CHARACTERIZATIONS OF BESOV SPACES
IN THE UNIT BALL

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Abstract. In this paper we obtain some new characterizations of Besov spaces on the unit ball of $\mathbb{C}^n$. These characterizations are also completely new even in the settings of the unit disk.

1. Introduction

Let $B$ be the unit ball of $\mathbb{C}^n$. Let $z = (z_1, \ldots, z_n)$ and $w = (w_1, \ldots, w_n)$ be points in $\mathbb{C}^n$, we write

$$\langle z, w \rangle = z_1w_1 + \cdots + z_nw_n, \ |z| = \sqrt{|z_1|^2 + \cdots + |z_n|^2}.$$ 

Thus $B = \{z \in \mathbb{C}^n : |z| < 1\}$. We denote the open unit disk in the complex plane by $D$. Let $\text{Aut}(B)$ be the group of all biholomorphic self-maps of $B$. It is well known that $\text{Aut}(B)$ is generated by the unitary operators on $\mathbb{C}^n$ and the involutions $\varphi_a$ of the form

$$\varphi_a(z) = \frac{a - P_\alpha z - s_a Q_\alpha z}{1 - \langle z, a \rangle},$$

where $s_a = \sqrt{1 - |a|^2}$, $P_a$ is the orthogonal projection into the space spanned by $a$ and $Q_a = I - P_a$ (see, e.g., [13]).

We denote by $H(B)$ the class of all holomorphic functions on the unit ball. For $f \in H(B)$, let $\mathcal{R}f$ denote the radial derivative of $f$, that is, $\mathcal{R}f(z) = \sum_{j=1}^n \bar{z}_j \frac{\partial f}{\partial z_j}(z)$. For $f \in H(B)$, let $\nabla f$ denote the complex gradient of $f$, i.e.,

$$\nabla f(z) = (\partial f/\partial z_1(z), \ldots, \partial f/\partial z_n(z)).$$

For $f \in C^1(B)$, let $\nabla f$ denote the invariant gradient on $B$, i.e.,

$$(\nabla f)(z) = \nabla (f \circ \varphi_z)(0).$$
For $f \in H(B)$ and $z \in B$, set

$$Q_f(z) = \sup_{w \in \mathbb{C}^n \setminus \{0\}} \frac{|\nabla f(z), \overline{w}|}{\sqrt{n+1} \left( \frac{1-|z|^2}{1-|w|^2} \right)^{n+1} |w|^2 + |w, z|^2}.$$ 

The Bloch space $B$ is the space of all $f \in H(B)$ such that (see, e.g., [10])

$$\|f\|_B = \sup_{z \in B} Q_f(z) < \infty.$$ 

Let $dv$ be the normalized Lebesgue measure of $B$. Suppose $0 < p < \infty$, recall that the Bergman space $A^p$ consists of those $f \in H(B)$ for which

$$\|f\|_{A^p} = \int_B |f(z)|^p dv(z) < \infty.$$ 

For $1 < p < \infty$ the Möbius invariant Besov space $B^p$ consists of those holomorphic functions $f$ such that

$$\|f\|_{B^p} = \int_B Q^p_f(z) d\lambda(z) < \infty,$$ 

where $d\lambda(z) = (1 - |z|^2)^{-n-1} dv(z)$ is a Möbius invariant measure, that is, for any $\psi \in \text{Aut}(B)$ and $f \in L^1(B)$,

$$\int_B f(z) d\lambda(z) = \int_B f \circ \psi(z) d\lambda(z).$$ 

From [1], the Besov space is nontrivial if and only if $p > 2n$ when $n > 1$.

It is very important to give more characterizations for a function space. Some new characterizations may be useful to study operator theory. For example, an $f \in H(B)$ is said to belong to the $\alpha$-Bloch space, denoted by $B^\alpha(B)$, if

$$\sup_{z \in B} (1 - |z|^2)^\alpha |\nabla f(z)| < \infty.$$ 

The last formula is simple. However it is difficult to study composition operators on $\alpha$-Bloch spaces by using this characterization. In [11], Zhang and Xu introduced the following metric (see [11] for more details)

$$F^\alpha_z(w) = \sqrt{n+1} \frac{\lambda_\alpha(|z|) |w|^2 + (1 - \lambda_\alpha(|z|)) |w, z|^2 / |z|^2}{(1 - |z|^2)^\alpha}.$$ 

Using this new metric, they obtained a new characterization for $\alpha$-Bloch space, i.e., they proved that $f \in B^\alpha(B)$ if and only if

$$\sup_{z, w \in \mathbb{C}^n \setminus \{0\}} \frac{|\nabla f(z)w|}{F^\alpha_z(w)} < \infty.$$ 

Using (1), the boundedness and compactness of composition operators on $\alpha$-Bloch spaces have been completely characterized.

In [7], the author proved that $f \in B_p$ if and only if

$$\int_B \int_B \left( \frac{|f(z) - f(w)|}{|w - P_{w,z} - s_w Q_{w,z}|} \right)^p (1 - |z|^2)^{p/2} (1 - |w|^2)^{p/2} d\lambda(z) d\lambda(w) < \infty.$$
Notice that when \( n = 1 \), we have that \( P_w = 1 \) and \( Q_w = 0 \), so that the denominator in (2) is exactly \(|w - z|\). Therefore (2) can be seen as a generalization of the result given by Stroethoff (see [9]). In [5], we proved that \( f \in B_p \) if and only if

\[
\int_B \int_B \left( \frac{|f(z) - f(w)|}{|1 - \langle z, w \rangle|} \right)^p (1 - |z|^2)^{n/2} (1 - |w|^2)^{n/2} d\lambda(z) d\lambda(w) < \infty.
\]

See [1, 2, 3, 4, 5, 7, 9, 12, 13] for more characterizations of the Besov space.

In this paper, we give some derivative-free characterizations for the Besov space in the unit ball of \( \mathbb{C}^n \). Some characterizations generalized the results in the literature. Furthermore, these characterizations are new even in the unit disk.

Throughout this paper, constants are denoted by \( C \), they are positive and may differ from one occurrence to the other. The notation \( A \asymp B \) means that there is a positive constant \( C \) such that \( B = C A \).

### 2. Main results and proofs

In this section, we state our main results and proofs. We begin with the following estimate.

**Lemma 1** ([8]). Let \(-1 < t < \infty\). If \( c > 0 \), then there is a positive constant \( C \) such that

\[
\int_B \frac{(1 - |z|^2)^t}{|1 - \langle z, w \rangle|^{n+1+\epsilon t}} d\nu(z) \leq \frac{C}{(1 - |w|^2)^c}
\]

for all \( w \in B \).

**Lemma 2** ([13]). Suppose \( p > 2n \) and \( f \in H(B) \). Then the following are equivalent.

(i) \( f \in B_p \);

(ii) \( \int_B (1 - |w|^2)^p |\nabla f(w)|^p d\lambda(w) < \infty \);

(iii) \( \int_B |\tilde{\nabla} f(w)|^p d\lambda(w) < \infty \).

**Lemma 3** ([6]). Assume that \( f \in H(B) \), \( 0 < p < \infty \), \(-1 < q < \infty \), \( 0 \leq s, t < \infty \) such that \( p + s > n \) and \( p + 2n > t \). Then for \( a \in B \),

\[
\int_B \left| \frac{|f(z) - f(0)|^p}{|z|^t} (1 - |z|^2)^q(1 - |\phi_a(z)|^2)^s d\nu(z) \right.
\]

\[
\leq C \int_B |\mathcal{R} f|^p (1 - |z|^2)^{p+q}(1 - |\phi_a(z)|^2)^s d\nu(z)
\]

\[
\leq C \int_B |\tilde{\nabla} f|^p (1 - |z|^2)^q(1 - |\phi_a(z)|^2)^s d\nu(z).
\]
Now we are in a position to state and prove the main results of this paper.

**Theorem 1.** Assume that $f \in H(B)$, $p > 2n$, $n + 1 \leq c, t < \infty$. Then $f \in B_p$ if and only if

$$
\int_B \int_B |f(z) - f(w)|^p \frac{(1 - |z|^2)^t(1 - |w|^2)^c}{|1 - \langle z, w \rangle|^{t+c}} d\lambda(z)d\lambda(w) < \infty.
$$

**Proof.** Suppose that (5) holds. For a fixed $r \in (0,1)$, let $E(a,r) = \{ z \in B : |\varphi_z(z)| < r \}$. Set $|E(a,r)| = v(E(a,r))$. From [7] or [13] we see that

$$
(1 - |a|^2)^{n+1} < (1 - |z|^2)^{n+1} < |1 - \langle a, z \rangle|^{n+1} = |E(a,r)|
$$

when $z \in E(a,r)$. It is easy to see (using Cauchy’s estimate for example) that there exists a positive constant $C$ such that

$$
|\nabla f(0)|^p \leq C \int_{E(0,r)} |f(w) - f(0)|^p dv(w)
$$

for all $f \in H(B)$, where $r$ is any fixed positive radius. Replacing $f$ by $f \circ \varphi_z$ and making a change of variable, we get

$$
|\nabla f(z)|^p \leq C \int_{E(z,r)} |f(z) - f(w)|^p \frac{(1 - |z|^2)^n}{|1 - \langle z, w \rangle|^{n+1}} dv(w)
$$

for all $f \in H(B)$ and $z \in B$. Combining the fact (6) with (7) we see that there is another positive constant $C > 0$ such that

$$
(1 - |z|^2)^n|\nabla f(z)|^p \leq C \int_{E(z,r)} |f(z) - f(w)|^p d\lambda(w)
$$

for all $f \in H(B)$ and $z \in B$. Then by (6) we have

$$
(1 - |w|^2)^n|\nabla f(w)|^p \leq C \int_{E(w,r)} \frac{|f(z) - f(w)|^p (1 - |z|^2)^{t(n+1)}(1 - |w|^2)^c}{|1 - \langle z, w \rangle|^{t+c}} dv(z).
$$

Hence

$$
\int_B (1 - |w|^2)^n|\nabla f(w)|^p d\lambda(w)
$$

$$
\leq C \int_B \int_{E(w,r)} |f(z) - f(w)|^p \frac{(1 - |z|^2)^n(1 - |w|^2)^c}{|1 - \langle z, w \rangle|^{t+c}} d\lambda(z)d\lambda(w)
$$

$$
\leq C \int_B \int_B |f(z) - f(w)|^p \frac{(1 - |z|^2)^n(1 - |w|^2)^c}{|1 - \langle z, w \rangle|^{t+c}} d\lambda(z)d\lambda(w) < \infty.
$$

It follows from Lemma 2 that $f \in B_p$.

Conversely, suppose that $f \in B_p$. From Theorem 3 of [7],

$$
\int_B \int_B \frac{|f(z) - f(w)|^p}{|1 - \langle z, w \rangle|^{t+c}} (1 - |z|^2)^t(1 - |w|^2)^c d\lambda(z)d\lambda(w)
$$

$$
= \int_B \int_B \frac{|f(z) - f(w)|^p}{|1 - \langle z, w \rangle|^{2(n+1)}} (1 - |z|^2)^{t(n+1)}(1 - |w|^2)^{-c(n+1)} dv(z)dv(w)
$$
Remark 1. Taking $t = c = n + 1$ in Theorem 1, we get Theorem 3 of [7]. Hence Theorem 1 can be regarded as an extension of the result in [7]. In particular, taking $t = c = p/2$ in Theorems 1 and 2, we get (3) and (2) respectively.

From Theorems 1 and 2, we get the following characterizations of the Besov space in the unit disk.
Corollary 1. Assume that $f \in H(D)$, $2 \leq c, t < \infty$ and $p > 2$. Then $f \in B_p(D)$ if and only if
\[
\int_D \int_D \frac{|f(z) - f(w)|^p}{|1 - wz|^{t+c}} (1 - |z|^2)^t(1 - |w|^2)^t d\tau(z) d\tau(w) < \infty,
\]
where $d\tau = (1 - |z|^2)^{-2} dA(z)$ is the Möbius invariant measure and $dA(z)$ is the Lebesgue measure on the unit disk.

Corollary 2. Assume that $f \in H(D)$, $2 \leq c, t < \infty$ and $p > \max\{2, t+c-2\}$. Then $f \in B_p(D)$ if and only if
\[
\int_D \int_D \frac{|f(z) - f(w)|^p}{|w - z|^{t+c}} (1 - |z|^2)^t(1 - |w|^2)^t d\tau(z) d\tau(w) < \infty.
\]

Theorem 3. Assume that $f \in H(B)$, $p > 2n$. Then $f \in B_p$ if and only if
\[
\int_B \int_B \frac{|f(z) - f(w)|^p}{|1 - \{z, w\}|^{2(n+1)}} (1 - |\varphi_a(z)|^2)^{\frac{n+1}{n}}(1 - |\varphi_a(w)|^2)^{\frac{n+1}{n}} d\lambda(z) d\lambda(w) < \infty.
\]

Proof. Suppose that (11) holds. For a fixed $r \in (0,1)$, when $z \in E(w,r)$, it holds
\[
(12) \quad |1 - \{z, a\}| \geq |1 - \{w, a\}|,
\]
for any $a \in B$ (see [13]). Using (12), from the proof of Theorem 1 we have
\[
(1 - |w|^2)^p |\nabla f(w)|^p (1 - |\varphi_a(w)|^2)^{n+1}
\leq C \int_{E(w,r)} \frac{|f(z) - f(w)|^p}{|1 - \{z, w\}|^{2(n+1)}} (1 - |w|^2)^{n+1}(1 - |\varphi_a(w)|^2)^{\frac{n+1}{n}}(1 - |\varphi_a(z)|^2)^{\frac{n+1}{n}} d\lambda(z) d\lambda(w).
\]

Hence
\[
\int_B (1 - |w|^2)^p |\nabla f(w)|^p (1 - |\varphi_a(w)|^2)^{n+1} d\lambda(w)
\leq C \int_B \int_B \frac{|f(z) - f(w)|^p}{|1 - \{z, w\}|^{2(n+1)}} (1 - |w|^2)^{n+1}(1 - |\varphi_a(w)|^2)^{\frac{n+1}{n}}(1 - |\varphi_a(z)|^2)^{\frac{n+1}{n}} d\lambda(z) d\lambda(w).
\]

Therefore
\[
\int_B (1 - |w|^2)^p |\nabla f(w)|^p d\lambda(w)
= \int_B (1 - |w|^2)^p |\nabla f(w)|^p d\lambda(w) \int_B (1 - |\varphi_a(w)|^2)^{n+1} d\lambda(a)
= \int_B \int_B (1 - |w|^2)^p |\nabla f(w)|^p (1 - |\varphi_a(w)|^2)^{n+1} d\lambda(w) d\lambda(a)
\leq C \int_B \int_B \frac{|f(z) - f(w)|^p}{|1 - \{z, w\}|^{2(n+1)}} (1 - |w|^2)^{n+1}(1 - |\varphi_a(w)|^2)^{\frac{n+1}{n}}(1 - |\varphi_a(z)|^2)^{\frac{n+1}{n}} d\lambda(z) d\lambda(w) d\lambda(a)
< \infty,
\]
i.e., $f \in B_p$, as desired.
Conversely, suppose that \( f \in B_p \). We first claim that for any \( g \in A^p \),

\[
Y = \int_B \int_B \frac{|g(z) - g(w)|^p}{|1 - \langle z, w \rangle|^{2(n+1)}} (1 - |z|^2)^{\frac{\alpha + 1}{2}} (1 - |w|^2)^{\frac{\alpha + 1}{2}} dv(\zeta) dv(\omega)
\]

\[\leq C \int_B |g(z)|^p dv(z) = \int_B |\nabla g(z)|^p dv(z) < \infty.
\]

In fact, using Lemma 1 we obtain

\[
Y \leq C \int_B \int_B \frac{|g(z)|^p}{|1 - \langle z, w \rangle|^{2(n+1)}} (1 - |z|^2)^{\frac{\alpha + 1}{2}} (1 - |w|^2)^{\frac{\alpha + 1}{2}} dv(\zeta) dv(\omega)
\]

\[\quad + C \int_B \int_B \frac{|g(w)|^p}{|1 - \langle z, w \rangle|^{2(n+1)}} (1 - |z|^2)^{\frac{\alpha + 1}{2}} (1 - |w|^2)^{\frac{\alpha + 1}{2}} dv(\zeta) dv(\omega)
\]

\[\leq C \int_B |g(z)|^p (1 - |z|^2)^{\frac{\alpha + 1}{2}} dv(z) \int_B \frac{(1 - |w|^2)^{\frac{\alpha + 1}{2}}}{|1 - \langle z, w \rangle|^{2(n+1)}} dv(w)
\]

\[\quad + C \int_B |g(w)|^p (1 - |w|^2)^{\frac{\alpha + 1}{2}} dv(w) \int_B \frac{(1 - |z|^2)^{\frac{\alpha + 1}{2}}}{|1 - \langle z, w \rangle|^{2(n+1)}} dv(z)
\]

\[\leq C \int_B |g(z)|^p dv(z) + C \int_B |g(w)|^p dv(w) \leq C \int_B |g(z)|^p dv(z).
\]

For \( f \in B_p \subset B \) and \( a \in B \), we have that \( f \circ \varphi_a - f(a) \in A^p \) (see, e.g., [13]). From (13) we have

\[
\int_B \int_B \frac{|f \circ \varphi_a(z) - f \circ \varphi_a(w)|^p}{|1 - \langle z, w \rangle|^{2(n+1)}} (1 - |z|^2)^{\frac{\alpha + 1}{2}} (1 - |w|^2)^{\frac{\alpha + 1}{2}} dv(\zeta) dv(\omega)
\]

\[\leq C \int_B |\nabla f(\varphi_a(z))|^p dv(z) = C \int_B |\nabla f(z)|^p (1 - |\varphi_a(z)|^2)^{n+1} d\lambda(z).
\]

Making the change of variables \( z \mapsto \varphi_a(z) \), \( w \mapsto \varphi_a(w) \) and using the following equality (see [13])

\[
\frac{(1 - |\varphi_a(z)|^2)(1 - |\varphi_a(w)|^2)}{|1 - \langle \varphi_a(z), \varphi_a(w) \rangle|^2} = 1 - |\varphi_\omega(z)|^2,
\]

we obtain

\[
\int_B \int_B \frac{|f(z) - f(w)|^p}{|1 - \langle z, w \rangle|^{2(n+1)}} (1 - |\varphi_a(z)|^2)^{\frac{\alpha + 1}{2}} (1 - |\varphi_a(w)|^2)^{\frac{\alpha + 1}{2}} dv(\zeta) dv(\omega) d\lambda(\alpha)
\]

\[\leq C \int_B \int_B |\nabla f(z)|^p (1 - |\varphi_a(z)|^2)^{n+1} d\lambda(z) d\lambda(\alpha)
\]

\[= C \int_B |\nabla f(z)|^p d\lambda(z) < \infty.
\]

The proof is finished. □
Theorem 4. Assume that \( f \in H(B) \), \( p > 2n \). Then \( f \in B_p \) if and only if
\[
\int_B \int_B |f(z) - f(w)|^p (1 - |\varphi_z(z)|^2)^{\frac{n+1}{2}} (1 - |\varphi_w(w)|^2)^{\frac{n+1}{2}} \frac{dz dv(w)}{|w - P_w z - s_w Q_w z|^{2(n+1)}} < \infty.
\]

Proof. Sufficiency. The result follows from Theorem 3 and (9).

Necessity. Suppose that \( f \in B_p \). For \( a \in B \), by Lemma 3 we have
\[
\int_B |f(z) - f(w)|^p (1 - |\varphi_z(z)|^2)^{\frac{n+1}{2}} (1 - |\varphi_w(w)|^2)^{\frac{n+1}{2}} \frac{dz dv(w)}{|w - P_w z - s_w Q_w z|^{2(n+1)}} \leq C \int_B \int_B |\nabla f \circ \varphi_w(w)|^p (1 - |\varphi_w(w)|^2)^{\frac{n+1}{2}} \frac{dz dv(w)}{|w - P_w z - s_w Q_w z|^{2(n+1)}} \leq C \frac{1}{|a|^{2(n+1)}} \int_B |f(z) - f(w)|^p (1 - |\varphi_z(z)|^2)^{\frac{n+1}{2}} (1 - |\varphi_w(w)|^2)^{\frac{n+1}{2}} \frac{dz dv(w)}{|w - P_w z - s_w Q_w z|^{2(n+1)}} \leq C \sup \int_B |\nabla f \circ \varphi_w(w)|^p (1 - |\varphi_w(w)|^2)^{\frac{n+1}{2}} \frac{dz dv(w)}{|w - P_w z - s_w Q_w z|^{2(n+1)}} \frac{1}{|a|^{2(n+1)}} \int_B \frac{dz dv(w)}{|w - P_w z - s_w Q_w z|^{2(n+1)}} \leq C \sup \int_B |\nabla f \circ \varphi_w(w)|^p (1 - |\varphi_w(w)|^2)^{\frac{n+1}{2}} \frac{dz dv(w)}{|w - P_w z - s_w Q_w z|^{2(n+1)}}.
\]

Here
\[
J = \sup a \in B \int_B \frac{1}{(1 - |\varphi_z(z)|^2)^{n+1}} (1 - |\varphi_w(w)|^2)^{\frac{n+1}{2}} d\lambda(a).
\]

Making the change of variables \( w \mapsto \varphi_z(u) \) and using the fact that \( |\varphi_z(u)| = |\varphi_z(z)| \) we have
\[
J = \sup a \in B \int_B \frac{1}{(1 - |\varphi_z(u)|^2)^{n+1}} (1 - |\varphi_w(u)|^2)^{\frac{n+1}{2}} d\lambda(u).
\]

From the exercises 1.24 of [13] we see that \( |\varphi_w(\varphi_z(u))| = |\varphi_{\varphi_z}(u)| \). It follows from Theorem 1.12 of [13] that
\[
J = \sup a \in B \int_B \frac{1}{(1 - |\varphi_z(u)|^2)^{n+1}} (1 - |\varphi_{\varphi_z}(u)|^2)^{\frac{n+1}{2}} d\lambda(u).
\]

Combining (15) with (16), we get
\[
\int_B \int_B |f(z) - f(w)|^p (1 - |\varphi_z(z)|^2)^{\frac{n+1}{2}} (1 - |\varphi_w(w)|^2)^{\frac{n+1}{2}} \frac{dz dv(w)}{|w - P_w z - s_w Q_w z|^{2(n+1)}} \frac{dz dv(w)}{|w - P_w z - s_w Q_w z|^{2(n+1)}} d\lambda(a)
\]

\[= \sup a \in B \int_B \frac{1}{(1 - |\varphi_z(u)|^2)^{n+1}} (1 - |\varphi_w(u)|^2)^{\frac{n+1}{2}} d\lambda(u).
\]
The proof is completed. □

Similarly, we have the following characterizations of the Besov space in the unit disk.

**Corollary 3.** Assume that \( f \in H(D) \) and \( p > 2 \). Then the following are equivalent.

(i) \( f \in B_p(D) \);

(ii) \[
\int_D \int_D \int_D \frac{|f(z) - f(w)|^p}{|1 - \overline{w}z|^2} (1 - |\varphi_u(z)|^2)(1 - |\varphi_u(w)|^2) dA(z)dA(w)d\tau(a) < \infty;
\]

(iii) \[
\int_D \int_D \int_D \frac{|f(z) - f(w)|^p}{|z - w|^2} (1 - |\varphi_u(z)|^2)(1 - |\varphi_u(w)|^2) dA(z)dA(w)d\tau(a) < \infty.
\]

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