 APPROXIMATION AND INTERPOLATION IN THE SPACE OF CONTINUOUS FUNCTIONS VANISHING AT INFINITY

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Abstract. We establish a result concerning simultaneous approximation and interpolation from certain uniformly dense subsets of the space of vector-valued continuous functions vanishing at infinity on locally compact Hausdorff spaces.

1. Introduction and preliminaries

Throughout this paper we shall assume, unless stated otherwise, that $X$ is a locally compact Hausdorff space and $(E, \| \cdot \|)$ is a normed vector space over $\mathbb{K}$, where $\mathbb{K}$ denotes either the field $\mathbb{R}$ of real numbers or the field $\mathbb{C}$ of complex numbers. We shall denote by $E^*$ the topological dual of $E$ and by $C(X; E)$ the vector space over $\mathbb{K}$ of all continuous functions from $X$ into $E$.

A continuous function $f$ from $X$ to $E$ is said to vanish at infinity if for every $\varepsilon > 0$ the set $\{ x \in X : \| f(x) \| \geq \varepsilon \}$ is compact. Let $C_0(X; E)$ be the vector space of all continuous functions from $X$ into $E$ vanishing at infinity and equipped with the supremum norm. The vector subspace of all functions in $C(X; E)$ with compact support is denoted by $C_c(X; E)$.

Let $A$ be a nonempty subset of $C_0(X; \mathbb{K})$. We denote by $A \otimes E$ the subset of $C_0(X; E)$ consisting of all functions of the form

$$f(x) = \sum_{i=1}^{n} \phi_i(x)v_i, \quad x \in X,$$

where $\phi_i \in A$, $v_i \in E$, $i = 1, \ldots, n$, $n \in \mathbb{N}$.

A subset $W \subset C_0(X; E)$ is an interpolating family for $C_0(X; E)$ if given any nonempty finite subset $S \subset X$ and any $f \in C_0(X; E)$, there exists $g \in W$ such that $g(x) = f(x)$ for all $x \in S$.

A nonempty subset $B$ of $C_0(X; E)$ is said to have the approximation-interpolation property on finite subsets (in short, the $SAI$ property) if for every $f \in C_0(X; E)$, every $\varepsilon > 0$ and every nonempty finite subset $S$ of $X$, there exists $g \in B$ such that $\| f - g \| < \varepsilon$ and $f(x) = g(x)$ for all $x \in S$. 

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The purpose of this paper is to present a result of simultaneous approximation and interpolation from certain subsets of $C_0(X; E)$. As a consequence, we obtain a generalization of a result by Prolla concerning simultaneous approximation and interpolation from vector subspaces of $C(X; E)$ when $X$ is a compact Hausdorff space.

2. Main result

Walsh (Theorem 6.5.1 [2]) proved the following result.

**Theorem 2.1.** Let $K$ be a compact set in the complex plane and let $z_1, \ldots, z_n$ be any set of $n$ points in $K$. If the function $f$ is defined on $K$ and can be uniformly approximated by polynomials there, then $f$ can be uniformly approximated by polynomials $p$ which also satisfy the auxiliary conditions $p(z_i) = f(z_i), i = 1, \ldots, n$.

Motivated by Walsh, we establish the result below.

**Theorem 2.2.** Let $A$ be an interpolating family for $C_0(X; K)$ and $B$ an uniformly dense subset of $C_0(X; E)$. If $(A \otimes E) + B \subset B$, then $B$ has the SAI property.

**Lemma 2.1.** If $X$ is a locally compact Hausdorff space and $\{x_1, \ldots, x_n\} \subset X$, then there exists $l_i \in C_c(X; \mathbb{R})$ such that $l_i(x_i) = 1$ and $l_i(x_j) = 0, j \neq i$.

**Proof.** Since $X$ is Hausdorff and $\{x_1, \ldots, x_n\}$ is finite there exists an open neighborhood $U_i$ of $x_i$ such that $x_j \notin U_i$ for all $j \neq i$, $j \in \{1, \ldots, n\}$. By Urysohn’s Lemma [8] there exists $l_i \in C_c(X; \mathbb{R})$, $0 \leq l_i \leq 1$, such that $l_i(x_i) = 1$ and $l_i(x) = 0$ if $x \notin U_i$, in particular, $l_i(x_j) = 0, j \neq i$. □

**Proof of Theorem 2.2.** Let $S = \{x_1, \ldots, x_n\}$ be a subset of $X$. Let $f \in C_0(X; E)$ and $\varepsilon > 0$.

By Lemma 2.1, for each $x_i \in S$ there exists $l_i \in C_c(X; \mathbb{R})$ such that

\[
\begin{align*}
l_i(x_i) &= 1 \\
l_i(x_j) &= 0; \quad j \neq i, \ x_j \in S.
\end{align*}
\]

Since $A$ is an interpolating family for $C_0(X; \mathbb{R})$, there exist $\phi_1, \ldots, \phi_n \in A$ such that

\[
\phi_i(x_j) = l_i(x_j); \quad 1 \leq i, j \leq n.
\]

Since $B$ is uniformly dense in $C_0(X; E)$ there exists $g \in B$ such that $\|f - g\| < \eta$ where $\eta := \varepsilon/(1 + \sum_{i=1}^n \|\phi_i\|)$.

The function $h : X \rightarrow E$ defined by

\[
h(x) = \sum_{i=1}^n \phi_i(x)(f(x_i) - g(x_i))
\]

belongs to $A \otimes E$ and $h(x_j) = f(x_j) - g(x_j)$ for $j = 1, \ldots, n$. 
Now the function \( p = h + g \) belongs to \( B \) and \( p(x_j) = f(x_j) \) for \( j = 1, \ldots, n \). Moreover,
\[
\|f - p\| \leq \|f - g\| + \|h\| < \eta + \eta \sum_{i=1}^{n} \|\phi_i\| = \varepsilon. \tag*{□}
\]

**Example 2.1.** The set of all continuous real-valued nowhere differentiable functions on \([a, b]\), denoted by \( ND[a, b] \), has the SAI property. Indeed, let \( P[a, b] \) be the set of all real polynomials on \([a, b]\). Note that
(a) \( P[a, b] \) is an interpolating subset of \( C([a, b]; \mathbb{R}) \) (take the Lagrange polynomials);
(b) \( ND[a, b] \) is uniformly dense in \( C([a, b]; \mathbb{R}) \);
(c) \( (P[a, b] \otimes \mathbb{R}) + ND[a, b] = P[a, b] + ND[a, b] \subset ND[a, b] \).
Hence, it follows from Theorem 2.2 that \( ND[a, b] \) has the SAI property.

**Lemma 2.2.** Every uniformly dense vector subspace of \( C_0(X; \mathbb{K}) \) is an interpolating family for \( C_0(X; \mathbb{K}) \).

**Proof.** Let \( S = \{x_1, \ldots, x_n\} \) be a subset of \( X \) and \( G \) be an uniformly dense vector subspace of \( C_0(X; \mathbb{K}) \). Consider the following continuous linear mapping
\[
T : C_0(X; \mathbb{K}) \rightarrow \mathbb{R}^n
\]
\[
f \mapsto (f(x_1), \ldots, f(x_n)).
\]
Note that \( T(G) \) is closed because it is a vector subspace of \( \mathbb{R}^n \). Then by density of \( G \) and continuity of \( T \), it follows that
\[
T(C_0(X; \mathbb{K})) = T(G) \subset \overline{T(G)} = T(G). \tag*{□}
\]
Therefore, for any \( f \in C_0(X; \mathbb{K}) \), there exists \( g \in G \) such that \( (f(x_1), \ldots, f(x_n)) = (g(x_1), \ldots, g(x_n)) \).

A subset \( M \) of \( C_0(X; \mathbb{K}) \) is dense-lineable or algebraically generic if \( M \cup \{0\} \) contains a vector space dense in \( C_0(X; \mathbb{K}) \). For more information, see [1].

**Corollary 2.1.** If \( M \) is a dense-lineable subset of \( C_0(X; \mathbb{K}) \), then \( M \cup \{0\} \) has the SAI property. In particular, all dense vector subspaces of \( C_0(X; \mathbb{K}) \) have the SAI property.

**Proof.** Since \( M \cup \{0\} \) contains a vector space \( A \) dense in \( C_0(X; \mathbb{K}) \), it follows from Lemma 2.2 that \( A \) is an interpolating family for \( C_0(X; \mathbb{K}) \). Moreover,
\[
(A \otimes \mathbb{K}) + A \subset A.
\]
Then, by Theorem 2.2, it follows that \( A \) has the SAI property. Since \( A \subset M \cup \{0\} \) we conclude that \( M \cup \{0\} \) has the SAI property. \( \square \)

The last corollary can also be proved by using Deutsch’s result [3].

In order to give a criterion to identify vector subspaces of \( C_0(X; E) \) which have the SAI property, we need the next two results.
Proposition 2.1. The vector subspace \( C_0(X; \mathbb{K}) \otimes E \) is uniformly dense in \( C_0(X; E) \).

Proof. It follows from Corollary 6.4 [5] that \( C_c(X; \mathbb{K}) \otimes E \) is uniformly dense in \( C_0(X; E) \). Since

\[ C_c(X; \mathbb{K}) \otimes E \subset C_0(X; \mathbb{K}) \otimes E \subset C_0(X; E), \]

we conclude that \( C_0(X; \mathbb{K}) \otimes E \) is uniformly dense in \( C_0(X; E) \).

Lemma 2.3. If \( A \) is an uniformly dense subset of \( C_0(X; \mathbb{K}) \), then \( A \otimes E \) is uniformly dense in \( C_0(X; E) \).

Proof. By Proposition 2.1, \( C_0(X; \mathbb{K}) \otimes E \) is uniformly dense in \( C_0(X; E) \). Then given \( f \in C_0(X; E) \) and \( \varepsilon > 0 \), there exists \( g \in C_0(X; \mathbb{K}) \otimes E \), say \( g(x) = \sum_{j=1}^{n} \psi_j(x)v_j, \psi_j \in C_0(X; \mathbb{K}), v_j \in E, j = 1, \ldots, n \), such that \( \|f - g\| < \varepsilon/2 \).

Since \( A \) is uniformly dense in \( C_0(X; \mathbb{K}) \), there exists \( a_j \in A \) such that

\[ \|\psi_j - a_j\| < \frac{\varepsilon}{2(\sum_{j=1}^{n} \|v_j\| + 1)}. \]

The function \( h := \sum_{j=1}^{n} a_j v_j \in A \otimes E \). Moreover,

\[
\|f - h\| \leq \|f - g\| + \|g - h\| < \frac{\varepsilon}{2} + \sum_{j=1}^{n} \|\psi_j - a_j\| \|v_j\| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2(\sum_{j=1}^{n} \|v_j\| + 1)} \sum_{j=1}^{n} \|v_j\| < \varepsilon. \]

We obtain the following result.

Theorem 2.3. If \( A \) is an uniformly dense vector subspace of \( C_0(X; \mathbb{K}) \) and \( B \) is a vector subspace of \( C_0(X; E) \) such that \( A \otimes E \subset B \), then \( B \) has the SAI property.

Proof. It follows from Lemma 2.2, Lemma 2.3 and Theorem 2.2.

Corollary 2.2 (Prolla [7], Theorem 7). Let \( X \) be a compact Hausdorff space and \( B \subset C(X; E) \) a vector subspace such that \( A := \{ \phi \circ g : \phi \in E^*, g \in B \} \) is uniformly dense in \( C(X; \mathbb{K}) \) and \( A \otimes E \subset B \). Then \( B \) has the SAI property.

Example 2.2. Let \( (X, d) \) be a compact metric space. A function \( f : X \mapsto E \) is called Lipschitzian if there is some constant \( K_f > 0 \) such that

\[ \|f(x) - f(y)\| \leq K_f d(x, y) \]

for every \( x, y \in X \). We denote by \( Lip(X; E) \) the subset of \( C(X; E) \) of all such functions. By Theorem 9 [6], the vector subspace \( Lip(X; \mathbb{K}) \) is uniformly
dense in $C(X; \mathbb{K})$. For any $f_1, \ldots, f_n \in \text{Lip}(X; \mathbb{K})$, $v_1, \ldots, v_n \in E$, there exist constants $k_1, \ldots, k_n > 0$ such that
\[
\left\| \sum_{j=1}^{n} f_j(x)v_j - f_j(y)v_j \right\| \leq \sum_{j=1}^{n} |f_j(x) - f_j(y)||v_j| \leq \left( \sum_{j=1}^{n} k_j \|v_j\| \right) d(x, y)
\]
for every $x, y \in X$. Hence, $\text{Lip}(X; \mathbb{K}) \otimes E \subset \text{Lip}(X; E)$. Then, by Theorem 2.3, $\text{Lip}(X; E)$ has the SAI property.

**Example 2.3.** Since $C_c(X; \mathbb{K})$ is uniformly dense in $C_0(X; \mathbb{K})$ (see Nachbin [4], p. 64) and $C_c(X; \mathbb{K}) \otimes E \subset C_c(X; E)$, it follows from Theorem 2.3 that $C_c(X; E)$ has the SAI property.

**Example 2.4.** Let $X_i$ be a locally compact Hausdorff space for $i = 1, \ldots, n$ and $X = X_1 \times \cdots \times X_n$.

Let $A$ be the set of all finite sums of functions of the form
\[
x = (x_1, \ldots, x_n) \mapsto f(x) = g_1(x_1) \cdots g_n(x_n),
\]
where $g_j \in C_0(X_j; \mathbb{K})$ for $j = 1, \ldots, n$. By the weighted Dieudonné theorem ([4] Theorem 1 p. 68), $A$ is an uniformly dense vector subspace of $C_0(X; \mathbb{K})$. From Theorem 2.3, $A \otimes E$ has the SAI property.

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**References**


