AMALGAMATED DUPLICATION OF SOME SPECIAL RINGS

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Abstract. Let $R$ be a commutative Noetherian ring and let $I$ be an ideal of $R$. In this paper we study the amalgamated duplication ring $R \triangleright ◁ I$ which is introduced by D’Anna and Fontana. It is shown that if $R$ is generically Cohen-Macaulay (resp. generically Gorenstein) and $I$ is generically maximal Cohen-Macaulay (resp. generically canonical module), then $R \triangleright I$ is generically Cohen-Macaulay (resp. generically Gorenstein). We also defined generically quasi-Gorenstein ring and we investigate when $R \triangleright ◁ I$ is generically quasi-Gorenstein. In addition, it is shown that $R \triangleright ◁ I$ is approximately Cohen-Macaulay if and only if $R$ is approximately Cohen-Macaulay, provided some special conditions. Finally it is shown that if $R$ is approximately Gorenstein, then $R \triangleright ◁ I$ is approximately Gorenstein.

1. Introduction

Throughout this paper all rings are considered commutative with identity element and all ring homomorphisms are unital. In [8], D’Anna and Fontana considered a different type of construction obtained involving a ring $R$ and an ideal $I \subset R$ that is denoted by $R \triangleright I$, called amalgamated duplication, and it is defined as the following subring of $R \times R$:

$$R \triangleright I = \{(r, r + i) \mid r \in R, i \in I\}.$$

In [6] D’Anna showed that if $R$ is a Noetherian local ring, then $R \triangleright I$ is Cohen-Macaulay if and only if $R$ is Cohen-Macaulay and $I$ is maximal Cohen-Macaulay. In [1] it is shown that if $R$ is a Noetherian local ring, then $R \triangleright I$ is Gorenstein if and only if $R$ is Cohen-Macaulay and $I$ is a canonical module for $R$, and then $R/I$ is Cohen-Macaulay of dimension $\dim (R) - 1$. In this paper it is shown that if $R \triangleright I$ is a Gorenstein ring where $I$ is a non-zero flat ideal of Noetherian zero dimensional ring $R$, then $R$ is Gorenstein (see Proposition 2.2). Recently, the authors in [4] showed that if $R$ is a Noetherian local ring and $I$ is a proper ideal of $R$ such that $\text{Ann}_R(I) = 0$, then $R \triangleright I$ is a quasi-Gorenstein...
ring if and only if $\hat{R}$ satisfies Serre’s condition ($S_2$) and $I$ is a canonical ideal of $R$.

Recall that a Noetherian ring $R$ is called generically Cohen-Macaulay (resp. generically Gorenstein) if the ring $R_p$ is Cohen-Macaulay (resp. Gorenstein) for all $p \in \text{Ass}(R)$. Every Cohen-Macaulay (resp. Gorenstein) ring is also generically Cohen-Macaulay (resp. generically Gorenstein) and every Artinian generically Cohen-Macaulay (resp. generically Gorenstein) ring is Cohen-Macaulay (resp. Gorenstein). In Section 2 we define a generically quasi-Gorenstein ring and we investigate when $R \bowtie I$ is a generically Cohen-Macaulay (resp. generically Gorenstein, generically quasi-Gorenstein) ring (see Theorem 2.8 and Proposition 2.9).

In [9] Goto defined approximately Cohen-Macaulay ring and in [13] the authors examined how this property transfers under flat maps and tensor product operations. In [10] Hochster defined approximately Gorenstein ring. In Section 3 we provide necessary and sufficient conditions which led $R \bowtie I$ be an approximately Cohen-Macaulay (resp. approximately Gorenstein) ring (see Proposition 3.2 and Theorem 3.4).

2. Generically Cohen-Macaulay, generically Gorenstein and generically quasi-Gorenstein rings

As general reference for terminology and well-known results, we refer the reader to [5]. This section deals with some general results about generically Cohen-Macaulay, generically Gorenstein and generically quasi-Gorenstein properties of a general construction, introduced in [8], called amalgamated duplication of a ring along an ideal.

Let $R$ be a commutative ring with unit element $1$ and let $I$ be a proper ideal of $R$. Set

$$R \bowtie I = \{(r,s) \mid r, s \in R, s - r \in I\}.$$ 

It is easy to check that $R \bowtie I$ is a subring, with unit element $(1,1)$, of $R \times R$ (with the usual componentwise operations) and that $R \bowtie I = \{(r,r + i) \mid r \in R, i \in I\}$. In the following we bring some main properties of the ring $R \bowtie I$ from [6].

**Proposition 2.1.** Let $R$ be a ring and let $I$ be an ideal of $R$. Then the following statements hold.

1. The map $f : R \oplus I \to R \bowtie I$ defined by $f((r,i)) = (r,r + i)$ is an $R$-isomorphism. Moreover, there is a split exact sequence of $R$-modules

   $$(a)\quad 0 \to R \xrightarrow{\varphi} R \bowtie I \xrightarrow{\psi} I \to 0,$$

   where $\varphi(r) = (r,r)$ for all $r \in R$, and $\psi((r,s)) = s - r$ for all $(r,s) \in R \bowtie I$. We also have the short exact sequence of $R$-modules:

   $$(b)\quad 0 \to I \xrightarrow{\psi'} R \bowtie I \xrightarrow{\varphi'} R \to 0,$$
where \( 
abla'(i) = (0, i) \) and \( \varphi'(r, s) = r \) for every \( r \in R \) and \( (r, s) \in R \cong I \). Note that the exact sequence (b) is also a sequence of \( R \cong I \)-module, while the other one is not.

(2) Let \( p \) be a prime ideal of \( R \) and set:

\[
\begin{align*}
p_0 &= \{(p, p + i) \mid p \in p, i \in I \cap p\}, \\
p_1 &= \{(p, p + i) \mid p \in p, i \in I\}, \text{ and} \\
p_2 &= \{(p + i, p) \mid p \in p, i \in I\}.
\end{align*}
\]

(a) If \( I \subseteq p \), then \( p_0 = p_1 = p_2 \) is a prime ideal of \( R \cong I \) and it is the unique prime ideal of \( R \cong I \) lying over \( p \) and \( (R \cong I)_{p_0} \cong R_0 \cong I_p \).

(b) If \( I \nsubseteq p \), then \( p_1 \neq p_2 \) and \( p_1 \cap p_2 = p_0 \). Moreover, \( p_1 \) and \( p_2 \) are the only prime ideals of \( R \cong I \) lying over \( p \), and \( (R \cong I)_{p_1} \cong R_p \cong (R \cong I)_{p_2} \).

(3) \( R \) and \( R \cong I \) have the same Krull dimension and if \( R \) is a local ring with maximal ideal \( m \), then \( R \cong I \) is local with maximal ideal \( m_0 = \{(r, r + i) \mid r \in m, i \in I\} \). Also, if \( R \) is a Noetherian ring, then \( R \cong I \) is a finitely generated \( R \)-module.

In [6, Discussion 10], D’Anna showed that if \( R \) is a local ring of dimension \( d \) and \( I \) is a non-unit ideal of \( R \), then the ring \( R \cong I \) is Cohen-Macaulay if and only if \( R \) is Cohen-Macaulay and \( I \) is a maximal Cohen-Macaulay \( R \)-module. Recently in [1, Theorem 1.8], it is shown that if \( R \) is a Noetherian local ring, then \( R \cong I \) is Gorenstein if and only if \( R \) is Cohen-Macaulay and \( I \) is a canonical module for \( R \), and then \( R/I \) is Cohen-Macaulay of dimension \( \dim (R) - 1 \). In the following proposition we suppose that \( R \cong I \) is Gorenstein and we would like to know when \( R \) is Gorenstein.

**Proposition 2.2.** Let \( I \) be a non-zero flat ideal of Noetherian zero dimensional ring \( R \). If \( R \cong I \) is a Gorenstein ring, then \( R \) is Gorenstein.

**Proof.** By Proposition 2.1(3), \( \dim (R \cong I) = \dim (R) = 0 \) and so \( R \cong I \) is self-injective. Hence by [14, Corollary 3.4], \( \id_R(R \cong I) = \id_R(R \cong I) \). Now by assumption \( I \) is a flat ideal of \( R \), so \( R \cong I \) is a flat \( R \)-module. Therefore \( R \cong I \) is an injective \( R \)-module and hence for every \( R \)-module \( M \) and every integer \( i \geq 1 \), we have

\[
0 = \Ext^i_R(M, R \cong I) \cong \Ext^i_R(M, R) \oplus \Ext^i_R(M, I).
\]

So for every \( R \)-module \( M \) and for all \( i \geq 1 \), we have \( \Ext^i_R(M, R) = 0 \). Hence \( R \) is self-injective and therefore \( R \) is Gorenstein, since \( \dim (R) = 0 \).

We recall the notion of quasi-Gorenstein ring due to Platte and Storch in [12].
Definition 2.3. A local ring $R$ is said to be a quasi-Gorenstein ring if a canonical module of $R$ exists and is a free $R$-module (of rank one). This is equivalent to saying that $\tilde{H}_m^d(R) \cong E_R(R/m)$, where $d = \dim R$ and $m$ is the maximal ideal of $R$.

The ring $R$ is Gorenstein if and only if it is quasi-Gorenstein and Cohen-Macaulay. In [4, Theorem 3.3], it is shown that if $R$ is a Noetherian local ring and $I$ is a proper ideal of $R$ such that $\text{Ann}_R(I) = 0$, then $R \triangleright I$ is a quasi-Gorenstein ring if and only if $\hat{R}$ satisfies Serre’s condition ($S_2$) and $I$ is a canonical ideal of $R$.

Recall that a Noetherian ring $R$ is called generically Cohen-Macaulay (resp. generically Gorenstein) if the ring $R_p$ is Cohen-Macaulay (resp. Gorenstein) for all $p \in \text{Ass } (R)$. Every Cohen-Macaulay (resp. Gorenstein) ring is also generically Cohen-Macaulay (resp. generically Gorenstein) and every Artinian generically Cohen-Macaulay (resp. generically Gorenstein) ring is Cohen-Macaulay (resp. Gorenstein). We are ready now to introduce generically quasi-Gorenstein ring.

Definition 2.4. Let $R$ be a Noetherian local ring. Then $R$ is called generically quasi-Gorenstein if the ring $R_p$ is quasi-Gorenstein for all $p \in \text{Ass } (R)$.

According to [2, Corollary 2.4], the localization of every quasi-Gorenstein ring is quasi-Gorenstein. Therefore every quasi-Gorenstein ring is generically quasi-Gorenstein. It is straightforward to see that if $R$ is a zero dimensional local ring, then $R$ is quasi-Gorenstein if and only if $R$ is generically quasi-Gorenstein. It is routine to show that a Noetherian local ring $R$ is generically Gorenstein if and only if $R$ is generically quasi-Gorenstein and generically Cohen-Macaulay.

We are interested in understanding when $R \triangleright I$ is generically Cohen-Macaulay (resp. generically Gorenstein, generically quasi-Gorenstein). In the following lemma we investigate the associated prime ideals of the ring $R \triangleright I$.

Lemma 2.5. Let $R$ be a Noetherian ring and let $I$ be a proper ideal of $R$. Consider the ring homomorphism $\varphi : R \to R \triangleright I$, where $\varphi(r) = (r, r)$. Then the following statements hold.

(i) If $p \in \text{Ass } (R \triangleright I)$, then $\varphi^{-1}(p) \in \text{Ass } (R)$.

(ii) If $q \in \text{Ass } (R)$, then there exists $p \in \text{Ass } (R \triangleright I)$ such that $\varphi^{-1}(p) = q$.

Proof. (i) The exact sequence $0 \to I \to R \to R \triangleright I \to 0$ of $R \triangleright I$-modules implies that

$$\text{Ass } (R \triangleright I) \subseteq \text{Ass }_{R \triangleright I}(I) \cup \text{Ass }_{R \triangleright I}(R) = \text{Ass }_{R \triangleright I}(R).$$

So by assumption $p \in \text{Ass }_{R \triangleright I}(R)$. By [11, Exercise 6.7] we have $\varphi^{-1}(p) \in \text{Ass } (R)$, since $R$ is a finitely generated $R \triangleright I$-module.
(ii) From the $R$-monomorphism $\varphi : R \to R \bowtie I$, we have $\text{Ass}_R(R) \subseteq \text{Ass}_R(R \bowtie I)$. So by assumption $q \in \text{Ass}_R(R \bowtie I)$ and by [11, Exercise 6.7] there exists $p \in \text{Ass}_R(R \bowtie I)$ such that $\varphi^{-1}(p) = q$. □

**Definition 2.6.** A finitely generated $R$-module $M$ is called generically maximal Cohen-Macaulay (resp. generically canonical module) if the $R_p$-module $M_p$ is maximal Cohen-Macaulay (resp. canonical module) for all $p \in \text{Ass}(R)$.

**Definition 2.7.** The ring $R$ is called generically ($S_n$) if $R$ satisfies Serre’s condition ($S_n$) for all $p \in \text{Ass}(R)$.

**Theorem 2.8.** Let $R$ be a Noetherian ring and let $I$ be a proper ideal of $R$. Then the following statements hold.

(i) If $R \bowtie I$ is generically Cohen-Macaulay, then $R$ is generically Cohen-Macaulay.

(ii) If $R$ is generically Cohen-Macaulay (resp. generically Gorenstein) and $I$ is generically maximal Cohen-Macaulay (resp. generically canonical module), then $R \bowtie I$ is generically Cohen-Macaulay (resp. generically Gorenstein).

(iii) If $R$ is generically quasi-Gorenstein and $I$ is a generically canonical ideal of $R$, then $R \bowtie I$ is generically quasi-Gorenstein.

(iv) If $\text{Ann}_R(I) = 0$, then $R$ is generically ($S_2$) provided that $R \bowtie I$ is generically quasi-Gorenstein.

**Proof.** We prove items (iii) and (iv). The proof of the others is similar.

(iii) Let $p \in \text{Ass}(R \bowtie I)$. By Lemma 2.5, $q = \varphi^{-1}(p) \in \text{Ass}(R)$. According to Proposition 2.1(2), we have the following two cases:

Case (1). If $I \subseteq q$, then $(R \bowtie I)_p = R_q \bowtie I_q$. By assumption $I_q$ is a canonical ideal and $R_q$ is quasi-Gorenstein. Therefore $R_q$ satisfies Serre’s condition ($S_2$) by [3, Remark 1.4]. Hence $R_q$ satisfies Serre’s condition ($S_2$) by [3, Proposition 1.2]. Now according to [4, Theorem 3.3], $(R \bowtie I)_p$ is quasi-Gorenstein.

Case (2). If $I \nsubseteq q$, then $(R \bowtie I)_p = R_q$. So $(R \bowtie I)_p$ is quasi-Gorenstein.

(iv) Let $q \in \text{Ass}(R)$. By Lemma 2.5, there exists $p \in \text{Ass}(R \bowtie I)$ such that $\varphi^{-1}(p) = q$ and, by Proposition 2.1(2), we have the following two cases:

Case (1). If $I \subseteq q$, then $(R \bowtie I)_p = R_q \bowtie I_q$. So by assumption $R_q \bowtie I_q$ is quasi-Gorenstein. Therefore by [4, Theorem 3.3], $R_q$ satisfies Serre’s condition ($S_2$) and so $R_q$ satisfies Serre’s condition ($S_2$) by [3, Proposition 1.2].

Case (2). If $I \nsubseteq q$, then $(R \bowtie I)_p = R_q$. So $R_q$ satisfies Serre’s condition ($S_2$), by [3, Remark 1.4]. □

**Proposition 2.9.** Let $R$ be a Cohen-Macaulay ring and let $I$ be a non-zero ideal of $R$ such that $I_q$ is a flat $R_q$-module for all $q \in \text{Ass}(R)$. If $R \bowtie I$ is generically Gorenstein, then $R$ is generically Gorenstein.

**Proof.** Note that $\dim(R_q) = 0$ for all $q \in \text{Ass}(R)$, since $R$ is Cohen-Macaulay. The assertion follows by Propositions 2.2 and 2.1(3). □
3. Approximately Cohen-Macaulay and approximately Gorenstein rings

In this section we study when $R ▷◁ I$ is approximately Cohen-Macaulay and when it is approximately Gorenstein. To state the first result of this section, we need the notion of approximately Cohen-Macaulay ring due to Goto in [9].

**Definition 3.1.** The local ring $(R; m)$ is called an approximately Cohen-Macaulay ring if either $\dim(R) = 0$ or there exists an element $a$ of $m$ such that $R/a^nR$ is a Cohen-Macaulay ring of dimension $\dim(R) - 1$ for every integer $n > 0$.

It is straightforward to see that a Cohen-Macaulay local ring $R$ is approximately Cohen-Macaulay and the converse is true when $\dim(R) = 0$. Also Goto in [9, Corollary 2.8], showed that if $(R; m)$ is an approximately Cohen-Macaulay local ring such that $\dim(R) \geq 2$ and that $H^i_m(R)$ is finitely generated $R$-module for all $i \neq \dim(R)$, then $R$ is Cohen-Macaulay.

The next result shows that $R ▷◁ I$ is approximately Cohen-Macaulay if and only if $R$ is approximately Cohen-Macaulay provided some special conditions.

**Proposition 3.2.** Let $(R, m)$ be a Noetherian local ring and let $I$ be a non-zero flat ideal of $R$. Assume that $R$ is not a Cohen-Macaulay ring such that $R$ is a homomorphic image of a Cohen-Macaulay local ring. Then $R ▷◁ I$ is approximately Cohen-Macaulay if and only if $R$ is approximately Cohen-Macaulay.

**Proof.** Note that $\varphi : R \to R ▷◁ I$ is a flat ring homomorphism. By [7, Proposition 5.1], we have $R ▷◁ I/m_0 \cong R/m$, where $m_0 = \{(r, r + i) \mid r \in m, i \in I\}$ is the maximal ideal of $R ▷◁ I$. So $R ▷◁ I/m_0$ is a Cohen-Macaulay ring. Now the assertion follows from [13, Theorem 6]. \qed

Before stating our main results of this section, we recall the definition of approximately Gorenstein ring due to Hochster in [10].

**Definition 3.3.** A Noetherian local ring $(R, m)$ is called approximately Gorenstein, if for every integer $n > 0$ there is an ideal $I \subseteq m^n$ such that $R/I$ is Gorenstein.

It is routine to see that every Gorenstein ring is approximately Gorenstein, and a zero dimensional ring is approximately Gorenstein if and only if it is Gorenstein. While approximately Gorenstein rings must have positive depth, they need not to be Cohen-Macaulay. In fact, every complete Noetherian domain is approximately Gorenstein [10, Theorem 1.6].

The next result shows that $R ▷◁ I$ is approximately Gorenstein provided some special conditions.

**Theorem 3.4.** Let $(R, m)$ be a Noetherian local ring and let $I$ be a proper ideal of $R$. Then the following statements hold.
(i) If $R$ is approximately Gorenstein, then $R \bowtie I$ is approximately Gorenstein.

(ii) If $R \bowtie I$ is Gorenstein and $R$ is generically Gorenstein, then $R$ is approximately Gorenstein.

Proof. (i) According to Proposition 2.1(3), $(R \bowtie I, m_0)$ is a Noetherian local ring. Let $n > 0$ be an integer. By assumption there exists an ideal $J \subseteq m^n$ such that $R/J$ is Gorenstein. By [7, Proposition 5.1], $J \bowtie I$ is an ideal of $R \bowtie I$ and

$$\frac{R \bowtie I}{J \bowtie I} \cong \frac{R}{J}.$$ 

It is straightforward to see that $J \bowtie I \subseteq m^n \bowtie I = m_0^n$ and so $(R \bowtie I)/(J \bowtie I)$ is Gorenstein, therefore the assertion is proved.

(ii) By [1, Theorem 1.8], $R$ is Cohen-Macaulay and $I$ is a canonical ideal of $R$. The assertion follows from [10, Remarks (4.8b)].

Corollary 3.5. Let $R$ be a generically Gorenstein local ring and let $I$ be a proper ideal of $R$. Assume that $R$ is Cohen-Macaulay with canonical module. Then $R \bowtie I$ is approximately Gorenstein.

Proof. According to [10, Remarks (4.8b)], $R$ is approximately Gorenstein, so $R \bowtie I$ is approximately Gorenstein by Theorem 3.4(i).

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