EXTREMUM PROPERTIES OF DUAL $L_p$-CENTROID BODY AND $L_p$-JOHN ELLIPSOID

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Abstract. For $0 < p \leq \infty$ and a convex body $K$ in $\mathbb{R}^n$, Lutwak, Yang and Zhang defined the concept of dual $L_p$-centroid body $\Gamma_{-p}K$ and $L_p$-John ellipsoid $E_pK$. In this paper, we prove the following two results:

(i) For any origin-symmetric convex body $K$, there exist an ellipsoid $E$ and a parallelotope $P$ such that for $1 \leq p \leq 2$ and $0 < q \leq \infty$,

$$E_q E \supseteq \Gamma_{-p}K \supseteq (nc_{n - 2, p})^{-\frac{1}{p}} E_q P \text{ and } V(E) = V(K) = V(P);$$

For $2 \leq p \leq \infty$ and $0 < q \leq \infty$,

$$2^{-\frac{1}{p}} \omega_n^{\frac{1}{p}} E_q E \supseteq \Gamma_{-p}K \supseteq 2\omega_n^{\frac{1}{p}} (nc_{n - 2, p})^{-\frac{1}{p}} E_q P \text{ and } V(E) = V(K) = V(P).$$

(ii) For any convex body $K$ whose John point is at the origin, there exists a simplex $T$ such that for $1 \leq p \leq \infty$ and $0 < q \leq \infty$,

$$\alpha_n (nc_{n - 2, p})^{-\frac{1}{p}} E_q T \supseteq \Gamma_{-p}K \supseteq (nc_{n - 2, p})^{-\frac{1}{p}} E_q T \text{ and } V(K) = V(T).$$

1. Introduction and main results

For each convex subset in $\mathbb{R}^n$, it is well-known that there is a unique ellipsoid with the following property: The moment of inertia of the ellipsoid and the moment of inertia of the convex set are the same about every 1-dimensional subspace of $\mathbb{R}^n$. This ellipsoid is called the Legendre ellipsoid of the convex set. The Legendre ellipsoid and its polar (the Binet ellipsoid) are well-known concepts from classical mechanics; see [4, 5, 13] for historical references.

It has slowly come to be recognized that along side the Brunn-Minkowski theory there is a dual theory. The Legendre ellipsoid (and Binet ellipsoid) is an object of this dual Brunn-Minkowski theory. A nature question is whether there is a dual analog of the classical Legendre ellipsoid in the Brunn-Minkowski theory. Applying the $L_p$-curvature theory ([6, 7]), Lutwak, Yang and Zhang

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demonstrated the existence of precisely this dual object. Further, some beautiful and deep properties for this dual analog of the Legendre ellipsoid have been discovered (see [8]).

An often used fact in convex geometry is that associated with each convex body $K$ is a unique ellipsoid $JK$ of maximal volume that is contained in $K$. The ellipsoid is called the John ellipsoid of $K$ and the center of this ellipsoid is called John point of $K$.

For the ease of use we first introduce some notations and concepts. Let $\mathcal{K}^n$ denote the set of convex bodies (compact, convex subsets with non-empty interiors) in Euclidean space $\mathbb{R}^n$, $\mathcal{K}_0^n$ and $\mathcal{K}_e^n$ denote the set of convex bodies containing origin in their interiors and the set of origin-symmetric convex bodies in $\mathbb{R}^n$, respectively. Let $S^{n-1}$ denote the unit sphere in $\mathbb{R}^n$, $V(K)$ denote the $n$-dimensional volume of body $K$. If $K$ is the standard unit ball $B_n$ in $\mathbb{R}^n$, then it is denoted as $\omega_n = V(B_n)$.

Let $K \in \mathcal{K}^n$, the support function $h_K$ of $K$ is defined by (see [1, 10])

$$h(K, u) = h_K(u) = \max\{\langle u, x \rangle : x \in K\}, \quad u \in S^{n-1},$$

where $\langle u, x \rangle$ denotes the standard inner product of $u$ and $x$ in $\mathbb{R}^n$.

For a compact subset $L$ of $\mathbb{R}^n$, which is star-shaped with respect to the origin, we shall use $\rho(L, \cdot)$ or $\rho_L(\cdot)$ to denote its radial function; i.e., for $u \in S^{n-1}$ (see [1, 10]),

$$\rho(L, u) = \rho_K(u) = \max\{\lambda > 0 : \lambda u \in L\}.$$

In [8], Lutwak, Yang and Zhang proposed the concept of the new ellipsoid as follows:

**Definition 1.1.** For $K \in \mathcal{K}^n$, the new ellipsoid $\Gamma_{-2}K$ was defined by

$$(1.1) \quad \rho_{\Gamma_{-2}K}(u) = \frac{1}{V(K)} \int_{S^{n-1}} |\langle u, v \rangle|^2 dS_2(K, v)$$

for all $u \in S^{n-1}$, where $S_2(K, \cdot)$ denotes the $L_2$-surface measure.

In [8], it was shown that $S_2(K, \cdot)$ is absolutely continuous with respect to the classical surface area measure $S_K := S(K, \cdot)$ and has the Radon-Nikodym derivative

$$\frac{dS_2(K, \cdot)}{dS_K} = \frac{1}{h_K}.$$

If $P$ is a convex polytope in $\mathbb{R}^n$ whose faces have outer unit normals $u_1, \ldots, u_N$, and $a_i$ denotes the area ($(n-1)$-dimensional volumes) of the face with outer normal $u_i$ and $h_i$ denotes the distance from the origin to this face, then the measure $S_2(P, \cdot)$ is concentrated at the points $u_1, \ldots, u_N \in S^{n-1}$ and $S_2(P, u_i) = a_i/h_i$. Thus, for the convex polytope $P$, we have for $u \in S^{n-1}$ (see [8]),

$$(1.2) \quad \rho_{\Gamma_{-2}P}(u) = \frac{1}{V(P)} \sum_{i=1}^N \langle u, u_i \rangle^2 \frac{a_i}{h_i}.$$
In 2005, Lutwak, Yang and Zhang further put forward the following concept of $L_p$-John ellipsoid of convex body $K$ which is a generalized of the John ellipsoid $JK$ and new ellipsoid $\Gamma_{-2}K$ (see [9]).

**Definition 1.2.** Suppose that $K \in \mathcal{K}^o_n$ and $0 < p \leq \infty$, for each origin-symmetric ellipsoid $E$, there exists a unique ellipsoid $E_pK$ that solves the constrained maximization problem

$$V(E_pK) = \max V(E) \text{ subject to } \nabla_p(K, E) \leq 1.$$ 

Then $E_p$ is called the $L_p$-John ellipsoid of $K$.

$$\nabla_p(K, L) = (V_p(K, L)/V(K))^{1/p},$$

$V_p(K, L)$ is the $L_p$-mixed volume of $K, L \in \mathcal{K}^o_n$.

When $p = 1$, the $E_1K$ is just the Petty ellipsoid; When $p = 2$, the $E_2K$ is just the new ellipsoid $\Gamma_{-2}K$; When $p = \infty$, the $E_\infty K$ is just the well-known classical John ellipsoid $JK$.

At the same time, Lutwak, Yang and Zhang put forward the following concept of the new geometry body $pK$ as a generalization of the new ellipsoid $2K$ (see [9]):

**Definition 1.3.** If $K \in \mathcal{K}^o_n$ and $p > 0$, then geometric body $\Gamma_{-p}K$ is an origin-symmetric body whose radial function is defined by

$$\rho_{\Gamma_{-p}K}(u) = \frac{1}{V(K)} \int_{S^{n-1}} |\langle u, v \rangle|^p dS_p(K, v)$$

for all $u \in S^{n-1}$. Note for $p \geq 1$, the geometric body $\Gamma_{-p}K$ is an origin-symmetric convex body. $S_p(K, \cdot)$ is a positive Borel measure on $S^{n-1}$, called the $L_p$-surface area measure of $K$. It turns out that the measure $S_p(K, \cdot)$ on $S^{n-1}$ is absolutely continuous with respect to $S_K$, and has the Radon-Nikodym derivative

$$\frac{dS_p(K, \cdot)}{dS(K, \cdot)} = h^{1-p}(K, \cdot).$$

And $L_p$-centroid bodies $\Gamma_pK$ corresponding to the new geometry $\Gamma_{-p}K$ might be called the dual $L_p$-centroid bodies.

Let $K$ be a convex body containing the origin, in [9] the authors have proved that $E_pK$ and $\Gamma_{-p}K$ have the affine invariant, i.e.,

**Theorem A.** If $K \in \mathcal{K}^o_n$ and $0 < p \leq \infty$, then for $\phi \in GL(n)$,

$$E_p\phi K = \phi E_p K,$$

$$\Gamma_{-p}\phi K = \phi \Gamma_{-p} K.$$ 

Where $GL(n)$ denotes non-singular affine (or linear) transformation group. Apparently, $E_pB_n = B_n$. And if $K$ is an ellipsoid that is centered at the origin, then

$$E_p E = E.$$
For $L_p$-John ellipsoid $E_pK$, we have that:

**Theorem B** (see [9]). If $K \in \mathcal{K}^n_\omega$ and $1 \leq p \leq \infty$, then

$$V(K) \geq V(E_pK),$$

with equality for $p > 1$ if and only if $K$ is an ellipsoid containing the origin, for $p = 1$ if and only if $K$ is an ellipsoid.

**Theorem C** (see [9]). If $K \in \mathcal{K}^n_\omega$ and $0 < p \leq \infty$, then

$$V(E_pK) \geq 2^{-n}\omega_n V(K),$$

with equality if and only if $K$ is a parallelotope.

We note that an equation as follows: If $K \in \mathcal{K}^n_\omega$, $p \geq 1$, then

$$\Gamma_{-p}K = \left(\frac{V(K)}{nc_{n-2,p}\omega_n}\right)^{\frac{1}{p}} \Pi_p^*K,$$

where $\Pi_p^*K$ denotes polar body of $L_p$-projection body $\Pi_pK$ of $K$, and

$$c_{n,p} = \frac{\omega_{n+p}}{\omega_2\omega_n\omega_{p-1}}.$$

Using Eq.(1.9) and Theorem 2 in [10], then the upper bound on $V(\Gamma_{-p})K$ is established the following results:

**Theorem D.** Suppose $K \in \mathcal{K}^n_\omega$ and $1 \leq p \leq \infty$, then

$$V(\Gamma_{-p}K) \leq (nc_{n-2,p})^{-\frac{1}{p}} V(K),$$

with equality for $p > 1$ if and only if $K$ is an ellipsoid centered at the origin, for $p = 1$ if and only if $K$ is an ellipsoid.

**Remark 1.1.** For $p = 2$ the inequality (1.11) is just the following well-known inequality by Lutwak, Yang and Zhang (see [8]):

$$V(\Gamma_{-2} K) \leq V(K),$$

with equality if and only if $K$ is an ellipsoid centered at the origin.

For $L_p$-John ellipsoid $E_pK$, Wang proved also the following results (see [16, p. 64]):

**Theorem E.** Suppose $K \subset \mathbb{R}^n$ is a convex body positioned so that its John point is at the origin, and $0 < p \leq \infty$, then

$$V(E_pK) \geq \frac{n!\omega_n}{n^\frac{n}{p}(n+1)^{\frac{n}{p}}} V(K),$$

with quality if and only if $K$ is a simplex.
Let $K$ be a convex body in $\mathbb{R}^n$. Then the difference body $DK$ of $K$ is defined by $DK = K + (-K)$. The Rogers-Shephard inequality of $DK$ is

$$V(DK) \leq \binom{2n}{n} V(K),$$

with equality if and only if $K$ is a simplex.

In [2] (also see Schneider’s review article [14]), Jonasson establish an enhanced version of the Rogers-Shephard inequality for the plane $\mathbb{R}^2$ as follows: Let $K$ be a convex body in $\mathbb{R}^2$. Then there exists a triangle $T$ such that

$$DK \subseteq DT, \ V(K) = V(T).$$

From $V(DK) \leq V(DT) = 6V(T) = 6V(K)$, we can obtain the Rogers-Shephard inequality.

The work of Jonasson and Yuan (see [17]) inspired us to further study the extremum properties of dual $L_p$-centroid body $p \Gamma K$ and the $L_p$-John ellipsoid $E_p K$ of convex body $K$ and have established the following two main theorems:

**Theorem 1.1.** For any origin-symmetric convex body $K$, there exist an ellipsoid $E$ and a parallelotope $P$ such that for $1 \leq p \leq 2$ and $0 < q \leq \infty$,

$$E_q E \supseteq \Gamma_{-p} K \supseteq (nc_{n-2,p})^{-\frac{1}{p}} E_q P \text{ and } V(E) = V(K) = V(P);$$

For $2 \leq p \leq \infty$ and $0 < q \leq \infty$,

$$2^{-1} \omega_n E_q E \subseteq \Gamma_{-p} K \subseteq 2\omega_n (nc_{n-2,p})^{-\frac{1}{p}} E_q P \text{ and } V(E) = V(K) = V(P).$$

**Theorem 1.2.** For any convex body $K$ whose John point is at the origin, there exists a simplex $T$ such that for $1 \leq p \leq \infty$ and $0 < q \leq \infty$,

$$\alpha_q (nc_{n-2,p})^{-\frac{1}{p}} E_q T \supseteq \Gamma_{-p} K \supseteq (nc_{n-2,p})^{-\frac{1}{p}} E_q T \text{ and } V(K) = V(T).$$

Where

$$\alpha_n = \left( \frac{n^2 (n+1)^{\frac{n+1}{n}}}{n! \omega_n} \right)^{\frac{1}{n}}.$$

**2. Preliminaries**

For $K, L \in K^n$ and real $p \geq 1$, the $L_p$-mixed volume $V_p(K, L)$ of the $K$ and $L$ is given by (see [6])

$$V_p(K, L) = \frac{1}{n} \int_{S^{n-1}} h_p^p(v) dS_p(K, v).$$

The $L_p$-Minkowski inequality states that (see [7]), if $K, L \in K^n$ and $p \geq 1$, then

$$V_p(K, L) \geq V(K)^{\frac{n-p}{p}} V(L)^{\frac{p}{p}},$$

with equality for $p = 1$ if and only if $K$ and $L$ are homothetic, for $p > 1$ if and only if $K$ and $L$ are dilates.
For $K, L \in S^n_0$ and real $p \geq 1$, the $L_p$-dual mixed volume $V_{-p}(K, L)$ of the $K$ and $L$ is given by (see [7])

$$V_{-p}(K, L) = \frac{1}{n} \int_{S^{n-1}} \rho_K^{n+p}(v) \rho_L^{-p}(v) dS(v),$$

where $S$ is the spherical Lebesgue measure on $S^{n-1}$. From Eq.(2.3), it follows immediately that for each $K \in S^n_0$ and $p \geq 1$,

$$V_{-p}(K, K) = V(K).$$

The $L_p$-Minkowski inequality for the dual mixed volume $V_{-p}(K, L)$ states that (see [7]), if $K, L \in S^n_0$ and $p \geq 1$, then

$$V_{-p}(K, L) \leq V(K)^{\frac{n+p}{n}} V(L)^{-\frac{p}{n}},$$

with equality if and only if $K$ and $L$ are dilates.

**Definition 2.1** (see [11]). For each compact star-shaped $K \subset \mathbb{R}^n$ about the origin and the real number $p \geq 1$, the $L_p$-centroid body $\Gamma_p K$ of $K$ is the origin-symmetric body whose support function is defined by

$$h_{\Gamma_p K}(u) = \frac{1}{c_{n,p} V(K)} \int_K |u \cdot x|^p dx$$

for all $u \in S^{n-1}$. From Eq.(2.6), we can easily get that for all $u \in S^{n-1}$,

$$h_{\Gamma_p K}(u) = \frac{1}{(n+p)c_{n,p} V(K)} \int_{S^{n-1}} |u \cdot v|^p \rho_K^{n+p}(v) dS(v).$$

If $E$ is an ellipsoid that is centered at the origin, then ([11])

$$\Gamma_p E = E.$$

**Lemma 2.1** (John [3]). Each convex body $K$ contains a unique ellipsoid $JK$ of maximal volume. This ellipsoid is the unit ball $B_n$ if and only if the following conditions are satisfied: $B_n \subset K$ and there are contact points $\{u_i\}^m_1$ and positive numbers $\{c_i\}^m_1$ so that

$$||x||^2 = \sum_{i=1}^m c_i u_i \otimes u_i = I_n,$$

where $u_i \otimes u_i$ is the rank-one orthogonal projection onto the span of $u_i$ and $I_n$ is the identity on $\mathbb{R}^n$.

The condition (2.9) shows that the $u_i$ behave like an orthonormal basis to the extent that, for each $x \in \mathbb{R}^n$,

$$||x||^2 = \sum_{i=1}^m c_i (x, u_i)^2.$$ 

A detailed discussion of these equivalent conditions, refer to K. Ball’s article [1].
The equality of the traces in (2.9) shows that
\[
\sum_{i=1}^{m} c_i = n.
\]

**Lemma 2.2** (see [9]). If \( K \in \mathcal{K}^n_p \), then for \( 0 < p \leq 2 \),
\[
E_p K \supseteq \Gamma_{-p} K \supseteq n^{\frac{1}{p} - \frac{1}{2}} E_p K.
\]
For \( 2 \leq p \leq \infty \),
\[
E_p K \subseteq \Gamma_{-p} K \subseteq n^{\frac{1}{p} - \frac{1}{2}} E_p K.
\]

**Lemma 2.3.** Let \( C \) be a cube centered at the origin in \( \mathbb{R}^n \). Then for \( 0 < p \leq \infty \),
\[
E_p C = JC.
\]
**Proof.** Without loss of generality, let \( C \) be cube \([-1,1]^n\) in \( \mathbb{R}^n \). By simple calculation shows that for cube \( C \), its \( L_p \)-John ellipsoid is the unit ball \( B_n \); see [9]. That is \( E_p C = B_n \). \( \square \)

For \( C \) the John ellipsoid \( JC \) is also \( B_n \). The contact points are the standard basis vectors \((e_1, \ldots, e_n)\) of \( \mathbb{R}^n \) and their negatives, and they satisfy
\[
\sum_{i=1}^{m} c_i e_i = I_n.
\]
That is, one can take all the weights \( c_i \) equal to 1 in (2.12).

**Lemma 2.4.** Let \( T \) be a simplex containing the origin in \( \mathbb{R}^n \). Then for \( 0 < p \leq \infty \),
\[
E_p T = JT.
\]
**Proof.** Since both \( E_p T \) and \( JK \) are affine invariant, it suffices to prove that \( E_p T = JT \) holds for a regular simplex. It is easy to verify that the John ellipsoid of a regular simplex is its inscribed ball.

Without loss of generality, we may assume that \( T \) is a regular simplex whose inscribed ball is \( B_n \). Let \( u_1, \ldots, u_{n+1} \) denote the outer unit normals and \( S \) the area of the face of \( T \). By (2.10) and (2.11), we have for each \( x \in \mathbb{R}^n \),
\[
||x||^2 = \sum_{i=1}^{n+1} c_i (x, u_i)^2, \quad \sum_{i=1}^{n+1} c_i = n.
\]
Take \( u_i \) for \( x \), and notice that \( (u_i, u_j) = -\frac{1}{n} \) for \( i \neq j \).
It follows that
\[
1 = \sum_{j=1}^{n+1} c_j (u_j, u_i)^2 = \frac{1}{n^2} \left( \sum_{j=1}^{n+1} c_j - c_i \right) + c_i.
\]
Hence
\[
c_i = \frac{n}{n+1}, \quad i = 1, 2, \ldots, n + 1.
\]
In addition, Theorem A show that it is sufficient to prove the $E_pT' = B_n$ when $T'$ is a regular simplex whose inscribed ball is $B_n$. In fact, let $T$ be a simplex that the John point is at the origin (e.g., if $\phi T$ is origin-symmetric), and $E_pT = E$ with $E$ is an ellipsoid that is centered at the origin, then there exists an affine transformation $\phi \in GL(n)$, such that $\phi T = T'$ is a regular simplex whose inscribed ball is $B_n$ and $\phi E = B_n$. From Theorem A we give

$$E_pT' = E_p\phi T = \phi E = B_n.$$  

In summary, we have that $E_pT = JT$.  

Lemma 2.5 (Fejes Tóth [12]). Let $T$ be a simplex in $\mathbb{R}^n$ with inscribed ball radius $r$. Then

$$\left(2.14\right) \quad V(T) \geq \frac{n^{\frac{p}{2}}(n+1)^{\frac{p+1}{2}}}{n!}r^n,$$

with equality if and only if $T$ is a regular simplex.

3. Proofs of theorem

In order to prove theorems, we need the following several lemmas.

Lemma 3.1. If $K \in S^n_0$, $L \in K^n_o$, and $p \geq 1$, then

$$\left(3.1\right) \quad V_p(L, \Gamma_pK) = \frac{V(L)}{nc_{n-2,p}V(K)}V_{-p}(K, \Gamma_{-p}L).$$

Proof. Using (2.1) and (2.7), and combining with (1.3) and (2.3), we have

$$V_p(L, \Gamma_pK) = \frac{1}{n} \int_{S^{n-1}} h_{p, K}(u) dS_p(L, u)$$

$$= \frac{1}{n(n+p)c_{n,p}V(K)} \int_{S^{n-1}} \int_{S^{n-1}} |u \cdot v|^p \rho_{K}^{n+p}(v) dS(v) dS_p(L, u)$$

$$= \frac{V(L)}{n(n+p)c_{n,p}V(K)} \int_{S^{n-1}} \rho_{K}^{n+p}(v) \rho_{-p}^{-L}(v) dS(v)$$

$$= \frac{V(L)}{(n+p)c_{n,p}V(K)}V_{-p}(K, \Gamma_{-p}L).$$

But the equality (1.10) gives $(n+p)c_{n,p} = nc_{n-2,p}$, so we get (3.1).  

We proved the following inequality associated with dual $L_p$-centroid bodies $\Gamma_{-p}K$ and $L_{-p}$-John ellipsoid $E_pK$:

Lemma 3.2. Suppose $K \in K^n_o$ and $1 \leq p \leq \infty$, then

$$\left(3.2\right) \quad V(\Gamma_{-p}K) \geq (nc_{n-2,p})^{-\frac{p}{2}}V(E_pK),$$

with equality if and only if $K$ is an ellipsoid centered at the origin or $p = 2$. 

Proof. From (2.8) we see that if $E$ is an ellipsoid centered at the origin, then $\Gamma_p E = E$. Thus for $L_p$-John ellipsoid $E_p K$, we have

\begin{equation}
\Gamma_p E_p K = E_p K.
\end{equation}

Consider $E_p K$ instead of $K$ in (3.1), and using (3.3), we have

$$V_p(L, E_p K) = V_p(L, \Gamma_p E_p K) = \frac{V(L)}{nc_{n-2,p} V(E_p K)} V_{-p}(E_p K, \Gamma_{-p} L).$$

Take $L = K$ in the above equation, and combined with inequality (2.5), we have

$$V_p(K, E_p K) = \frac{V(K)}{nc_{n-2,p} V(E_p K)} V_{-p}(E_p K, \Gamma_{-p} K) \geq \frac{V(K)}{nc_{n-2,p} V(E_p K)} V(E_p K)^{-\frac{n+p}{n-p}} V(\Gamma_{-p} K)^{-\frac{n}{n+p}}
= \frac{V(K)}{nc_{n-2,p}} V(E_p K)^{\frac{n}{n+p}} V(\Gamma_{-p} K)^{-\frac{n}{n+p}},$$

namely

\begin{equation}
V_p(K, E_p K) \geq \frac{V(K)}{nc_{n-2,p}} V(E_p K)^{\frac{n}{n+p}} V(\Gamma_{-p} K)^{-\frac{n}{n+p}}.
\end{equation}

According to the condition of equality holds in inequality (2.5), we know the equality holds in the inequality (3.4) if and only if $\Gamma_{-p} K$ and $E_p K$ are dilates, thus the equality holds in the inequality (3.4) if and only if $\Gamma_{-p} K$ is an ellipsoid centered at the origin.

From Definition 1.2 we know that $L_p$-John ellipsoid $E_p K$ satisfy the condition as follows:

\begin{equation}
V(K) \geq V_p(K, E_p K),
\end{equation}

with equality if and only if $K$ is an ellipsoid centered at the origin.

Combining with (3.4) and (3.5), we immediately obtain (3.2).

According to the condition of equality holds in inequalities (3.4) and (3.5), we know the equality holds in (3.2) if and only if both $\Gamma_{-p} K$ and $K$ are ellipsoid centered at the origin, namely $K$ must be an ellipsoid centered at the origin in (3.2). In addition, we note that for $p = 2$ the equality holds in (3.2). The proof of Lemma 3.2 is completed. 

Combined with the inequality (3.2) and the inequality (1.8), we immediately obtain:

Lemma 3.3. If $K \in \mathcal{K}_c^n$ and $1 \leq p \leq \infty$, then

\begin{equation}
V(\Gamma_{-p} K) \geq 2^{-n} \omega_n(nc_{n-2,p})^{-\frac{n}{p}} V(K),
\end{equation}

with equality for $p = 2$ if and only if $K$ is a paralleotope, for $p \neq 2$ if and only if $n = 1$ and $K$ is an origin-symmetrical line segment or $n = 1$ and $p \to \infty$. 

Remark 3.1. When $p = 2$, note that $nc_{n-2, 2} = 1$, then (3.6) is just the following inequality by Lutwak, Yang and Zhang (see [8]):

(3.7) $V(\Gamma_{-2}K) \geq 2^{-n} \omega_n V(K)$,

with quality if and only if $K$ is a parallelotope.

Combined with the inequality (3.2) and the inequality (1.13), we immediately obtain the following inequality:

Lemma 3.4. Suppose $K \subset \mathbb{R}^n$ is a convex body positioned so that its John point is at the origin, and $1 \leq p \leq \infty$, then

(3.8) $V(\Gamma_p K) \geq \frac{n! \omega_n}{n^\frac{n}{2} (n + 1)^{\frac{n}{2}} (nc_{n-2, p})^{\frac{n}{2}}} V(K),$

with quality for $p = 2$ if and only if $K$ is a simplex, for $p \neq 2$ if and only if $n = 1$ and $K$ is an origin-symmetrical line segment or $n = 1$ and $p \to \infty$.

Now we give the proofs of theorems.

Proof of Theorem 1.1. First step: we prove that the left inclusion. Let $V(E_p K) = \lambda V(K)$ and $K \in K^n_c$. From Theorem B and Theorem C, we have

$$\frac{1}{2^n} \omega_n \leq \lambda \leq 1.$$ 

Put that $E = \lambda^{-\frac{1}{p}} E_p K$,

we obtain

$$V(E) = V(\lambda^{-\frac{1}{p}} E_p K) = \lambda^{-1} V(E_p K) = V(K).$$

By Lemma 2.2 and note that $1 \leq \lambda^{-\frac{1}{p}} \leq 2 \omega_n^{-\frac{1}{p}}$, we have that for $0 < p \leq 2$ and $0 < q \leq \infty$,

$$E_q E = E = \lambda^{-\frac{1}{p}} E_p K \supseteq E_p K \supseteq \Gamma_p K,$$

namely, for $K \in K^n_c$ and $0 < p \leq 2$ and $0 < q \leq \infty$,

(3.9) $E_q E \supseteq \Gamma_p K$ and $V(K) = V(E)$.

For $2 \leq p \leq \infty$ and $0 < q \leq \infty$,

$$E_q E = E = \lambda^{-\frac{1}{p}} E_p K \subseteq 2 \omega_n^{-\frac{1}{p}} E_p K \subseteq 2 \omega_n^{-\frac{1}{p}} \Gamma_p K,$$

namely, for $K \in K^n_c, 2 \leq p \leq \infty$ and $0 < q \leq \infty$,

(3.10) $\Gamma_p K \supseteq 2^{-1} \omega_n^{\frac{1}{p}} E_p E$ and $V(K) = V(E)$.

The second step: we prove that the right inclusion of Theorem 1.1.

Since for $p \geq 1$, $\Gamma_p K$ is an origin-symmetric convex body, then there exists an equivalent affine transformation $\phi \in SL(n)$ such that $\phi(\Gamma_p K)$ is a ball, that is,

(3.11) $\phi(\Gamma_p K) = \left(\frac{V(\Gamma_p K)}{\omega_n}\right)^{\frac{1}{p}} B_n.$
Now we construct a parallelotope $P$ with its known condition is satisfied. Let $C$ be the cube centered at the origin with the side length $V(K)^{\frac{1}{n}}$, so $V(C) = V(K)$.

For $0 < q \leq \infty$, by using Lemma 2.3, we know that $E_q C = JC$. So, we have

\[(3.12)\quad E_q C = JC = \frac{1}{2} V(K)^{\frac{1}{n}} B_n.\]

Therefore, if $K \in K^n_\epsilon, 1 \leq p \leq \infty$ and $0 < q \leq \infty$, from (3.11), (3.12) and Theorem D, we have

\[\phi(\Gamma_{-p} K) = \left(\frac{V(\Gamma_{-p} K)}{\omega_n}\right)^{\frac{1}{n}} B_n = 2 \left(\frac{V(\Gamma_{-p} K)}{\omega_n V(K)}\right)^{\frac{1}{n}} E_q C \supseteq 2 \omega_n^{\frac{1}{n}} (nc_{n-2,p})^{-\frac{1}{n}} E_q C,\]

namely,

\[(3.13)\quad \phi(\Gamma_{-p} K) \supseteq 2 \omega_n^{\frac{1}{n}} (nc_{n-2,p})^{-\frac{1}{n}} E_q C.\]

Using (3.13) and Theorem A, we have

\[
\phi(\Gamma_{-p} K) \supseteq 2 \omega_n^{\frac{1}{n}} (nc_{n-2,p})^{-\frac{1}{n}} E_q (\phi^{-1} C) = 2 \omega_n^{\frac{1}{n}} (nc_{n-2,p})^{-\frac{1}{n}} E_q (\phi^{-1} C) = \phi(2 \omega_n^{\frac{1}{n}} (nc_{n-2,p})^{-\frac{1}{n}} E_q (\phi^{-1} C)),
\]

it follows that

\[\Gamma_{-p} K \supseteq 2 \omega_n^{\frac{1}{n}} (nc_{n-2,p})^{-\frac{1}{n}} E_q (\phi^{-1} C).\]

Now put $P = \phi^{-1} C$, then for $K \in K^n_\epsilon, 1 \leq p \leq \infty$ and $0 < q \leq \infty$, we have

\[(3.14)\quad \Gamma_{-p} K \supseteq 2 \omega_n^{\frac{1}{n}} (nc_{n-2,p})^{-\frac{1}{n}} E_q P, \text{ and } V(K) = V(C) = V(P).\]

On the other hand, for $K \in K^n_\epsilon, 1 \leq p \leq \infty$ and $0 < q \leq \infty$, using the same argument as in the first part of the proof and Lemma 3.3, we get

\[\phi(\Gamma_{-p} K) = \left(\frac{V(\Gamma_{-p} K)}{\omega_n}\right)^{\frac{1}{n}} B_n = 2 \left(\frac{V(\Gamma_{-p} K)}{\omega_n V(K)}\right)^{\frac{1}{n}} E_q C \supseteq (nc_{n-2,p})^{-\frac{1}{n}} E_q C,\]

namely

\[\phi(\Gamma_{-p} K) \supseteq (nc_{n-2,p})^{-\frac{1}{n}} E_q C.\]
From this and Theorem A, we get
\[
\phi(\Gamma_{-p}K) \supseteq (nc_{n-2,p})^{-\frac{1}{p}} E_q(\phi^{-1}C) \\
= (nc_{n-2,p})^{-\frac{1}{p}} \phi(E_q\phi^{-1}C) \\
= \phi((nc_{n-2,p})^{-\frac{1}{p}} E_q(\phi^{-1}C)).
\]

It follows that
\[
\Gamma_{-p}K \supseteq (nc_{n-2,p})^{-\frac{1}{p}} E_q(\phi^{-1}C).
\]
Let \( P = \phi^{-1}C \). Then for \( K \in K^n, 1 \leq p \leq \infty \) and \( 0 < q \leq \infty \), we have
\[
(3.15) \quad \Gamma_{-p}K \supseteq (nc_{n-2,p})^{-\frac{1}{p}} E_qP, \text{ and } V(K) = V(C) = V(P).
\]
Combination of the above two steps, we know that the proof of Theorem 1.1 is completed. \( \square \)

**Proof of Theorem 1.2.** As in the proof of Theorem 1.1, there exists \( \phi \in SL(n) \) such that \( \phi(\Gamma_{-p}K) \) is a ball, that is,
\[
(3.16) \quad \phi(\Gamma_{-p}K) = \left( \frac{V(\Gamma_{-p}K)}{\omega_n} \right)^{\frac{1}{p}} B_n,
\]
where \( p \geq 1 \). Construct a regular simplex \( T' \) with inscribed ball radius
\[
(3.17) \quad r = \left( \frac{n!V(K)}{n^{\frac{n}{2}}(n + 1)^{\frac{n+1}{2}}} \right)^{\frac{1}{n}}.
\]

By Lemma 2.5, we know \( V(T') = V(K) \).

From Eq.(3.17) and Lemma 2.4, we have
\[
(3.18) \quad E_q T' = JT' = rB_n = \left( \frac{n!V(K)}{n^{\frac{n}{2}}(n + 1)^{\frac{n+1}{2}}} \right)^{\frac{1}{p}} B_n.
\]

Further, from Theorem D, (3.17) and (3.18), we have
\[
\phi(\Gamma_{-p}K) = \left( \frac{V(\Gamma_{-p}K)}{\omega_n} \right)^{\frac{1}{p}} B_n \\
= \frac{1}{r} \left( \frac{V(\Gamma_{-p}K)}{\omega_n} \right)^{\frac{1}{p}} rB_n \\
= \left( \frac{n^{\frac{n}{2}}(n + 1)^{\frac{n+1}{2}}}{n!} \right)^{\frac{1}{p}} \left( \frac{V(\Gamma_{-p}K)}{\omega_n V(K)} \right)^{\frac{1}{p}} rB_n \\
\subseteq \left( \frac{n^{\frac{n}{2}}(n + 1)^{\frac{n+1}{2}}}{n!\omega_n} \right)^{\frac{1}{p}} (nc_{n-2,p})^{-\frac{1}{p}} E_q T',
\]

namely
\[
\phi(\Gamma_{-p}K) \subseteq \alpha_n(nc_{n-2,p})^{-\frac{1}{p}} E_q T',
\]
Corollary 4.1. For any origin-symmetric convex body \( K \), there exist an ellipsoid \( E \) and a parallelotope \( P \) such that for \( 0 < q \leq \infty \),

\[
2^{-1} \omega_n^{1/2} E_q E \subseteq K \subseteq 2 \omega_n^{-1/2} E_q P \text{ and } V(E) = V(K) = V(P).
\]
Take $q \to \infty$ in Theorem 1.1, and note that
\[
\lim_{q \to \infty} E_q P = JP,
\]
we get that:

**Corollary 4.2.** For any origin-symmetric convex body $K$, there exist an ellipsoid $E$ and a parallelotope $P$ such that for $0 < p \leq 2$,
\[
JE \supseteq \Gamma_{-p} K \supseteq (nc_{n-2,p})^{-\frac{1}{p}} JP \text{ and } V(E) = V(K) = V(P).
\]
For $2 \leq p \leq \infty$,
\[
2^{-1} \omega_n^\frac{1}{p} JE \subseteq \Gamma_{-p} K \subseteq 2\omega_n^\frac{1}{p} (nc_{n-2,p})^{-\frac{1}{p}} JP \text{ and } V(E) = V(K) = V(P).
\]
Take $p = q = 2$ in Theorem 1.1 and note that $E_2 K = \Gamma_{-2} K$, we immediately have that:

**Corollary 4.3.** For any origin-symmetric convex body $K$, there exist an ellipsoid $E$ and a parallelotope $P$ such that
\[
\Gamma_{-2} E \supseteq \Gamma_{-2} K \supseteq \Gamma_{-2} P \text{ and } V(E) = V(K) = V(P),
\]
\[
2^{-1} \omega_n^\frac{1}{p} \Gamma_{-2} E \subseteq \Gamma_{-2} K \subseteq 2\omega_n^\frac{1}{p} \Gamma_{-2} P \text{ and } V(E) = V(K) = V(P).
\]

Take $p \to \infty$ or $q \to \infty$ in Theorem 1.2, then we have that:

**Corollary 4.4.** For any convex body $K$ whose John point is at the origin, there exists a simplex $T$ such that for $0 < q \leq \infty$,
\[
\alpha_n E_q T \supseteq K \supseteq E_q T \text{ and } V(K) = V(T).
\]
For $1 \leq p \leq \infty$,
\[
\alpha_n (nc_{n-2,p})^{-\frac{1}{p}} JT \supseteq \Gamma_{-p} K \supseteq (nc_{n-2,p})^{-\frac{1}{p}} JT \text{ and } V(K) = V(T).
\]
Take $p \to \infty$ and $q \to \infty$ in Theorem 1.2, we immediately have that:

**Corollary 4.5.** For any convex body $K$ whose John point is at the origin, there exists a simplex $T$ such that
\[
\alpha_n JT \supseteq K \supseteq JT \text{ and } V(K) = V(T).
\]
Take $p = q = 2$ in Theorem 1.2, we immediately have that:

**Corollary 4.6.** For any convex body $K$ whose John point is at the origin, there exists a simplex $T$ such that
\[
\Gamma_{-2} T \subseteq K \subseteq \alpha_n \Gamma_{-2} T \text{ and } V(T) = V(K).
\]

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