NORMAL FAMILIES AND SHARED HOLOMORPHIC FUNCTIONS

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Abstract. In this paper, we study the problem of normal families and deduce some results, which improve and generalize several related theorems obtained by Pang [7], Fang and Xu [3], Lu, Xu, and Yi [6]. Meanwhile, some examples are given to show the sharpness of our results.

1. Introduction and main results

Let $f$, $g$ and $a$ be three holomorphic functions in a domain $D \subset \mathbb{C}$. Here, we denote the condition that $f(z) - a(z) = 0$ implies $g(z) - a(z) = 0$ by $f(z) = a(z) \Rightarrow g(z) = a(z)$. If $f(z) = a(z) \Rightarrow g(z) = a(z)$ and $g(z) = a(z) \Rightarrow f(z) = a(z)$, we write $f(z) = a(z) \iff g(z) = a(z)$. In what follows, we assume that the reader is familiar with the basic notations and results in Nevanlinna value distribution theory (see, [14, 15]).

One important subject in the theory of normal family is to find sufficient conditions for normality. According to Bloch’s principle, a lot of normality criteria have been obtained by starting from Picard type theorems (see, [1, 2, 4, 8, 9, 10]). The first attempt was made by Schwick [11] in 1992.

In a different way, Pang [7] and Xu [12] proved the following result.

**Theorem A.** Let $F$ be a family of holomorphic functions in a domain $D$, and $a$, $b$ be distinct finite complex numbers. If $f(z) = a \iff f'(z) = a$ and $f(z) = b \iff f'(z) = b$ in $D$ for every $f \in F$, then $F$ is normal in $D$.

The following result was obtained by Fang and Xu [3] in 2002. They replaced the condition $f(z) = b \iff f'(z) = b$ by $f(z) = b \Rightarrow f'(z) = b$.

**Theorem B.** Let $F$ be a family of holomorphic functions in a domain $D$, and $a$, $b$ be distinct finite complex numbers. If $f(z) = a \iff f'(z) = a$ and $f(z) = b \Rightarrow f'(z) = b$ in $D$ for every $f \in F$, then $F$ is normal in $D$.

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In 2009, Lü, Xu and Yi [6] improved Theorem B. They pointed out that Theorem B still holds if the condition \( f(z) = a \Leftrightarrow f'(z) = a \) is weakened to \( f(z) = a \Rightarrow f'(z) = a \).

**Theorem C.** Let \( \mathcal{F} \) be a family of holomorphic functions in a domain \( D \), let \( a \) and \( b \) be two distinct complex numbers. If for all \( f \in \mathcal{F} \), \( f(z) = a \Rightarrow f'(z) = a \) and \( f(z) = b \Rightarrow f'(z) = b \), then \( \mathcal{F} \) is normal in \( D \).

By studying the above theorems, we naturally ask what could happen if \( f' \) is replaced by a linear differential polynomial in \( f \) with holomorphic coefficients?

In order to state our main results, we need the notation

\[
L[f] = a_0 f' + a_1 f
\]

for a linear differential polynomial in \( f \), where \( a_0, a_1 \) are holomorphic functions with \( a_0(z) \neq 0 \).

In the paper, by considering the above question, we obtain a result as follows, which is an improvement of the previous theorems.

**Theorem 1.1.** Let \( \mathcal{F} \) be a family of holomorphic functions in a domain \( D \), let \( L[f] \) be defined as in (1.1), and let \( a, b \) be two holomorphic functions in \( D \). For each \( f \in \mathcal{F} \), if

1. \( a \neq b \);
2. \( a - a_1 a - a_0 a' \neq 0 \);
3. \( a - a_1 a - a_0 a' \) and \( b - a_1 b - a_0 b' \) have no common zeros;
4. \( f(z) = a(z) \Rightarrow L[f](z) = a(z) \) and \( f(z) = b(z) \Rightarrow L[f](z) = b(z) \),

then \( \mathcal{F} \) is normal in \( D \).

**Remark 1.** Clearly, Theorem 1.1 is an improvement of the previous results. The following example shows that the condition (3) is necessary in Theorem 1.1.

**Example 1.** Let \( D = \{ z : |z| < 1 \} \) and \( k \geq 2 \) be an integer, let \( a(z) = z^k \) and \( b(z) = 2z^k \), and let

\[
\mathcal{F} = \{ f_n(z) = nz^k : n = 3, 4, \ldots ; z \in D \}
\]

Suppose that \( a_0 = 1 \) and \( a_1 = 0 \). Then \( L[f_n] = f_n' \). For each \( f_n \in \mathcal{F} \), we have that \( f_n(z) = a(z) \Rightarrow L[f_n](z) = a(z) \) and \( f_n(z) = b(z) \Rightarrow L[f_n](z) = b(z) \). Moreover,

\[
a(z) - a_1(z)a(z) - a_0(z)a'(z) = a(z) - a'(z) = z^{k-1}(z - k)
\]

and

\[
b(z) - a_1(z)b(z) - a_0(z)b'(z) = b(z) - b'(z) = 2z^{k-1}(z - k).
\]

So \( a - a_1 a - a_0 a' \) and \( b - a_1 b - a_0 b' \) have a common zero \( z = 0 \). Obviously, \( \mathcal{F} \) is not normal in \( D \).

Suppose that \( a_0 = 1 \) and \( a_1 = 0 \) in (1.1). Then the following corollary is an immediate consequence of Theorem 1.1.
Corollary 1.2. Let $F$ be a family of holomorphic functions in a domain $D$, and let $a$, $b$ be two holomorphic functions in $D$. For each $f \in F$, if
(1) $a \neq b$ and $a - a' \neq 0$;
(2) $a - a'$ and $b - b'$ have no common zeros;
(3) $f(z) = a(z) \Rightarrow f'(z) = a(z)$ and $f(z) = b(z) \Rightarrow f'(z) = b(z)$,
then $F$ is normal in $D$.

Remark 2. The following example shows that Corollary 1.2 is not valid for a family of meromorphic functions.

Example 2. Let $D = \{ z : |z| < 1 \}$, let $a = 1$ and $b = 0$, and let
$$F = \{ f_n(z) = \frac{(2nz - 1)^{2n}}{(2nz - 1)^{2n} - 1} : n = 1, 2, \ldots ; z \in D \}.$$  
Clearly, for each $f_n \in F$, we have that $f_n(z) = 0 \Rightarrow f_n'(z) = 0$, $f_n(z) \neq 1$ and $a(z) \neq b(z)$. But $f_n'(0) = 4n^2 \to \infty$ as $n \to \infty$. It follows from Marty criterion that $F$ is not normal in $D$.

Remark 3. Recently, Xu and Qiu [13] derived a similar result to Theorem 1.1. The proof of our result has roots in their work and [5]. Some of the above examples can be found in [13].

2. The lemma

To prove our result, we need the well-known Zalcman lemma. For the proof of our result, Zalcman lemma is essential.

Zalcman Lemma ([16]). Let $F$ be a family of functions holomorphic in a domain $D$. If $F$ is not normal at $z_0 \in D$, then there exist
(a) points $z_n \in D$, $z_n \to z_0$;
(b) functions $f_n \in F$, and
(c) positive number $\rho_n \to 0$ such that $f_n(z_n + \rho_n \xi) = g_n(\xi) \to g(\xi)$ locally uniformly, where $g$ is a non-constant entire function.

3. The proof of Theorem 1.1

Since normality is a local property, it is sufficient to show that $F$ is normal at $\forall z_0 \in D$. We now distinguish between two cases.

Case 1. $a(z_0) \neq b(z_0)$ and $a - a_1 a - a_0 a'|_{z=z_0} \neq 0$.

Suppose, to the contrary, that $F$ is not normal at $z_0$. By Zalcman lemma, there exist a sequence of functions $f_n \in F$, a sequence of complex numbers $z_n \to z_0$ and a sequence of positive numbers $\rho_n \to 0$, such that
$$g_n(\xi) = f_n(z_n + \rho_n \xi) \to g(\xi)$$
converges locally uniformly in $\mathbb{C}$, where $g$ is a non-constant entire function. Noting that $\rho_n \to 0$, $z_n \to z_0$ and (3.1), we deduce that
$$f_n(z_n + \rho_n \xi) - a(z_n + \rho_n \xi) \to g(\xi) - a(z_0)$$
and
\begin{equation}
(3.3) \quad f_n(z_n + \rho_n \xi) - b(z_n + \rho_n \xi) \to g(\xi) - b(z_0).
\end{equation}

It follows from (3.1) that
\begin{equation}
(3.4) \quad g_n'(\xi) = \rho_n f_n'(z_n + \rho_n \xi) \to g'(\xi).
\end{equation}

Combining (3.1), (3.4) and $a_0(z) \neq 0$ yields that
\begin{equation}
(3.5) \quad \rho_n \frac{L[f_n](z_n + \rho_n \xi)}{a_0(z_n + \rho_n \xi)} = \rho_n f_n'(z_n + \rho_n \xi) + \rho_n \frac{a_1(z_n + \rho_n \xi) f_n(z_n + \rho_n \xi)}{a_0(z_n + \rho_n \xi)} \to g'(\xi).
\end{equation}

Next, we will prove that $g - a(z_0)$ and $g - b(z_0)$ have only multiple zeros.

Suppose that $g(\eta_0) - a(z_0) = 0$. Noting that $g - a(z_0) \neq 0$, Hurwitz’s theorem and (3.2), there exists a sequence $\eta_n \to \eta_0$ such that (for $n$ large enough)
\begin{equation}
(3.6) \quad f_n(z_n + \rho_n \eta_n) = a(z_n + \rho_n \eta_n).
\end{equation}

Then, the assumption $f(z) = a(z) \Rightarrow L[f](z) = a(z)$ leads to $L[f_n](z_n + \rho_n \eta_n) = a(z_n + \rho_n \eta_n)$. Furthermore, it follows from (3.5) that
\begin{equation}
(3.7) \quad g'(\eta_0) = \lim_{n \to \infty} \rho_n \frac{L[f_n](z_n + a_n \eta_n)}{a_0(z_n + a_n \eta_n)} = \lim_{n \to \infty} \rho_n \frac{a(z_n + a_n \eta_n)}{a_0(z_n + a_n \eta_n)} = 0,
\end{equation}

which implies that $g - a(z_0)$ has only multiple zeros. Similarly, we can derive that $g - b(z_0)$ has only multiple zeros.

We claim that $g(\xi) \neq a(z_0)$, which is proved as follows.

Suppose that $\xi_0$ is a zero of $g - a(z_0)$ with multiplicity $m$. Then $g^{(m)}(\xi_0) \neq 0$. Clearly, $m \geq 2$. So there exists a positive number $\delta_1$ such that
\begin{equation}
(3.8) \quad g(\xi) \neq 0, \quad g'(\xi) \neq 0, \quad g^{(m)}(\xi) \neq 0
\end{equation}
in $D_{\delta_1}^\circ = \{ z : 0 < |\xi - \xi_0| < \delta_1 \}$.

Noting that $g \neq a(z_0)$, Rouché theorem and (3.2), there exist $\xi_{n,j}$ ($j = 1, 2, \ldots, m$) on $D_{\delta_1/2} = \{ \xi : |\xi - \xi_0| < \delta_1/2 \}$ such that
\begin{equation}
(3.9) \quad f_n(z_n + \rho_n \xi_{n,j}) = a(z_n + \rho_n \xi_{n,j}).
\end{equation}

Then, we have
\begin{equation}
(3.10) \quad L[f_n](z_n + \rho_n \xi_{n,j}) = a(z_n + \rho_n \xi_{n,j}) \quad (j = 1, 2, \ldots, m).
\end{equation}

Let $A$ be defined as
\begin{equation}
(3.11) \quad A = \frac{a - a_1 a}{a_0}.
\end{equation}

Obviously, $A$ is holomorphic in $D$. Combining (3.7), (3.8) and the form of $L[f_n]$ yields
\begin{equation}
(3.12) \quad f_n'(z_n + \rho_n \xi_{n,j}) = A(z_n + \rho_n \xi_{n,j}) \quad (j = 1, 2, \ldots, m).
\end{equation}

Set
\begin{equation}
(3.13) \quad G_n(\xi) = f_n(z_n + \rho_n \xi) - a(z_n + \rho_n \xi).
\end{equation}

Then $G_n(\xi_{n,j}) = 0$ ($j = 1, 2, \ldots, m$).
Observing that \( a - a_1a - a_0a' \bigg|_{z=z_0} \neq 0 \), we obtain (for \( n \) large enough)
\[
a - a_1a - a_0a' \bigg|_{z=z_0+\rho_n \xi_n,j} \neq 0.
\]
Furthermore, we deduce that (for \( n \) large enough)
\[
G'_n(\xi_{n,j}) = \rho_n(f'_n(z_n + \rho_n \xi_{n,j}) - a'(z_n + \rho_n \xi_{n,j}))
= \rho_n(A(z_n + \rho_n \xi_{n,j}) - a'(z_n + \rho_n \xi_{n,j}))
= \rho_n \frac{a - a_1a - a_0a'}{a_0} \bigg|_{z=z_0+\rho_n \xi_{n,j}} \neq 0,
\]
which implies that each \( \xi_{n,j} \) is a simple zero of \( G_n \). That is \( \xi_{n,j} \neq \xi_{n,i} \) (1 ≤ \( i \neq j \leq m \)).

Set
\[
K_n(\xi) = \rho_n \frac{L[f_n](z_n + \rho_n \xi) - a(z_n + \rho_n \xi)}{a_0(z_n + \rho_n \xi)}.
\]
Then
\[
(3.10) \quad K_n(\xi) \rightarrow g'(\xi)
\]
and \( K_n(\xi_{n,j}) = 0 \) (\( j = 1, 2, \ldots, m \)). From (3.6), we have
\[
\lim_{n \rightarrow \infty} \xi_{n,j} = \xi_0 \quad (j = 1, 2, \ldots, m).
\]
By (3.6), (3.10) and the fact that \( K_n(\xi) \) has \( m \) zeros \( \xi_{n,j} \) (\( j = 1, 2, \ldots, m \)) in \( D_{1/2} \), \( \xi_0 \) is a zero of \( g' \) with multiplicity \( m \), and thus \( g^{(m)}(\xi_0) = 0 \). This is a contradiction and hence, the claim is proved.

By Nevanlinna’s first and second fundamental theorems, we derive that
\[
T(r, g) \leq \overline{N}(r, \frac{1}{g-a(z_0)}) + \overline{N}(r, \frac{1}{g-b(z_0)}) + S(r, g)
\leq \frac{1}{2} \overline{N}(r, \frac{1}{g-b(z_0)}) + S(r, g) \leq \frac{1}{2} T(r, g) + S(r, g),
\]
which indicates that \( T(r, g) = S(r, g) \), a contradiction. Thus, \( \mathcal{F} \) is normal at \( z_0 \) and the proof of Case 1 is finished.

**Case 2.** \( a(z_0) = b(z_0) \) or \( a - a_1a - a_0a' \bigg|_{z=z_0} = 0 \).

Since \( a \neq b \) and \( a - a_1a - a_0a' \neq 0 \), then there exists \( r > 0 \) such that \( a(z) \neq b(z) \) and \( a(z) - a_1(z)a(z) - a_0(z)a'(z) \neq 0 \) in \( D'(z_0, r) = \{ z : 0 < |z - z_0| < r \} \subset D \).

It follows from Case 1 that \( \mathcal{F} \) is normal in \( D'(z_0, r) \). Then for any sequence \( \{ f_n \} \subset \mathcal{F} \), there exists a subsequence \( \{ f_{n,j} \} \) such that \( \{ f_{n,j} \} \) converges locally uniformly to a function \( h \) in \( D'(z_0, r) \), where \( h \) is either holomorphic or identically infinite in \( D'(z_0, r) \).

In the following, we consider two subcases.

**Subcase 2.1.** \( h \) is holomorphic in \( D'(z_0, r) \).

Then, there exists a positive number \( M \) such that \( |h(z)| \leq M \) in \( |z - z_0| = r/2 \). It follows that \( |f_{n,j}(z)| \leq 2M \) on \( |z - z_0| = r/2 \) for large \( j \). By the
maximum principle, we have \(|f_{n,j}(z)| \leq 2M\) in \(D(z_0, r/2) = \{ z : |z - z_0| \leq r/2 \}\). Then \(h\) is bounded in \(D(z_0, r/2)\), and \(h\) extends to be holomorphic in \(D(z_0, r/2)\). Again by the maximum principle, we have \(f_{n,j}(z) \to h(z)\) in \(D(z_0, r/2)\).

**Subcase 2.2.** \(h = \infty\).

We consider again two subcases.

**Subcase 2.2.1.** \(a - a_1a - a_0a' \big|_{z=z_0} = 0\).

Since \(a - a_1a - a_0a'\) and \(b - a_1b - a_0b'\) have no common zeros, then \(b - a_1b - a_0b' \neq 0\). So, there exists a positive number \(r' < r\) such that

\[
|b(z) - a_1(z)b(z) - a_0(z)b'(z)| \neq 0
\]

in \(D(z_0, r') = \{ z : |z - z_0| < r' \} \subset D\). Suppose that \(z_n\) is a zero of \(f_{n,j} - b\) in \(D(z_0, r')\). Then, we have \(f_{n,j}(z_n) = b(z_n)\) and \(L[f_{n,j}](z_n) = b(z_n)\). In view of \(L[f] = a_0f' + a_1f\), we deduce

\[
f_{n,j}'(z_n) = \frac{b - a_1b - a_0b'}{a_0}|_{z=z_n}.
\]

Let \(H_{n,j} = f_{n,j} - b\). Then \(H_{n,j}(z_n) = 0\) and

\[
H_{n,j}'(z_n) = f_{n,j}'(z_n) - b'(z_n) = \frac{b - a_1b - a_0b'}{a_0}|_{z=z_n} \neq 0,
\]

which implies that \(f_{n,j} - b\) just has simple zeros in \(D(z_0, r')\).

So the function \(\frac{L[f_{n,j}]-b}{f_{n,j}-b}\) is holomorphic in \(D(z_0, r')\). Let \(0 < r_1 < r'\) and \(\Gamma := \{ z : |z - z_0| = r_1 \}\). By Cauchy theorem we conclude that

\[
\int_{\Gamma} \frac{L[f_{n,j}](z) - b(z)}{f_{n,j}(z) - b(z)} \, dz = 0.
\]

Noting that \(f_{n,j} - b \to \infty\) on \(\Gamma\), we derive that (for sufficiently large \(n\))

\[
\left| \int_{\Gamma} \frac{a_1(z)b(z) + a_0(z)b'(z) - b(z)}{f_{n,j}(z) - b(z)} \, dz \right| \leq \pi.
\]

By \(n(\Gamma, \frac{1}{f_{n,j} - b})\) we denote the number of zeros of \(f_{n,j} - b\) in \(D(z_0, r) = \{ z : |z - z_0| < r \}\). From the argument principle, (3.13) and (3.14) (for sufficiently large \(n\)), we obtain that

\[
n(\Gamma, \frac{1}{f_{n,j} - b}) = \frac{1}{2\pi i} \left| \int_{\Gamma} \frac{f_{n,j}'(z) - b'(z)}{f_{n,j}(z) - b(z)} \, dz \right| = \frac{1}{2\pi i} \left| \int_{\Gamma} \frac{a_0(z)f_{n,j}'(z) - a_0(z)b'(z)}{a_0(z)f_{n,j}(z) - a_0(z)b(z)} \, dz \right| = \frac{1}{2\pi i} \left| \int_{\Gamma} \frac{L[f_{n,j}](z) - b(z) - a_1(z)f_{n,j}(z) + b(z) - a_0(z)b'(z)}{a_0(z)f_{n,j}(z) - a_0(z)b(z)} \, dz \right|
\]
\[ \leq \left| \frac{1}{2\pi i} \int_{\Gamma} L[f_{n,j}](z) - b(z) \right| + \left| \frac{1}{2\pi i} \int_{\Gamma} a_1(z)[f_{n,j}(z) - b(z)] \right| + \left| \frac{1}{2\pi i} \int_{\Gamma} a_1(z)b(z) + a_0(z)b'(z) - b(z) \right| \leq \frac{1}{2}, \]

which implies that
\[ n(\Gamma, \frac{1}{f_{n,j} - b}) = 0. \]

So \( f_{n,j} - b \) has no zeros in \( D(z_0, r_1) \). Thus, \( \frac{1}{f_{n,j} - b} \) is holomorphic and \( \frac{1}{f_{n,j} - b} \to 0 \) on \( D'(z_0, r_1) \). Similarly as in Case 2.1, we can deduce \( f_{n,j} \to \infty \) in \( D(z_0, r_1) \).

**Subcase 2.2.2.** \( a - a_1a - a_0a' \mid_{z=z_0} \neq 0 \).

Then, there exists a positive number \( r'' < r \) such that
\[ a(z) - a_1(z)a(z) - a_0(z)a'(z) \neq 0 \]

in \( D(z_0, r'') = \{ z : |z - z_0| < r'' \} \subset D \). Furthermore, in a similar way as in Subcase 2.2.1, it is easy to deduce that \( f_{n,j}(z) \to \infty \) in \( D(z_0, r'') \).

Thus, the proof of Case 2 is finished. Combining Case 1 and 2 yields that \( \mathcal{F} \) is normal at \( z_0 \), which completes the proof of Theorem 1.1.

References


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