. Let $R$ be a ring and $N$ be an ideal of $R$ such that $I(R/N)$ is commuting. If $ae = ea$ for all $e \in I(R)$ and all $a \in N$, then $I(R)$ is commuting.

Proof. Let $e, f \in I(R)$ be arbitrary. Since $I(R/N)$ is commuting, $(e+N)(f+N) = (f+N)(e+N)$, and so $ef - fe \in N$. By assumption, $e(ef - fe) = (ef - fe)e$, and then $ef = efe = fe$. Hence $I(R)$ is commuting.

3. A decomposition of a semiperfect ring

Recall that a ring $R$ is called semiperfect if $R/J$ is left Artinian, and every idempotent in $R/J$ can be lifted to $R$. In [1], it was shown that every element in a semiperfect ring $R$ can be expressed as a sum of a unit and an idempotent in $R$ (also refer [3]). Recall that a minimal idempotent in a semiperfect ring is local by Proposition 23.5 in [4].

Proposition 3.1. Let $R$ be a ring. If $I(R)$ is commuting, then for all $e \in I(R)$, $eGe$ is contained in the group of units in $eRe$.

Proof. Let $ege \in eGe(ge \in G)$ be arbitrary. Since $I(R)$ is commuting, every idempotent $e \in I(R)$ is central by Lemma 2.1. Hence $e = (ege)(eg^{-1}e) = (eg^{-1}e)(ege)$, that is, $ege$ is a unit in $eRe$.

In [2], Dolžan has shown that for $e, f \in I_m(R)$ by defining $e \preceq_1 f$ if $efe = e$ and also defining $e \sim f$ if $e \preceq_1 f$ and $f \preceq_1 e \in I_m(R)$ is an equivalence relation on $I_m(R)$ provided $I_m(R)$ is closed under multiplication.

Lemma 3.2. Let $R$ be a ring such that $I_m(R)$ is closed under multiplication, and let $e, f, g, h \in I_m(R)$. If $[e] = [f]$ and $[g] = [h]$, then $[eg] = [fh]$, where $[a]$ is an equivalence class containing $a \in I_m(R)$ under the equivalence relation $\sim$.

Proof. Refer [2, Lemma 2.5].

Theorem 3.3. Let $R$ be a ring, $e \in E(R)$ and $a \in G$. Then $eae$ is a unit of $eRe$ if and only if $e \sim aea^{-1}$.

Proof. Refer [2, Theorem 3.2].

Theorem 3.4. Let $R$ be a semiperfect ring such that $2$ is a unit in $R$. Then the following are equivalent:

1. $I(R)$ is closed under multiplication;
2. $R$ is a direct sum of local rings;
3. $I_m(R)$ is closed under multiplication and $eGe$ is contained in the group of units of $eRe$.

Proof. (1) $\Rightarrow$ (2). Suppose that $I(R)$ is closed under multiplication. Since $R$ is semiperfect, there exists a finite mutually orthogonal set of local idempotents $\{e_1, \ldots, e_n\}$ such that $1 = e_1 + \cdots + e_n$ by [4, Theorem 23.6]. For all $a \in R$, $a = ae_1 + \cdots + ae_n \in Re_1 + \cdots + Re_n$. Since all $e_i$ are central for all $i = 1, \ldots, n$ by Lemma 2.1, $Re_i \cap Re_j = \{0\}$ for all $i, j = 1, \ldots, n (i \neq j)$. Hence $R = \bigoplus_{i=1}^n Re_i$ is a direct sum of local rings since each $Re_i(= e_iRe_i)$ is a local ring.
some loss of generality, we can let $n$.

**Proof.** Let $R$ be an idempotent and let $I(R)$ be a direct sum of local rings. Since $aR=0$ for all orthogonal minimal idempotents $e$, $f(e \neq f)$. Hence $I(R)f=0$. If $e=1, \ldots, n$, we have $R=\oplus_{i=1}^{n} e_i R e_i$ is a direct sum of local rings.

**Remark 2.** Note that for a semiperfect ring $R$ such that $2$ is a unit in $R$ and $I(R)$ is commutative, $R$ is a direct sum of local rings by Theorem 3.4, and then the number of summands of local rings is equal to the maximal number of mutually orthogonal minimal idempotents in $R$.

**Lemma 3.5.** Let $n \geq 2$ be a positive integer and $R$ be the $n \times n$ matrix ring over a division ring. Then $I_m(R)$ is not closed under multiplication.

**Proof.** The proof is similar to the one in the [2, Lemma 5.4]. Choose two idempotents $e = E_{11} + E_{12} + \cdots + E_{1n}$ and $f = E_{nm}$ where $E_{ij}$ is the matrix such that $(i,j)$-entry is $1$ and otherwise $0$. Then $e$ and $f$ are minimal idempotents of $R$ and $e f \neq 0$ with $(e f)^2 = 0$.

**Proposition 3.6.** Let $e \in R$ be an idempotent and let $N \subseteq J$ be an ideal of $R$. If $e$ is primitive (equivalently, minimal) in $R/N$, then $e$ is primitive in $R$. The converse holds if idempotents of $R/N$ can be lifted to $R$.

**Proof.** Refer [4, Proposition 21.22].

**Theorem 3.7.** Let $R$ be a semiperfect ring in which $J$ is a nil ideal. If $I_n(R)$ is closed under multiplication, then $eG e$ is contained in the group of units of $eRe$.

**Proof.** First, we will show that $R/J$ is a direct sum of division rings. Since $R$ is semiperfect, $R/J \cong \bigoplus_{i=1}^{n} M_i(D_i)$ where $M_i(D_i)$ is the full matrix ring of all $n_i \times n_i$ matrices over a division ring $D_i$ for each $i = 1, \ldots, m$. Without loss of generality, we can let $R/J = \bigoplus_{i=1}^{n} M_i(D_i)$. Assume that $n_i \geq 2$ for some $i$. Consider two minimal idempotents $e_i = E_{11} + E_{12} + \cdots + E_{1n_i}, f_i = E_{n_i, n_i} \in M_i(D_i)$ from Lemma 3.5. Note that $(0_1, \ldots, 0_{i-1}, e_i, 0_{i+1}, \ldots, 0_n)$ and
$(0_1, \ldots, 0_{i-1}, f_i, 0_{i+1}, \ldots, 0_n) \in I(R/J)$ are also minimal idempotents in $R/J$ where $0_j$ is the additive identity of $M_j(D_j)$ for all $j = 1, \ldots, n$. Since $R$ is semiperfect, every idempotent of $R/J$ can be lifted to $R$, and so there exist idempotents $e, f$ such that $e = (0_1, \ldots, 0_{i-1}, e_i, 0_{i+1}, \ldots, 0_n)$, $f = (0_1, \ldots, 0_{i-1}, f_i, 0_{i+1}, \ldots, 0_n)$. By Proposition 3.6, $e$ and $f$ are minimal idempotents of $R$. Since $e_i f_i \neq 0$ with $(e_i f_i)^2 = 0$, that is, $ef \notin J$ with $(ef)^2 \in J$. Thus $ef$ is a nonzero nilpotent of $R$, and so $I_m(R)$ is not closed under multiplication, which is a contradiction. Therefore, $R/J$ is a direct sum of division rings. Let $e \in I(R)$ be arbitrary, and let $g$ be a unit in $G$. Since every idempotent in $R/J$ is central, we have that $(e+J)(g+J)(e+J) = (e+J)(g^{-1}+J)(e+J)(g+J)(e+J) = e+J$. Thus $(ege)(eg^{-1}e) = e + j_1 = e + ej_1 e$ and $(eg^{-1}e)(ege) = e + j_2 = e + ej_2 e$ for some $j_1, j_2 \in J$. Since $ej_1 e, ej_2 e \in J$ and $J$ is a nil ideal of $R$, $ej_1 e$ and $ej_2 e$ are nilpotents in $eRe$, and hence $e+ej_1 e$ and $e+ej_2 e$ are units in $eRe$. Therefore, $ege$ is a unit in $eRe$. □

Corollary 3.8. Let $R$ be a left Artinian ring with identity 1 such that 2 is a unit in $R$. Then the following are equivalent:

1) $I(R)$ is closed under multiplication;
2) $R$ is a direct sum of local rings;
3) $I_m(R)$ is closed under multiplication.

Proof. It follows from Theorem 3.4 and Theorem 3.7. □

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