REGULARITY AND GREEN’S RELATIONS ON SEMIGROUPS OF TRANSFORMATION PRESERVING ORDER AND COMPRESSION

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Abstract. Let \([n] = \{1, 2, \ldots, n\}\), and let \(PO_n\) be the partial order-preserving transformation semigroup on \([n]\). Let \(CPO_n = \{\alpha \in PO_n : (\forall x, y \in \text{dom}\alpha), |x\alpha - y\alpha| \leq |x - y|\}\). Then \(CPO_n\) is a subsemigroup of \(PO_n\). In this paper, we characterize Green’s relations and the regularity of elements for \(CPO_n\).

1. Introduction

Let \(S\) be a semigroup, \(a, b \in S\). If \(a\) and \(b\) generate the same left principal ideal, that is, \(S^1a = S^1b\), then we say that \(a\) and \(b\) are \(L\) equivalent and write \(aLb\) or \((a, b) \in L\). If \(a\) and \(b\) generate the same right principal ideal, that is, \(aS^1 = bS^1\), then we say that \(a\) and \(b\) are \(R\) equivalent and write \(aRb\) or \((a, b) \in R\). If \(a\) and \(b\) generate the same principal ideal, that is, \(S_1^1aS_1 = S_1^1bS_1\), then we say that \(a\) and \(b\) are \(J\) equivalent and write \(aJb\) or \((a, b) \in J\). Let \(H = L \cap R\), \(D = L \circ R\), then \(H, D\) are equivalences on \(S\), too. It is well known that in a finite semigroup \(J = D\). These five equivalences are usually called Green’s equivalences on \(S\). They were introduced by J. A. Green in [2], and have a great importance in the study of the algebraic structure of semigroups.

Let \([n] = \{1, 2, \ldots, n\}\) ordered in the standard way. We denote by \(PT_n\) the semigroup of all partial transformations on \([n]\). We say \(\alpha \in PT_n\) is order-preserving if, for all \(x, y \in \text{dom} \alpha\), \(x \leq y\) implies \(x\alpha \leq y\alpha\), and \(\alpha\) is compressing-preserving if, for all \(x, y \in \text{dom} \alpha\), \(|x\alpha - y\alpha| \leq |x - y|\). We denote by \(PO_n\) the subsemigroup of \(PT_n\) of all partial order-preserving transformations (excluding the identity mapping). Let

\[CPO_n = \{\alpha \in PO_n : (\forall x, y \in \text{dom}\alpha), |x\alpha - y\alpha| \leq |x - y|\}\]
be the set of $PO_n$ consisting of all partial order-preserving and compressing-preserving mappings on $[n]$. It is easily verified that $CPO_n$ is a subsemigroup of $PO_n$.

The Green’s relations on various special subsemigroups of $PT_n$ have been studied by many authors; see for example, [1], [3], [4], [5], [6], [7], [8], [9], [10], [11]. The purpose of this paper is to investigate regularity of elements and Green’s relations for the new subsemigroup $CPO_n$ of $PT_n$. Accordingly, in Section 2, the condition under which an element $a \in CPO_n$ is regular is analyzed, and a necessary and sufficient condition is established. In Section 3, Green’s equivalences on $CPO_n$ are considered and the relations $L$, $R$ and $D$ are described for arbitrary elements.

2. Regular elements in $CPO_n$

In this section we investigate the condition under which an elements of $CPO_n$ is regular and describe some properties of regular elements. We first need the following terminology and notation.

Let $x, y \in [n]$, $x < y$. The set $[x, y] = \{ z \in [n] : x \leq z \leq y \}$ of $[n]$ is called a closed interval. Similarly, we can define the intervals of other kinds, such as $(x, y]$, $(x, y)$ and $[x, y)$. Let $P, Q$ be two subsets of $[n]$. If $a < b$ holds for arbitrary $a \in P$ and $b \in Q$, then we say that $P$ is less than $Q$, and write $P < Q$. For any $a \in PO_n$, it is obvious that $xa^{-1} < ya^{-1}$ if $x < y (x, y \in ima)$. Then every $a \in CPO_n$ can be expressed as

$$
(2.1) \quad a = \left( \begin{array}{cccc}
A_1 & A_2 & \cdots & A_s \\
a_1 & a_2 & \cdots & a_s
\end{array} \right),
$$

where $a_1 < a_2 < \cdots < a_s$, $A_1 < A_2 < \cdots < A_s$, $a_i - a_{i-1} \leq \min A_i - \max A_{i-1}$, $i = 2, \ldots, s$ (if $s \geq 2$).

**Definition 2.1.** For any two elements of $CPO_n$ with the same rank:

$$
\alpha = \left( \begin{array}{cccc}
A_1 & A_2 & \cdots & A_s \\
a_1 & a_2 & \cdots & a_s
\end{array} \right), \quad \beta = \left( \begin{array}{cccc}
B_1 & B_2 & \cdots & B_s \\
b_1 & b_2 & \cdots & b_s
\end{array} \right),
$$

where $a_1 < a_2 < \cdots < a_s$, $A_1 < A_2 < \cdots < A_s$, $a_i - a_{i-1} \leq \min A_i - \max A_{i-1}$, $b_1 < b_2 < \cdots < b_s$, $B_1 < B_2 < \cdots < B_s$, $b_i - b_{i-1} \leq \min B_i - \max B_{i-1}$, $i = 2, \ldots, s$, $s \geq 2$. Let $d = \max A_1 - \max B_1$. If $\min A_s - \min B_s = d$ and $A_i = B_i + d$ or $B_i = A_i - d$, $i \in [2, s - 1]$ (if $s \geq 3$), then $\alpha, \beta$ are called same kernel-type, denoted $\alpha \overset{K}{=} \beta$. If $|a_i - a_j| = |b_i - b_j|$ for all $i, j \in [1, s]$, then $\alpha, \beta$ are called same image-type, denoted $\alpha \overset{I}{=} \beta$.

Now we investigate the condition under which an element in $CPO_n$ is regular.

**Theorem 2.2.** Let $\alpha \in CPO_n$ be as defined in (2.1) and let $d = \max A_1 - a_1$.

1. If $|ima| = 1$, then $\alpha$ is regular.
2. If $|ima| = 2$, then $\alpha$ is regular if and only if $\max A_1 - a_1 = \min A_2 - a_2$. 
(3) If |ima| ≥ 3, then α is regular if and only if minA_i - a_i = d and A_i = {a_i + d}, i ∈ [2, s - 1].

Proof. (1) Note that α = (A_i/a_i). Let x ∈ A_1. Define β by

$$β = \begin{pmatrix} a_1 \\ x \end{pmatrix}.$$ 

Then β ∈ CPO_n and α = αβα.

(2) Note that α = (A_1, A_2). Suppose that α is regular. Then there exists β ∈ CPO_n such that α = αβα. Thus a_i = A_iα = (A_1α)βα = (a_iβ)α (i = 1, 2) and so a_iβ ∈ A_i (i = 1, 2). Note that a_1 < a_2, A_1 < A_2 and α, β ∈ CPO_n. It follows that

$$a_2 - a_1 = |a_2 - a_1| = |A_2α - A_1α|$$

$$= |(\text{min}A_2)α - (\text{max}A_1)α|$$

$$≤ |\text{min}A_2 - \text{max}A_1| = \text{min}A_2 - \text{max}A_1$$

$$≤ a_2ββ - a_1ββ = |a_2β - a_1ββ|$$

$$≤ |a_2 - a_1| = a_2 - a_1.$$ 

Thus a_2 - a_1 = minA_2 - maxA_1 and so maxA_1 - a_1 = minA_2 - a_2.

Conversely, suppose that maxA_1 - a_1 = minA_2 - a_2. Define β by

$$β = \begin{pmatrix} a_1 \\ \text{max}A_1 \\ a_2 \\ \text{min}A_2 \end{pmatrix}.$$ 

Then β ∈ CPO_n and α = αβα.

(3) Suppose that α is regular. Then there exists β ∈ CPO_n such that α = αβα. Thus, for any i, j ∈ [1, s],

$$a_i = A_iα = (A_1α)βα = (a_iβ)α, \ a_j = A_jα = (A_jα)βα = (a_jβ)α,$$

and so

$$a_iβ ∈ A_i, \ a_jβ ∈ A_j.$$ 

Take any i, j ∈ [1, s] with i < j. Note that a_i < a_j, A_i < A_j and (2.2). Since α, β ∈ CPO_n, we have

$$a_j - a_i = |a_j - a_i| = |A_jα - A_iα|$$

$$= |(\text{min}A_j)α - (\text{max}A_i)α|$$

$$≤ |\text{min}A_j - \text{max}A_i| = \text{min}A_j - \text{max}A_i$$

$$≤ a_jββ - a_iββ = |a_jβ - a_iββ|$$

$$≤ |a_j - a_i| = a_j - a_i.$$ 

Thus

$$a_j - a_i = \text{min}A_j - \text{max}A_i, \ i < j, \ i, j ∈ [1, s],$$

and so

$$\text{min}A_j - a_j = \text{max}A_i - a_i, \ i < j, \ i, j ∈ [1, s].$$
Clearly, \( \min A_s - a_s = \max A_1 - a_1 = d \). By (2.3), we have
\[
\min A_s - a_s = \max A_i - a_i, \min A_i - a_i = \max A_1 - a_1, i \in [2, s-1].
\]
Thus \( \max A_i - a_i = \min A_i - a_i = d \) and so \( \max A_i = \min A_i = a_i + d \). It follows that \( A_i = \{a_i + d\} \).

Conversely, suppose that \( \min A_s - a_s = d \) and \( A_i = \{a_i + d\}, i \in [2, s-1] \).

Define \( \beta \) by
\[
\beta = \left( \begin{array}{cccc}
a_1 & a_2 & \cdots & a_{s-1} & a_s \\
\max A_1 & a_2 + d & \cdots & a_{s-1} + d & \min A_s
\end{array} \right).
\]
Then \( \beta \in \text{CPO}_n \) and \( \alpha = \alpha \beta \alpha \).

Remark 2.3. If \( n \geq 5 \), then the semigroup \( \text{CPO}_n \) is not regular. In fact, let
\[
\alpha = \left( \begin{array}{ccc}1 & 1 & \{2, 3\} \\
2 & 4 & 5
\end{array} \right) \in \text{CPO}_n.
\]
Then, by Theorem 2.2(3), \( \alpha \) is not regular in \( \text{CPO}_n \).

Next we observe two properties for regular elements in the semigroup \( \text{CPO}_n \).

Theorem 2.4. Let \( \alpha, \beta \in \text{CPO}_n \) be regular elements with \( \text{Im} \alpha = \text{Im} \beta \ (|\text{Im} \alpha| \geq 2) \). Then \( \alpha \overleftarrow{\text{Ker}} \beta \).

Proof. Let
\[
\alpha = \left( \begin{array}{cccc}A_1 & A_2 & \cdots & A_s \\
\frac{a_1}{a_2} & a_2 & \cdots & a_s
\end{array} \right), \quad \beta = \left( \begin{array}{cccc}B_1 & B_2 & \cdots & B_s \\
\frac{a_1}{a_2} & a_2 & \cdots & a_s
\end{array} \right),
\]
where \( a_1 < a_2 < \cdots < a_s, A_1 < A_2 < \cdots < A_s, B_1 < B_2 < \cdots < B_s \).

Let \( d_1 = \max A_1 - a_1, d_2 = \max B_1 - a_1 \) and \( d = \max A_1 - \max B_1 \). Then \( d = d_1 - d_2 \).

Since \( \alpha, \beta \) are regular, by Theorem 2.2, we have
\[
\min A_s - a_s = \max A_1 - a_1 = d_1, \min B_s - a_s = \max B_1 - a_1 = d_2,
\]
\[
A_i = \{a_i + d_1\}, B_i = \{a_i + d_2\}, i \in [2, s-1] \ (\text{if } s \geq 3).
\]

It follows that
\[
\min A_s - \min B_s = \max A_1 - \max B_1 = d,
\]
\[
A_i = \{a_i + d_2 + d\} = B_i + d, i \in [2, s-1] \ (\text{if } s \geq 3).
\]
Thus \( \alpha \overleftarrow{\text{Ker}} \beta \).

Theorem 2.5. Let \( \alpha, \beta \in \text{CPO}_n \) be regular elements with \( \text{Ker} \alpha = \text{Ker} \beta \ (|\text{Ker} \alpha| \geq 2) \). Then \( \alpha \overleftarrow{\text{Im}} \beta \).

Proof. Let
\[
\alpha = \left( \begin{array}{cccc}A_1 & A_2 & \cdots & A_s \\
\frac{a_1}{a_2} & a_2 & \cdots & a_s
\end{array} \right), \quad \beta = \left( \begin{array}{cccc}B_1 & A_2 & \cdots & A_s \\
\frac{b_1}{b_2} & b_2 & \cdots & b_s
\end{array} \right),
\]
Theorem 3.1. Let \( L \) the Green relations known that \( D = \text{im} \), \( A_i = \{a_i + d_1\}, A_i = \{b_i + d_2\}, i \in [2, s - 1] \) (if \( s \geq 3 \)). Then
\[
A_i - b_i = d_2 - d_1, \quad i \in [2, s - 1] \quad \text{(if \( s \geq 3 \))}
\]
and so
\[
a_i - b_i = d_2 - d_1, \quad i \in [1, s].
\]
It follows that for all \( i, j \in [1, s] \),
\[
|a_i - a_j| = |(b_i + d_2 - d_1) - (b_j + d_2 - d_1)| = |b_i - b_j|.
\]
Thus \( \alpha \overline{\beta} \).

3. Green’s relations on \( CPO_n \)

In this section, we describe Green’s relations on \( CPO_n \). Since it is well known that \( D = F \) in every finite semigroup and \( H = L \wedge R \), we only consider the Green relations \( L, R \) and \( D \). We begin with the relation \( L \).

Theorem 3.1. Let \( \alpha, \beta \in CPO_n \).

(1) If \( |\text{im} \alpha| = 1 \), then \( (\alpha, \beta) \in L \) if and only if \( \text{im} \alpha = \text{im} \beta \).

(2) If \( |\text{im} \alpha| \geq 2 \), then \( (\alpha, \beta) \in L \) if and only if \( \text{im} \alpha = \text{im} \beta \) and \( \alpha \overline{\beta} \).

Proof. (1) Suppose that \( (\alpha, \beta) \in L \). Then there exist \( \delta, \gamma \in (CPO_n)^1 \) such that \( \alpha = \delta \beta, \beta = \gamma \alpha \). It follows easily that \( \text{im} \alpha = \text{im} \beta \). Conversely, suppose that \( \text{im} \alpha = \text{im} \beta \). Let
\[
\alpha = \begin{pmatrix} A \\ a \end{pmatrix}, \quad \beta = \begin{pmatrix} B \\ a \end{pmatrix}.
\]
Let \( c \in B \) and \( d \in A \). Define \( \delta, \gamma \) by
\[
\delta = \begin{pmatrix} A \\ c \end{pmatrix}, \quad \gamma = \begin{pmatrix} B \\ d \end{pmatrix}.
\]
Clearly, \( \delta, \gamma \in CPO_n \) and \( \alpha = \delta \beta, \beta = \gamma \alpha \). Thus \( (\alpha, \beta) \in L \).

(2) Suppose that \( (\alpha, \beta) \in L \). Then there exist \( \delta, \gamma \in (CPO_n)^1 \) such that \( \alpha = \delta \beta, \beta = \gamma \alpha \). It follows easily that \( \text{im} \alpha = \text{im} \beta \). Let
\[
\alpha = \begin{pmatrix} A_1 & A_2 & \cdots & A_s \\ a_1 & a_2 & \cdots & a_s \end{pmatrix}, \quad \beta = \begin{pmatrix} B_1 & B_2 & \cdots & B_s \\ a_1 & a_2 & \cdots & a_s \end{pmatrix},
\]
where \( a_1 < a_2 < \cdots < a_s, A_1 < A_2 < \cdots < A_s, B_1 < B_2 < \cdots < B_s \). Let
\[
d = \max A_1 - \max B_1.
\]
Note that
\[
A_i \delta \beta = A_i \alpha = a_i = B_i \beta, B_i \gamma \alpha = B_i \beta = a_i = A_i \alpha, \quad i \in [1, s].
\]
We obtain that
\[(3.1) \quad A_i \delta \subseteq B_i, \quad B_i \gamma \subseteq A_i, \quad i \in [1, s].\]
Take any \(i, j \in [1, s]\) with \(i < j\). From (3.1) we know that
\[(3.2) \quad (\min A_i) \delta \in B_j, \quad (\max A_i) \delta \in B_i, \quad (\min B_j) \gamma \in A_j, \quad (\max B_i) \gamma \in A_i.
\]
Note that
\[(3.3) \quad A_i < A_j, B_i < B_j.
\]
Since \(\delta, \gamma \in (CPO_n)^1\), we have
\[
\begin{align*}
\min B_j - \max B_i & \leq (\min A_j) \delta - (\max A_i) \delta = |(\min A_j) \delta - (\max A_i) \delta| \\
& \leq |\min A_j - \max A_i| = \min A_j - \max A_i,
\end{align*}
\]
\[
\begin{align*}
\min A_j - \max A_i & \leq (\min B_j) \gamma - (\max B_i) \gamma = |(\min B_j) \gamma - (\max B_i) \gamma| \\
& \leq |\min B_j - \max B_i| = \min B_j - \max B_i.
\end{align*}
\]
It follows from (3.2) that
\[(3.4) \quad \min A_j - \min B_j = \max A_i - \max B_i,
\]
\[(3.5) \quad (\min A_j) \delta = \min B_j, \quad (\max A_i) \delta = \max B_i,
\]
\[(3.6) \quad (\min B_j) \gamma = \min A_j, \quad (\max B_i) \gamma = \max A_i.
\]
If \(|\text{ima}| = 2\), then, by (3.4), \(\min A_2 - \min B_2 = \max A_1 - \max B_1 = d\). Thus \(\alpha \kappa \beta\). If \(|\text{ima}| \geq 3\), we claim that
\[(3.7) \quad x \delta = x - d, \quad \forall x \in A_i, \quad i \in [2, s - 1],
\]
\[(3.8) \quad x \gamma = x + d, \quad \forall x \in B_i, \quad i \in [2, s - 1].
\]
Note that (3.1) and (3.3). Since \(\delta \in (CPO_n)^1\), we have
\[
\begin{align*}
x \delta - (\max A_i) \delta & = |x \delta - (\max A_i) \delta| \leq |x - \max A_i| = x - \max A_i, \\
(\min A_s) \delta - x \delta & = |(\min A_s) \delta - x \delta| \leq |\min A_s - x| = \min A_s - x.
\end{align*}
\]
Thus
\[\max A_1 - (\max A_i) \delta \leq x - x \delta \leq \min A_s - (\min A_s) \delta.\]
It follows from (3.4) and (3.5) that
\[d = \max A_1 - \max B_1 = \max A_1 - (\max A_i) \delta \leq x - x \delta \leq \min A_s - (\min A_s) \delta = \min A_s - \min B_s = \max A_1 - \max B_1 = d.
\]
Thus \(x \delta = x - d\). Similarly, we can prove that (3.8) holds. Now, we shall prove that \(\alpha \kappa \beta\). By (3.4), we have
\[\min A_s - \min B_s = \max A_1 - \max B_1 = d.
\]
From (3.7) and (3.8) we know that \(\delta|_{A_i}\) and \(\gamma|_{B_i}\) are mutually inverse bijections from \(A_i\) onto \(A_i \delta\) and \(B_i\) onto \(B_i \gamma\), respectively. Then, by (3.1),
\[
|A_i| = |A_i \delta| \leq |B_i|, \quad |B_i| = |B_i \gamma| \leq |A_i|, \quad i \in [2, s - 1]
\]
and so \(|A_i| = |A_\delta| = |B_\gamma| = |B_\delta|\). Thus \(A_i \delta = B_i\) and \(B_i \gamma = A_i\) by (3.1).
Moreover, by (3.7) and (3.8), we have
\[
A_i = B_i + d, \quad i \in [2, s - 1],
\]
\[
B_i = A_i - d, \quad i \in [2, s - 1].
\]
Thus \(A K x B\).

Conversely, suppose that \(ima = im\beta\) and \(\alpha K x \beta\). Let
\[
\alpha = \begin{pmatrix} A_1 & A_2 & \cdots & A_s \\ a_1 & a_2 & \cdots & a_s \end{pmatrix}, \quad \beta = \begin{pmatrix} B_1 & B_2 & \cdots & B_s \\ a_1 & a_2 & \cdots & a_s \end{pmatrix},
\]
where \(a_1 < a_2 < \cdots < a_s, A_1 < A_2 < \cdots < A_s, B_1 < B_2 < \cdots < B_s\). Let \(d = \max A_1 - \max B_1\). If \(|ima| = 2 (s = 2)\), Define \(\delta, \gamma\) by
\[
\delta = \begin{pmatrix} A_1 & A_2 \\ \max B_1 & \min B_2 \end{pmatrix}, \quad \gamma = \begin{pmatrix} B_1 & B_2 \\ \max A_1 & \min A_2 \end{pmatrix}.
\]
Clearly, \(\delta, \gamma \in P O_n\), \(\alpha = \delta \beta\) and \(\beta = \gamma \alpha\). Since \(\alpha K x \beta\), we have \(\max A_1 - \max B_1 = \min A_2 - \min B_2 (= d)\) and so \(\min A_2 - \max A_1 = \min B_2 - \max B_1\). It easily follows that \(\delta, \gamma \in C P O_n\). Thus \((\alpha, \beta) \in \mathcal{L}\).
If \(|ima| \geq 3 (s \geq 3)\), then since \(\alpha K x \beta\), we have
\[
(3.9) \quad \min A_s - \min B_s = d, \quad A_i = B_i + d.
\]
Define \(\delta, \gamma\) by
\[
\begin{align*}
\delta &= \begin{cases} 
\max B_1, & \text{if } x \in A_1, \\
-x - d, & \text{if } x \in A_i, i \in [2, s - 1], \\
\min B_s, & \text{if } x \in A_s,
\end{cases} \\
\gamma &= \begin{cases} 
\max A_1, & \text{if } x \in B_1, \\
-x + d, & \text{if } x \in B_i, i \in [2, s - 1], \\
\min A_s, & \text{if } x \in B_s.
\end{cases}
\end{align*}
\]
Clearly, \(\delta, \gamma \in P O_n\), \(\alpha = \delta \beta\) and \(\beta = \gamma \alpha\). Take any \(x, y \in \text{dom}\delta\) with \(x \leq y\).
Note that \(\text{dom}\delta = A_1 \cup A_2 \cup \cdots \cup A_s\). We distinguish five cases.

Case 1: \(x, y \in A_i, i \in \{1, s\}\). Clearly, \(x \delta - y \delta = 0\). Thus \(|x \delta - y \delta| \leq |x - y|\).

Case 2: \(x \in A_1, y \in A_i, i \in [2, s - 1]\). Since \(A_1 < A_i\), we have
\[
|x \delta - y \delta| = |\max B_1 - (y - d)| = |y - (d + \max B_1)| = |y - \max A_1| \leq |x - y|.
\]

Case 3: \(x \in A_1, y \in A_s\). Note that \(A_1 < A_s\). By (3.9), we have
\[
|x \delta - y \delta| = |\max B_1 - \min B_s| = |(\max A_1 - d) - (\min A_s - d)| = |\max A_1 - \min A_s| \leq |x - y|.
\]

Case 4: \(x \in A_i, y \in A_j, i, j \in [2, s - 1], i \neq j\). Clearly, \(|x \delta - y \delta| = |(x - d) - (y - d)| = |x - y|\). Thus \(|x \delta - y \delta| \leq |x - y|\).

Case 5: \(x \in A_i, i \in [2, s - 1], y \in A_s\). Note that \(A_i < A_s\). By (3.9), we have
\[
|x \delta - y \delta| = |(x - d) - \min B_s| = |(\min B_s + d) - x| = |\min A_s - x| \leq |x - y|.
\]
Form the discussion above, we have $\delta \in CPO_n$. Similarly, we can prove that $\gamma \in CPO_n$. Thus $(\alpha, \beta) \in \mathcal{L}$. □

**Theorem 3.2.** Let $\alpha, \beta \in CPO_n$.

1. If $|\text{Im}\alpha| = 1$, then $(\alpha, \beta) \in \mathcal{R}$ if and only if $\text{Ker}\alpha = \text{Ker}\beta$.
2. If $|\text{Im}\alpha| \geq 2$, then $(\alpha, \beta) \in \mathcal{R}$ if and only if $\text{Ker}\alpha = \text{Ker}\beta$ and $\alpha \triangleleft \beta$.

**Proof.** (1) Suppose that $(\alpha, \beta) \in \mathcal{R}$. Then there exist $\delta, \gamma \in (CPO_n)^1$ such that $\alpha = \beta\delta$, $\beta = \alpha\gamma$. For each $(x, y) \in \text{Ker}\alpha$, we have

$$x = x(\alpha\gamma) = (x\alpha)\gamma = y(\alpha\gamma) = y\beta.$$ 

Then $\text{Ker}\alpha \subseteq \text{Ker}\beta$. Similarly, we can prove that $\text{Ker}\beta \subseteq \text{Ker}\alpha$. Thus $\text{Ker}\alpha = \text{Ker}\beta$. Conversely, suppose that $\text{Ker}\alpha = \text{Ker}\beta$. Let

$$\alpha = \begin{pmatrix} A \\ a \end{pmatrix}, \quad \beta = \begin{pmatrix} A \\ b \end{pmatrix}.$$ 

Define $\delta, \gamma$ by

$$\delta = \begin{pmatrix} b \\ a \end{pmatrix}, \quad \gamma = \begin{pmatrix} a \\ b \end{pmatrix}.$$ 

Clearly, $\delta, \gamma \in CPO_n$ and $\alpha = \beta\delta$, $\beta = \alpha\gamma$. Thus $(\alpha, \beta) \in \mathcal{R}$.

(2) Suppose that $(\alpha, \beta) \in \mathcal{R}$. Then there exist $\delta, \gamma \in (CPO_n)^1$ such that $\alpha = \beta\delta$, $\beta = \alpha\gamma$. Thus by the same proof as given for (1), we have $\text{Ker}\alpha = \text{Ker}\beta$. Let

$$\alpha = \begin{pmatrix} A_1 & A_2 & \cdots & A_s \\ a_1 & a_2 & \cdots & a_s \end{pmatrix}, \quad \beta = \begin{pmatrix} A_1 & A_2 & \cdots & A_s \\ b_1 & b_2 & \cdots & b_s \end{pmatrix},$$ 

where $a_1 < a_2 < \cdots < a_s$, $b_1 < b_2 < \cdots < b_s$, $A_1 < A_2 < \cdots < A_s$. Note that for any $i, j \in [1, s]$,

$$a_i = A_i\alpha = A_i(\beta\delta) = (A_i\beta)d_i = b_i\delta, \quad a_j = A_j\alpha = A_j(\beta\delta) = (A_j\beta)d_j = b_j\delta,$$

$$b_i = A_i\beta = A_i(a\gamma) = (A_i\alpha)\gamma = a_i\gamma, \quad b_j = A_j\beta = A_j(a\gamma) = (A_j\alpha)\gamma = a_j\gamma.$$ 

Since $\delta, \gamma \in CPO_n$, we have

$$|a_i - a_j| = |b_i\delta - b_j\delta| \leq |b_i - b_j|,$$

$$|b_i - b_j| = |a_i\gamma - a_j\gamma| \leq |a_i - a_j|.$$ 

Thus $|a_i - a_j| = |b_i - b_j|$ and so $\alpha \triangleleft \beta$.

Conversely, suppose that $\text{Ker}\alpha = \text{Ker}\beta$ and $\alpha \triangleleft \beta$. Let

$$\alpha = \begin{pmatrix} A_1 & A_2 & \cdots & A_s \\ a_1 & a_2 & \cdots & a_s \end{pmatrix}, \quad \beta = \begin{pmatrix} A_1 & A_2 & \cdots & A_s \\ b_1 & b_2 & \cdots & b_s \end{pmatrix},$$ 

where $a_1 < a_2 < \cdots < a_s$, $b_1 < b_2 < \cdots < b_s$, $A_1 < A_2 < \cdots < A_s$. Define $\delta, \gamma$ by

$$b_i\delta = a_i, \quad i \in [1, s],$$

$$a_i\gamma = b_i, \quad i \in [1, s].$$ 

Clearly, $\delta, \gamma \in PO_n$, $\alpha = \beta\delta$ and $\beta = \alpha\gamma$. Since $\alpha \triangleleft \beta$, we have $|a_i - a_j| = |b_i - b_j|$. Thus $\delta, \gamma \in CPO_n$ and so $(\alpha, \beta) \in \mathcal{R}$. □
Theorem 3.3. Let \( \alpha, \beta \in CPO_n \).

(1) If \( |ima| = 1 \), then \( (\alpha, \beta) \in D \) if and only if \( |ima| = |im\beta| \).

(2) If \( |ima| \geq 2 \), then \( (\alpha, \beta) \in D \) if and only if \( |ima| = |im\beta| \), \( \alpha K_{\text{err}} \beta \) and \( \alpha L_{\text{im}} \beta \).

Proof. (1) Suppose that \( (\alpha, \beta) \in D \). Then there exists \( \gamma \in (CPO_n)^1 \) such that \( \alpha L_{\gamma} R_{\beta} \). By Theorems 3.1 and 3.2, we have \( ima = im\gamma \), \( Ker\gamma = Ker\beta \). It follows easily that \( |ima| = |im\beta| \). Conversely, suppose that \( |ima| = |im\beta| \).

Let
\[
\alpha = \begin{pmatrix} A \\ a \end{pmatrix}, \quad \beta = \begin{pmatrix} B \\ b \end{pmatrix}.
\]
Define \( \gamma = \begin{pmatrix} B \\ a \end{pmatrix} \).

Clearly, \( \gamma \in CPO_n \) and \( ima = im\gamma \), \( Ker\gamma = Ker\beta \). Then, by Theorems 3.1 and 3.2, \( \alpha L_{\gamma} R_{\beta} \). Thus \( (\alpha, \beta) \in D \).

(2) Suppose that \( (\alpha, \beta) \in D \). Then there exists \( \gamma \in (CPO_n)^1 \) such that \( \alpha L_{\gamma} R_{\beta} \). By Theorems 3.1 and 3.2, we have \( ima = im\gamma \), \( \alpha K_{\text{err}} \gamma \) and \( Ker\gamma = Ker\beta \), \( \gamma L_{\text{im}} \beta \). It follows easily that \( |ima| = |im\beta| \) and \( \alpha K_{\text{err}} \beta \), \( \alpha L_{\text{im}} \beta \).

Conversely, suppose that \( |ima| = |im\beta| \), \( \alpha K_{\text{err}} \beta \) and \( \alpha L_{\text{im}} \beta \). Let
\[
\alpha = \begin{pmatrix} A_1 & A_2 & \cdots & A_s \\ a_1 & a_2 & \cdots & a_s \end{pmatrix}, \quad \beta = \begin{pmatrix} B_1 & B_2 & \cdots & B_s \\ b_1 & b_2 & \cdots & b_s \end{pmatrix},
\]
where \( a_1 < a_2 < \cdots < a_s \), \( A_1 < A_2 < \cdots < A_s \), \( b_1 < b_2 < \cdots < b_s \), \( B_1 < B_2 < \cdots < B_s \). Define \( \gamma \) by
\[
\gamma = \begin{pmatrix} B_1 & B_2 & \cdots & B_s \\ a_1 & a_2 & \cdots & a_s \end{pmatrix}.
\]
Clearly, \( ima = im\gamma \) and \( Ker\gamma = Ker\beta \). Since \( \alpha K_{\text{err}} \beta \) and \( \alpha L_{\text{im}} \beta \), we have that \( \alpha K_{\text{err}} \gamma \) and \( \gamma L_{\text{im}} \beta \). Then, by Theorems 3.1 and 3.2, \( \alpha L_{\gamma} \beta \) and \( \gamma R_{\beta} \). Thus \( (\alpha, \beta) \in D \). \( \square \)

As a consequence of Theorems 2.3, 2.4, 3.1 and 3.2, the following result follows readily.

Corollary 3.4. Let \( \alpha, \beta \in CPO_n \) be regular elements. Then the following statements hold:

(1) \( (\alpha, \beta) \in L \) if and only if \( ima = im\beta \).

(2) \( (\alpha, \beta) \in R \) if and only if \( Ker\alpha = Ker\beta \).

For the condition under which two regular elements in \( CPO_n \) are \( D \) equivalent, we have:

Theorem 3.5. Let \( \alpha, \beta \in CPO_n \) be regular elements.

(1) If \( |ima| = 1 \), then \( (\alpha, \beta) \in D \) if and only if \( |ima| = |im\beta| \).

(2) If \( |ima| \geq 2 \), then \( (\alpha, \beta) \in D \) if and only if \( |ima| = |im\beta| \) and \( \alpha L_{\text{im}} \beta \).
Proof. (1) It is an immediate consequence of Theorem 3.3.
(2) Suppose that \((\alpha, \beta) \in \mathcal{D}\). Then, by Theorem 3.3, \(|i\alpha a| = |im\beta|\) and \(\alpha \text{Im} \beta\).

Conversely, suppose that \(|i\alpha a| = |im\beta|\) and \(\alpha \text{Im} \beta\). Let

\[
\alpha = \begin{pmatrix} A_1 & A_2 & \cdots & A_s \\ a_1 & a_2 & \cdots & a_s \end{pmatrix}, \quad \beta = \begin{pmatrix} B_1 & B_2 & \cdots & B_s \\ b_1 & b_2 & \cdots & b_s \end{pmatrix},
\]

where \(a_1 < a_2 < \cdots < a_s, A_1 < A_2 < \cdots < A_s, b_1 < b_2 < \cdots < b_s, B_1 < B_2 < \cdots < B_s\). Since \(\alpha \text{Im} \beta\), we have

\[
|a_i - a_j| = |b_i - b_j|, i, j \in [1, s].
\]

Note that \(a_i \geq a_j, b_i \geq b_j\) if \(i \geq j\); \(a_i \leq a_j, b_i \leq b_j\) if \(i \leq j\). It follows easily from (3.10) that

\[
a_i - b_i = a_j - b_j, i, j \in [1, s].
\]

Let \(d_1 = \max A_1 - a_1, d_2 = \max B_1 - b_1\) and \(d = \max A_1 - \max B_1\). Then

\[
a_i - b_i = (\max A_1 - \max B_1) + d_2 - d_1 = d + d_2 - d_1.
\]

By (3.11), we have

\[
a_i - b_i = a_1 - b_1 = d + d_2 - d_1, i \in [1, s].
\]

Since \(\alpha, \beta\) are regular, by Theorem 3.1, we have

\[
\min A_i - a_s = \max A_i - a_1 = d_1, \min B_i - b_s = \max B_i - b_1 = d_2,
\]

\[
A_i = \{a_i + d_1\}, \quad B_i = \{b_i + d_2\}, \quad i \in [2, s-1] \quad (\text{if } s \geq 3).
\]

Then, by (3.12) that

\[
\min A_i - \min B_i = (d_1 + a_s) - (d_2 + b_s) = (a_s - b_s) + (d_1 - d_2) = d,
\]

\[
A_i = \{a_i + d_1\} = \{(d + d_2 - d_1 + b_i) + d_1\} = \{(b_i + d_2) + d\} = B_i + d.
\]

Thus \(\alpha \text{Reg} \beta\) and so, by Theorem 3.3, \((\alpha, \beta) \in \mathcal{D}\). \(\square\)

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