VALUE SHARING RESULTS OF A MEROMORPHIC FUNCTION \( f(z) \) AND \( f(qz) \)

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Abstract. In this paper, we investigate sharing value problems related to a meromorphic function \( f(z) \) and \( f(qz) \), where \( q \) is a non-zero constant. It is shown, for instance, that if \( f(z) \) is zero-order and shares two values CM and one value IM with \( f(qz) \), then \( f(z) = f(qz) \).

1. Introduction

In what follows, a meromorphic function will mean meromorphic in the whole complex plane. We say that two meromorphic functions \( f \) and \( g \) share a value \( a \) \( \in \mathbb{C} \cup \{\infty\} \) IM (ignoring multiplicities) when \( f - a \) and \( g - a \) have the same zeros. If \( f - a \) and \( g - a \) have the same zeros with the same multiplicities, then we say that \( f \) and \( g \) share the value \( a \) CM (counting multiplicities). We assume that the reader is familiar with the standard symbols and fundamental results of Nevanlinna theory, as found in [5, 10].

As usual, by \( S(r, f) \) we denote any quantity satisfying \( S(r, f) = o(T(r, f)) \) for all \( r \) outside of a possible exceptional set of finite linear measure. In addition, denote by \( S(f) \) the family of all meromorphic functions \( a(z) \) that satisfy \( T(r, a) = o(T(r, f)) \), for \( r \to \infty \) outside a possible exceptional set of finite logarithmic measure. In particular, we denote by \( S_1(r, f) \) any quality satisfying \( S_1(r, f) = o(T(r, f)) \) for all \( r \) on a set of logarithmic density 1.

The classical results due to Nevanlinna [9] in the uniqueness theory of meromorphic functions are the five-point, resp. four-point, theorems:

**Theorem A.** If two meromorphic functions \( f \) and \( g \) share five distinct values \( a_1, a_2, a_3, a_4, a_5 \in \mathbb{C} \cup \{\infty\} \) IM, then \( f \equiv g \).

**Theorem B.** If two meromorphic functions \( f \) and \( g \) share four distinct values \( a_1, a_2, a_3, a_4 \in \mathbb{C} \cup \{\infty\} \) CM, then \( f \equiv g \) or \( f \equiv T \circ g \), where \( T \) is a Möbius transformation.

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It is well-known that 4 CM can not be improved to 4 IM, see [3]. Further, Gundersen [4, Theorem 1] has improved the assumption 4 CM to 2 CM+2 IM, while 1 CM+3 IM is still an open problem.

In recent papers [6], Heittokangas et al. started to consider the uniqueness of a finite order meromorphic function sharing values with its shift. They concluded that:

**Theorem C.** Let $f$ be a meromorphic function of finite order, let $c \in \mathbb{C}$, and let $a_1, a_2, a_3 \in S(f) \cup \{\infty\}$ be three distinct periodic functions with period $c$. If $f(z)$ and $f(z+c)$ share $a_1, a_2$ CM and $a_3$ IM, then $f(z) = f(z+c)$ for all $z \in \mathbb{C}$.

Closely related to difference expressions are $q$-difference expressions, where the usual shift $f(z+c)$ of a meromorphic function will be replaced by the $q$-shift $f(qz)$, $q \in \mathbb{C} \setminus \{0\}$. The Nevanlinna theory of $q$-difference expressions and its applications to $q$-difference equations have recently been considered, see [1, 7]. In addition, some results about solutions of zero-order for complex $q$-difference equations, can be found in the introduction in [1].

A natural question is: what is the uniqueness result in the case when $f(z)$ shares values with $f(qz)$ for a zero-order meromorphic function $f(z)$. Corresponding to this question, we get the following result:

**Theorem 1.1.** Let $f$ be a zero-order meromorphic function, and $q \in \mathbb{C} \setminus \{0\}$, and let $a_1, a_2, a_3 \in \mathbb{C} \cup \{\infty\}$ be two distinct values. If $f(z)$ and $f(qz)$ share $a_1$ and $a_2$ IM, then $f(z) = f(qz)$.

**Remark 1.** Indeed, from the proof of Theorem 1.1, we know the assumption that share $a_3$ IM can be replaced by one of the following assumptions:

1. if there exists a point $z_0$ such that $f(z_0) = f(qz_0) = a_3$; or
2. if $a_3$ is a Picard exceptional value of $f$.

However, we give Theorem 1.1 just as a $q$-difference analogue of Theorem C.

If $f$ is an entire function in Theorem 1.1, then the conclusion will be improved.

**Theorem 1.2.** Let $f$ be a zero-order entire function, $q \in \mathbb{C} \setminus \{0\}$, and let $a_1, a_2 \in \mathbb{C}$ be two distinct values. If $f(z)$ and $f(qz)$ share $a_1$ and $a_2$ IM, then $f(z) = f(qz)$.

**Remark 2.** As a corollary of Theorem 1.1, we just know that $f(z) = f(qz)$ provided that $f(z)$ and $f(qz)$ share values under the condition that “1 CM + 1 IM”.

In the following, we consider the value sharing problems relative to $F(z) = f^n$ and $F(qz)$, and we obtain the following results:
Theorem 1.3. Let $f$ be a zero-order meromorphic function, and $q \in \mathbb{C} \setminus \{0\}$, $n \geq 4$ be an integer, and let $F = f^n$. If $F(z)$ and $F(qz)$ share $a \in \mathbb{C} \setminus \{0\}$ and $\infty$ CM, then $f(z) = tf(qz)$ for a constant $t$ that satisfies $t^n = 1$.

Remark 3. Theorem 1.3 is not true, if $a = 0$. This can be seen by considering $f(z) = z$ and $f(\frac{1}{2}z) = \frac{1}{2}z$. Then $f(z)^n$ and $f(\frac{1}{2}z)^n$ share 0 and $\infty$ CM, however, $f(z) = 2f(\frac{1}{2}z)$, $2^n \neq 1$, where $n$ is a positive integer.

Corollary 1.4. Let $f$ be a zero-order entire function, and $q \in \mathbb{C} \setminus \{0\}$, $n \geq 3$ be an integer, and let $F = f^n$. If $F(z)$ and $F(qz)$ share 1 CM, then $f(z) = tf(qz)$ for a constant $t$ that satisfies $t^n = 1$.

Corollary 1.5. Let $f$ be a zero-order meromorphic function, and $q \in \mathbb{C} \setminus \{0\}$, $n \geq 4$ be an integer, and let $F = f^n$. If $F(z)$ and $F(qz)$ share 0 and 1 CM, then $f(z) = tf(qz)$ for a constant $t$ that satisfies $t^n = 1$.

Remark 4. By simply calculations, we get $|q| = 1$ in above results. And some ideas of this paper are from [8].

2. Some lemmas

Lemma 2.1 ([1, Theorem 1.1]). Let $f$ be a zero-order meromorphic function, and $q \in \mathbb{C} \setminus \{0\}$. Then

$$m \left( r, \frac{f(qz)}{f(z)} \right) = S_1(r, f).$$

Lemma 2.2 ([1, Theorem 2.1]). Let $f$ be a zero-order meromorphic function, let $q \in \mathbb{C} \setminus \{0, 1\}$, and let $a_1, \ldots, a_p \in \mathbb{C}$, $p \geq 2$, be distinct points. Then

$$m(r, f) + \sum_{k=1}^{p} m \left( r, \frac{1}{f - a_k} \right) \leq 2T(r, f) - N_{\text{pair}}(r, f) + S_1(r, f),$$

where

$$N_{\text{pair}}(r, f) = 2N(r, f) - N(r, \Delta_q f) + N \left( r, \frac{1}{\Delta_q f} \right)$$

and $\Delta_q f = f(qz) - f(z)$.

Lemma 2.3 ([11, Theorem 1.1 and Theorem 1.3]). Let $f$ be a zero-order meromorphic function, and $q \in \mathbb{C} \setminus \{0\}$. Then

(2.1) $$T(r, f(qz)) = (1 + o(1))T(r, f(z))$$

and

(2.2) $$N(r, f(qz)) = (1 + o(1))N(r, f(z))$$

on a set of lower logarithmic density 1.

Remark. From Remark 1 after Theorem 1.1 in [11], we know that $f(z)$ and $f(qz)$ are simultaneously of order zero.
**Lemma 2.4** ([10, Theorem 2.17]). Let \( f \) and \( g \) be meromorphic functions, and the order of \( f \) and \( g \) is less than 1. If \( f \) and \( g \) share 0 and \( \infty \) CM, then \( f \equiv kg \), where \( k \) is a non-zero constant.

### 3. Proof of Theorem 1.1

If \( q = 1 \), then the conclusion holds. Now we consider the case that \( q \neq 1 \). Suppose first that \( a_1, a_2, a_3 \in \mathbb{C} \). Denote

\[
g(z) = \frac{f(z) - a_1 a_3 - a_2}{f(z) - a_2 a_3 - a_1},
\]

then

\[
g(qz) = \frac{f(qz) - a_1 a_3 - a_2}{f(qz) - a_2 a_3 - a_1}.
\]

From the assumption of Theorem 1.1, we know \( g(z) \) and \( g(qz) \) share 0, \( \infty \) CM.

Suppose first that 1 is not a Picard exceptional value of \( g(z) \) and \( g(qz) \). Assume that \( g(z) \neq g(qz) \), and from Lemma 2.2, we obtain

\[
m(r, g) + m \left( r, \frac{1}{g} \right) + m \left( r, \frac{1}{g - 1} \right) \\
\leq 2T(r, g) - 2N(r, g) + N(r, \Delta g) - N \left( r, \frac{1}{\Delta g} \right) + S_1(r, g),
\]

and so

\[
T(r, g) \leq N(r, g) + N \left( r, \frac{1}{g} \right) + N \left( r, \frac{1}{g - 1} \right) + N(r, g(qz)) \\
+ N(r, g) - 2N(r, g) - N \left( r, \frac{1}{\Delta g} \right) + S_1(r, g).
\]

Since 1 is a Picard exceptional value of \( g(z) \), by combining (2.2) and (3.1), it follows that

\[
T(r, g) \leq N(r, g) + N \left( r, \frac{1}{g} \right) - N \left( r, \frac{1}{\Delta g} \right) + S_1(r, g).
\]

Since \( g(z) \) and \( g(qz) \) share 0, \( \infty \) CM, we get

\[
N(r, g) + N \left( r, \frac{1}{g} \right) \leq N \left( r, \frac{1}{\Delta g} \right).
\]

From (3.2) and (3.3), we conclude that

\[
T(r, g) = S_1(r, g),
\]

which is impossible. Hence, we conclude that \( f(z) = f(qz) \).
It remains to consider the case that one of $a_j (j = 1, 2, 3)$ is infinite. Without loss of generality, we suppose that $a_1 = \infty$, while $a_2, a_3 \in \mathbb{C}$. Take $d \in \mathbb{C} \setminus \{a_2, a_3\}$ and denote $h(z) = \frac{1}{f(z) - d}$, $b_2 = \frac{1}{a_2 - d}$ and $b_3 = \frac{1}{a_3 - d}$. Then $b_2, b_3 \in \mathbb{C} \setminus \{0\}$ are two distinct values. Hence $h(z)$ and $h(qz)$ share 0, $b_2$ CM and $b_3$ IM. By the above argument, we get $h(z) = h(qz)$, and therefore $f(z) = f(qz)$.

4. Proof of Theorem 1.2

From the fact that a non-constant meromorphic function of zero-order can have at most one Picard exceptional value (see, e.g., [2, p. 114]), we obtain that $N(r, \frac{1}{f(z) - a_1}) \neq 0$ and $N(r, \frac{1}{f(z) - a_2}) \neq 0$. Let

$$F(z) = \frac{f(z) - a_1}{a_2 - a_1} \quad \text{and} \quad F(qz) = \frac{f(qz) - a_1}{a_2 - a_1}$$

Then $F(z)$ and $F(qz)$ share 0 and 1 IM. Clearly, neither 0 nor 1 is a Picard exceptional value of $F(z)$. From Lemma 2.3, we obtain that

$$T(r, F(qz)) = T(r, F(z)) + S_1(r, F).$$

Denote

$$V(z) = \frac{F'(z)(F(qz) - F(z))}{F(z)(F(z) - 1)}.$$ 

Lemma 2.1 and the lemma on logarithmic derivative yield that $m(r, V) = S_1(r, F)$. From (4.3), we know the poles of $V(z)$ are at the zeros and 1-points of $F(z)$. Since $F(z)$ and $F(z + c)$ share 0 and 1, we get $N(r, V) = S(r, F)$. Therefore, $T(r, V) = S_1(r, F)$.

Case 1. If $V \neq 0$, then $F(z) \neq F(qz)$. From (4.3) and Lemma 2.1, we have

$$N \left( r, \frac{1}{F(z)} \right) + N \left( r, \frac{1}{F(z) - 1} \right)$$

$$= N \left( r, \frac{F'(z)}{F(z)(F(z) - 1)} \right) + S(r, F)$$

$$= N \left( r, \frac{V}{F(qz) - F(z)} \right) + S(r, F)$$

$$\leq T(r, F(qz) - F(z)) + S_1(r, F) = m(r, F(qz) - F(z)) + S_1(r, F)$$

$$\leq m \left( r, \frac{F(qz) - F(z)}{F(z)} \right) + m(r, F(z)) + S_1(r, F)$$

$$\leq T(r, F) + S_1(r, F).$$

According to second main theorem and above inequality, we get

$$T(r, F) = N \left( r, \frac{1}{F} \right) + N \left( r, \frac{1}{F - 1} \right) + S_1(r, F).$$
Now we define
\begin{equation}
U(z) = \frac{F'(qz)(F(qz) - F(z))}{F(qz)(F(qz) - 1)}.
\end{equation}
Using the same argument as above, we know that \( T(r, U) = S_1(r, F(qz)) = S_1(r, F(z)) \).

In what follows, we denote \( S_{f\sim g(m,n)}(a) \) for the set of those points \( z \in \mathbb{C} \) such that \( z \) is an \( a \)-point of \( f \) with multiplicity \( m \) and an \( a \)-point of \( g \) with multiplicity \( n \). Let \( N_{(m,n)}(r, \frac{1}{F(qz)}) \) and \( N_{(m,n)}(r, \frac{1}{F(z)}) \) denote the counting function and reduced counting function of \( f \) with respect to the set \( S_{f\sim g(m,n)}(a) \), respectively.

For any point \( z_0 \in S_{F(z)\sim F(qz)(m,n)}(0) \), we have \( mn \neq 0 \), since 0 is not a Picard exceptional value of \( F(z) \) as we discuss above. From (4.3), (4.5) and the Taylor expansion of \( F(z) \) and \( F(qz) \) at \( z_0 \), by calculating carefully, we get
\begin{equation}
-V(z_0) = m \left( \frac{F'(qz_0) - F'(z_0)}{n} \right),
\end{equation}
and
\begin{equation}
-U(z_0) = n \left( \frac{F'(qz_0) - F'(z_0)}{m} \right).
\end{equation}
From (4.6) and (4.7), we know \( nV(z_0) = mU(z_0) \).

If \( nV = mU \), then we deduce that
\begin{equation}
n \left( \frac{F'(z)}{F(z) - 1} - \frac{F'(z)}{F(z)} \right) = m \left( \frac{F'(qz)}{F(qz) - 1} - \frac{F'(qz)}{F(qz)} \right),
\end{equation}
which implies that
\begin{equation}
\left( \frac{F - 1}{F} \right)^n = b \left( \frac{F(qz) - 1}{F(qz)} \right)^m,
\end{equation}
where \( b \) is a non-zero constant. If \( m \neq n \), then we get from above equality and (4.2) that
\begin{equation}
n T(r, F(z)) = m T(r, F(qz)) + S_1(r, F) = m T(r, F(z)) + S_1(r, F),
\end{equation}
which is a contradiction. If \( m = n \), then we get
\begin{equation}
\left( \frac{F'(z)}{F(z) - 1} - \frac{F'(z)}{F(z)} \right) = \left( \frac{F'(qz)}{F(qz) - 1} - \frac{F'(qz)}{F(qz)} \right).
\end{equation}
Hence
\begin{equation}
\frac{F(z) - 1}{F(z)} = d \frac{F(qz) - 1}{F(qz)},
\end{equation}
where \( d \) is a non-zero constant. If \( d = 1 \), then we obtain \( F(z) = F(qz) \), which contradicts the assumption of Case 1. It remains to consider the case that
It follows from (4.8) that
\[ \frac{d - 1}{d} \frac{F(z)}{F(z)} + \frac{1}{d} = \frac{1}{F(qz)}. \]
Since \( N(r, F(z)) = N(r, F(qz)) = 0 \), we get \( N(r, \frac{1}{F(z)} - \frac{1}{F(qz)}) = 0 \). Clearly, 
\( \frac{1}{F(z)} \neq 0 \) and \( \frac{1}{F(qz)} \neq 1 \), then apply the second main theorem, resulting in
\[ 2T(r, F) \leq \mathcal{N} \left( r, \frac{1}{F} \right) + \mathcal{N} \left( r, \frac{1}{F - 1} \right) + S(r, F), \]
which contradicts (4.4).

Hence \( nV \neq mU \). By the above argument, we know any point \( z_0 \in S_{F(z)} \sim F(qz)(m,n)(0) \) satisfies that \( nV(z_0) = mU(z_0) \). Therefore,
\[ \mathcal{N}_{(m,n)} \left( r, \frac{1}{F} \right) \leq N \left( r, \frac{1}{nU - mV} \right) = S_1(r, F). \]
Using the same reason, we get
\[ \mathcal{N}_{(m,n)} \left( r, \frac{1}{F - 1} \right) \leq N \left( r, \frac{1}{nU - mV} \right) = S_1(r, F). \]
It follows that
\[ (4.9) \quad \mathcal{N}_{(m,n)} \left( r, \frac{1}{F} \right) + \mathcal{N}_{(m,n)} \left( r, \frac{1}{F - 1} \right) = S_1(r, F). \]
From Lemma 2.3, (4.4) and (4.9), we obtain that
\[ T(r, F) = \mathcal{N} \left( r, \frac{1}{F} \right) + \mathcal{N} \left( r, \frac{1}{F - 1} \right) + S_1(r, F) \]
\[ = \sum_{m,n \geq 5} (\mathcal{N}_{(m,n)}(r, \frac{1}{F}) + \mathcal{N}_{(m,n)}(r, \frac{1}{F - 1})) + S_1(r, F) \]
\[ = \sum_{m+n \geq 5} (\mathcal{N}_{(m,n)}(r, \frac{1}{F}) + \mathcal{N}_{(m,n)}(r, \frac{1}{F - 1})) + S_1(r, F) \]
\[ \leq \frac{1}{5} \sum_{m+n \geq 5} (N_{(m,n)}(r, \frac{1}{F}) + N_{(m,n)}(r, \frac{1}{F - 1})) \]
\[ + N_{(m,n)}(r, \frac{1}{F(qz)}) + N_{(m,n)}(r, \frac{1}{F(qz) - 1})) + S_1(r, F) \]
\[ \leq \frac{2}{5} T(r, F) + \frac{2}{5} T(r, F(qz)) + S_1(r, F) \]
\[ = \frac{4}{5} T(r, F) + S_1(r, F), \]
which is a contradiction.

Case 2. If \( V = 0 \), then \( F(z) = F(qz) \). Clearly, \( f(z) = f(qz) \). This completes the proof of Theorem 1.2.
5. Proof of Theorem 1.3

Let $G(z) = \frac{f(z)}{z}$, then we know $G(z)$ and $G(qz)$ share 1 and $\infty$ CM, and since the order of $f$ is zero, it follows that

$$\frac{G(qz) - 1}{G(z) - 1} = \tau,$$

where $\tau$ is a non-zero constant. Rewriting the above equation, gives

$$G(z) + \frac{1}{\tau} - 1 = \frac{G(qz)}{\tau}. \quad (5.1)$$

Assume that $\tau \neq 1$. Noting (2.2) and (5.1), the second main theorem yields

$$nT(r, f(z)) = T(r, G(z)) \leq N(r, G(z)) + N \left( r, \frac{1}{G(z)} \right)$$

$$+ N \left( r, \frac{1}{G(z) - 1 + \frac{1}{\tau}} \right) + S(r, f)$$

$$\leq N(r, f(z)) + N \left( r, \frac{1}{f(z)} \right) + N \left( r, \frac{1}{f(qz)} \right) + S(r, f)$$

$$\leq N(r, f(z)) + 2N \left( r, \frac{1}{f(z)} \right) + S_1(r, f)$$

$$\leq 3T(r, f(z)) + S_1(r, f), \quad (5.2)$$

which contradicts the assumption that $n \geq 4$. Hence, we get $\tau = 1$, which implies that $G(z) = G(qz)$, that is, $f^n(z) = f^n(qz)$. So we have $f(z) = tf(qz)$ for a constant $t$ with $t^n = 1$.

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