EINSTEIN HALF LIGHTLIKE SUBMANIFOLDS WITH SPECIAL CONFORMALITIES

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Abstract. In this paper, we study the geometry of Einstein half lightlike submanifolds \( M \) of a semi-Riemannian space form \( \bar{M}(c) \) subject to the conditions: (a) \( M \) is screen conformal, and (b) the coscreen distribution of \( M \) is a conformal Killing one. The main result is a classification theorem for screen conformal Einstein half lightlike submanifolds of a Lorentzian space form with a conformal Killing coscreen distribution.

1. Introduction

A submanifold \( M \) of a semi-Riemannian manifold \( (\bar{M}, \bar{g}) \) is called a lightlike submanifold of \( \bar{M} \) if its radical distribution \( \text{Rad}(TM) = TM \cap TM^\perp \) is a vector subbundle of the tangent bundle \( TM \), of rank \( r > 0 \). A codimension 2 lightlike submanifold \( M \) is called a half lightlike submanifold if \( \text{rank}(\text{Rad}(TM)) = 1 \).

Then there exists two complementary non-degenerate distributions \( S(TM) \) and \( S(TM^\perp) \) of \( \text{Rad}(TM) \) in \( TM \) and \( TM^\perp \) respectively, which called the screen and coscreen distribution on \( M \), such that

\[
TM = \text{Rad}(TM) \oplus_{\text{orth}} S(TM), \quad TM^\perp = \text{Rad}(TM) \oplus_{\text{orth}} S(TM^\perp),
\]

where the symbol \( \oplus_{\text{orth}} \) denotes the orthogonal direct sum. We denote such a half lightlike submanifold by \( M = (M, \bar{g}, S(TM)) \). Denote by \( F(M) \) the algebra of smooth functions on \( M \) and by \( \Gamma(E) \) the \( F(M) \) module of smooth sections of any vector bundle \( E \) over \( M \). Then there exist a non-null section \( u \) on \( S(TM^\perp) \) and a null section \( \xi \) on \( \text{Rad}(TM) \) such that

\[
\bar{g}(u, u) = \epsilon, \quad \bar{g}(\xi, v) = 0, \quad \forall v \in \Gamma(TM^\perp),
\]

where \( \epsilon = \pm 1 \). Consider the orthogonal complementary distribution \( S(TM^\perp) \) to \( S(TM) \) in \( TM \). Certainly \( \xi \) and \( u \) belong to \( \Gamma(S(TM)^\perp) \). Thus we have

\[
S(TM)^\perp = S(TM^\perp) \oplus_{\text{orth}} S(TM^\perp)^\perp,
\]

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where $S(TM^\perp)$ is the orthogonal complementary to $S(TM^\perp)$ in $S(TM)$. For any null section $\xi$ of $\text{Rad}(TM)$ on a coordinate neighborhood $U \subset M$, there exists a uniquely defined vector field $N \in \Gamma(\text{ltr}(TM))$ \cite{4} satisfying
\begin{equation}
\bar{g}(\xi, N) = 1, \quad \bar{g}(N, N) = \bar{g}(N, X) = \bar{g}(N, u) = 0, \quad \forall X \in \Gamma(S(TM)).
\end{equation}
We call $\text{ltr}(TM)$, $N$ and $\text{tr}(TM) = S(TM^\perp) \oplus_{\text{orth}} \text{ltr}(TM)$ the lightlike transversal vector bundle, lightlike transversal vector field and transversal vector bundle of $M$ with respect to $S(TM)$ respectively. Then the tangent bundle $TM$ of the ambient manifold $M$ is decomposed as follows:
\begin{equation}
TM = TM \oplus \text{tr}(TM) = \{\text{Rad}(TM) \oplus \text{tr}(TM)\} \oplus_{\text{orth}} S(TM) = \{\text{Rad}(TM) \oplus \text{ltr}(TM)\} \oplus_{\text{orth}} S(TM^\perp) \oplus_{\text{orth}} S(TM).
\end{equation}

Example 1. Suppose $M$ is a surface of $R^4_1$ given by the equations
\begin{align*}
x_3 &= \sqrt{x_1^2 - x_2^2}, \quad x_4 = \sqrt{1 + x_1^2}.
\end{align*}
Then we derive $TM = \text{Span}\{\xi, U\}$ and $TM^\perp = \text{Span}\{\xi, u\}$, where
\begin{align*}
U &= x_3 x_4 \partial_1 + x_1 x_4 \partial_3 + x_1 x_3 \partial_4, \\
\xi &= x_1 \partial_1 + x_2 \partial_2 + x_3 \partial_3, \quad u = x_1 \partial_1 + x_4 \partial_4.
\end{align*}
It follows that $\text{Rad}(TM)$ is a distribution on $M$ of rank 1 spanned by $\xi$. Hence $M$ is a half-lightlike submanifold of $R^4_1$ such that $S(TM) = \text{Span}\{U\}$ and $S(TM^\perp) = \text{Span}\{u\}$. Then the lightlike transversal bundle $\text{ltr}(TM)$ and the transversal bundle $\text{tr}(TM)$ with respect to the screen distribution $S(TM)$ are given by $\text{ltr}(TM) = \text{Span}\{N\}$ and $\text{tr}(TM) = \text{Span}\{N, u\}$, where
\begin{align*}
N &= -\frac{1}{2x_1^2}(x_1 \partial_1 - x_2 \partial_2 - x_3 \partial_3).
\end{align*}

The classification of Einstein hypersurfaces $M$ in Euclidean spaces $R^{n+1}$ was first studied by Fialkow \cite{7} and Thomas \cite{14} in the middle of 1930’s. It was proved that if $M$ is a connected Einstein hypersurface ($n \geq 3$) such that $\text{Ric} = \gamma g$ for some constant $\gamma$, then $\gamma$ is non-negative. Moreover,
\begin{enumerate}
\item if $\gamma > 0$, then $M$ is contained in an $n$-sphere and
\item if $\gamma = 0$, then $M$ is locally isometric to $R^n$.
\end{enumerate}
The objective of this paper is the study of half lightlike version of above classical results. For this reason, we consider only screen conformal half lightlike submanifolds with a conformal Killing coscreen distribution. In Section 2, we investigate geometric properties for screen conformal half lightlike submanifolds $M$ of a semi-Riemannian space form $(M^{m+3}(c), \bar{g})$, $m > 2$, with a conformal Killing coscreen distribution. In the last Section 3, we prove our main classification theorem for screen conformal Einstein half lightlike submanifolds $M$ of a Lorentzian space form with a conformal Killing coscreen distribution (Theorem 3.2). Recall the following structure equations.
Let $\nabla$ be the Levi-Civita connection of $M$ and $P$ the projection morphism of $TM$ on $S(TM)$ with respect to the decomposition (1.1). Then the local Gauss and Weingarten formulas $M$ and $S(TM)$ are given respectively by

\begin{align}
\nabla_X Y &= \nabla_X Y + B(X, Y) N + D(X, Y) u, \\
\nabla_X N &= -A_X X + \tau(X) N + \rho(X) u, \\
\nabla_X u &= -A_X X + \phi(X) N; \\
\nabla_X PY &= \nabla_X^p PY + C(X, PY) \xi, \\
\nabla_X \xi &= -A_X X - \tau(X) \xi, \\
\end{align}

for any $X, Y \in \Gamma(TM)$, where $\nabla$ and $\nabla^*$ are induced linear connections on $TM$ and $S(TM)$ respectively, the bilinear forms $B$ and $D$ on $TM$ are called the local lightlike and screen second fundamental forms of $M$ respectively, $C$ is called the local radical second fundamental form on $S(TM)$. $A_X, A^*_X$ and $A_u$ are linear operators on $\Gamma(TM)$ and $\tau, \rho$ and $\phi$ are 1-forms on $TM$.

Since $\nabla$ is torsion-free, $\nabla^*$ is also torsion-free, and $B$ and $D$ are symmetric. From the facts $B(X, Y) = \bar{g}(\nabla_X Y, \xi)$ and $D(X, Y) = \epsilon \bar{g}(\nabla_X Y, u)$, we know that $B$ and $D$ are independent of the choice of a screen distribution and satisfy

\begin{align}
B(X, \xi) = 0, \quad D(X, \xi) = -\epsilon \phi(X), \quad \forall X \in \Gamma(TM).
\end{align}

The induced connection $\nabla$ on $M$ is not metric and satisfies

\begin{align}
(\nabla_X g)(Y, Z) = B(X, Y) \eta(Z) + B(X, Z) \eta(Y)
\end{align}

for all $X, Y, Z \in \Gamma(TM)$, where $\eta$ is a 1-form on $TM$ such that

\begin{align}
\eta(X) = \bar{g}(X, N), \quad \forall X \in \Gamma(TM).
\end{align}

But we show that $\nabla^*$ is metric. The above three local second fundamental forms on $TM$ and $S(TM)$ are related to their shape operators by

\begin{align}
B(X, Y) &= g(A^*_X Y, Y), \quad \bar{g}(A^*_X X, N) = 0, \\
C(X, PY) &= g(A_N X, PY), \quad \bar{g}(A_N X, N) = 0, \\
\epsilon D(X, PY) &= g(A_u X, PY), \quad \bar{g}(A_u X, N) = \epsilon \rho(X), \\
\epsilon D(X, Y) &= g(A_u X, Y) - \phi(X) \eta(Y).
\end{align}

From (1.12), $A^*_X$ is $S(TM)$-valued and self-adjoint on $\Gamma(TM)$ such that

\begin{align}
A^*_X \xi = 0.
\end{align}

We denote by $\bar{R}, R$ and $R^*$ the curvature tensors of the Levi-Civita connection $\nabla$ of $M$, the induced connection $\nabla$ of $M$ and the induced connection $\nabla^*$ on $S(TM)$ respectively. Using the Gauss-Weingarten equations for $M$ and $S(TM)$, we obtain the Gauss-Codazzi equations for $M$ and $S(TM)$:

\begin{align}
\bar{g}(\bar{R}(X, Y) Z, PW) &= g(R(X, Y) Z, PW) \\
&+ B(X, Z) C(Y, PW) - B(Y, Z) C(X, PW) \\
&+ \epsilon \{ D(X, Z) D(Y, PW) - D(Y, Z) D(X, PW) \},
\end{align}
(1.18) \( \bar{g}(\bar{\mathcal{R}}(X,Y)Z, \xi) = (\nabla_X B)(Y,Z) - (\nabla_Y B)(X,Z) + B(Y,Z)\tau(X) - B(X,Z)\tau(Y) + D(Y,Z)\phi(X) - D(X,Z)\phi(Y), \)
(1.19) \( \bar{g}(\bar{\mathcal{R}}(X,Y)Z, N) = \bar{g}(\bar{\mathcal{R}}(X,Y)Z, N) + \epsilon\{D(X,Z)\rho(Y) - D(Y,Z)\rho(X)\}, \)
(1.20) \( \bar{g}(\bar{\mathcal{R}}(X,Y)\xi, N) = g(A^*_n X, A_n Y) - g(A^*_n Y, A_n X) + \rho(X)\phi(Y) - \rho(Y)\phi(X) - 2d\tau(X,Y), \)
(1.21) \( \bar{g}(\bar{\mathcal{R}}(X,Y)Z, u) = \epsilon\{[\nabla_X D](Y,Z) - [\nabla_Y D](X,Z) + B(Y,Z)\rho(X) - B(X,Z)\rho(Y)\}, \)
(1.22) \( \bar{g}(\bar{\mathcal{R}}(X,Y)PZ, PW) = g(R^*(X,Y)PZ, PW) + C(X,PZ)B(Y,PW) - C(Y,PZ)B(X,PW), \)
(1.23) \( g(\bar{\mathcal{R}}(X,Y)PZ, N) = (\nabla_X C)(Y,PZ) - (\nabla_Y C)(X,PZ) + C(X,PZ)\tau(Y) - C(Y,PZ)\tau(X) \)

for all \( X, Y, Z \in \Gamma(TM) \). The Ricci curvature tensor \( \bar{\text{Ric}} \) of \( \bar{M} \) and the induced Ricci type tensor \( R^{(0,2)} \) of \( M \) are defined by

\[
\text{Ric}(X,Y) = \text{trace}\{Z \to \bar{\mathcal{R}}(Z,X)Y\}, \quad \forall X, Y \in \Gamma(TM),
\]
\[
R^{(0,2)}(X,Y) = \text{trace}\{Z \to \bar{\mathcal{R}}(Z,X)Y\}, \quad \forall X, Y \in \Gamma(TM).
\]

Consider the induced quasi-orthonormal frame fields \( \{\xi; W_a\} \) on \( M \) such that \( \text{RadTM} = \text{Span}\{\xi\} \) and \( S(TM) = \text{Span}\{W_a\} \) and let \( E = \{\xi, W_a; u, N\} \) be the corresponding frame fields on \( \bar{M} \). Let \( \epsilon_a = g(W_a, W_a) \) be the sign of \( W_a \). Using this quasi-orthonormal frame, (1.24) and (1.25) reduce respectively to

\[
\bar{\text{Ric}}(X,Y) = \sum_{a=1}^{m} \epsilon_a \bar{g}(\bar{\mathcal{R}}(W_a,X)Y, W_a) + \bar{g}(\bar{\mathcal{R}}(\xi,X)Y, N) + \epsilon \bar{g}(\bar{\mathcal{R}}(u,X)Y, u) + \bar{g}(\bar{\mathcal{R}}(N,X)Y, \xi),
\]
\[
R^{(0,2)}(X,Y) = \sum_{a=1}^{m} \epsilon_a g(R(W_a,X)Y, W_a) + \bar{g}(R(\xi,X)Y, N) \]

for any \( X, Y \in \Gamma(TM) \). Substituting (1.17) and (1.19) in (1.26) and then, using (1.12), (1.13) and (1.27), we obtain

\[
R^{(0,2)}(X,Y) = \bar{\text{Ric}}(X,Y) + B(X,Y)\tau A_u + D(X,Y)\tau A_u - g(A_n X, A^*_n Y) - \epsilon g(A_n X, A_n Y) + \rho(X)\phi(Y) - \bar{g}(\bar{\mathcal{R}}(\xi,Y)X, N) - \epsilon \bar{g}(\bar{\mathcal{R}}(u,Y)X, u) \]

for any \( X, Y \in \Gamma(TM) \). A tensor field \( R^{(0,2)} \) of \( M \) is called its induced Ricci tensor if it is symmetric. A symmetric \( R^{(0,2)} \) tensor will be denoted by \( \bar{\text{Ric}} \).
Then we have

\[ T \text{ M} \]

1. Using (1.20), (1.28) and the first Bianchi’s identity, we obtain

\[ S \]

and the transversal bundle

\[ 1 \]

spanned by \( \xi \).

By direct calculations we check that

\[ R \]

It follows that

\[ i.e., \ d\tau = 0 \]

on any \( U \subset M \) [6]. Therefore, suppose \( R^{(0,2)} \) is symmetric, then there exists a smooth function \( f \) on \( U \) such that \( \tau = df \). Consequently we get \( \tau(X) = X(f) \). If we take \( \xi = \alpha \xi \), it follows that \( \tau(X) = \tau(X) + X \ln \alpha \).

Setting \( \alpha = \exp(f) \) in this equation, we get \( \tau(X) = 0 \) for any \( X \in \Gamma(TM_{U}) \).

In the sequel, we call the pair \( \{ \xi, N \} \) on \( U \) such that the corresponding 1-form \( \tau \) vanishes the canonical null pair [9] of \( M \).

2. Screen conformal submanifolds

Definition. A half lightlike submanifold \( (M, g, S(TM)) \) of \( M \) is said to be screen conformal [1] if there exists a non-vanishing smooth function \( \varphi \) on a neighborhood \( U \) in \( M \) such that \( A_{N} = \varphi A_{\xi} \), or equivalently,

\[ C(X, PY) = \varphi B(X, Y), \ \forall \ X, Y \in \Gamma(TM). \]

(2.1)

In general, \( S(TM) \) is not necessarily integrable. From (1.7) and (1.13), we get

\[ g(A_{x}X, Y) - g(X, A_{x}Y) = C(X, Y) - C(Y, X) = \eta([X, Y]) \]

for all \( X, Y \in \Gamma(S(TM)) \). Thus \( A_{x} \) is self-adjoint on \( S(TM) \) with respect to \( g \) if and only if \( C \) is symmetric on \( S(TM) \) if and only if \( \eta([X, Y]) = 0 \) for all \( X, Y \in \Gamma(S(TM)) \), i.e., \( S(TM) \) is integrable [4].

Note 2. For a screen conformal \( M \), since \( C \) is symmetric on \( S(TM) \), the screen distribution \( S(TM) \) is integrable. Thus \( M \) is locally a product manifold \( L \times M^{\ast} \) where \( L \) is a null curve and \( M^{\ast} \) is a leaf of \( S(TM) \) [5].

Example 2. Consider a surface \( M \) in \( R_{3}^{5} \) given by the equation

\[ x_{4} = \sqrt{x_{1}^{2} + x_{2}^{2}}, \quad x_{5} = \sqrt{1 - x_{3}^{2}}. \]

Then we have \( TM = \text{Span}\{\xi, U, V\} \) and \( TM^{\perp} = \text{Span}\{\xi, u\} \), where

\[ U = x_{4}\partial_{1} + x_{1}\partial_{4}, \quad V = x_{5}\partial_{3} - x_{3}\partial_{5}, \]

\[ \xi = x_{1}\partial_{1} + x_{2}\partial_{2} + x_{4}\partial_{4}, \quad u = x_{3}\partial_{3} + x_{5}\partial_{5}. \]

By direct calculations we check that \( \text{Rad}(TM) \) is a distribution on \( M \) of rank 1 spanned by \( \xi \). Hence \( M \) is a half-lightlike submanifold of \( R_{3}^{5} \) such that \( S(TM) = \text{Span}\{U, V\} \) and \( S(TM^{\perp}) = \text{Span}\{u\} \). Then the lightlike transversal bundle \( ltr(TM) \) of the screen \( S(TM) \) is given by

\[ ltr(TM) = \text{Span}\left\{ N = \frac{1}{2x_{4}^{2}}(x_{1}\partial_{1} - x_{2}\partial_{2} + x_{4}\partial_{4}) \right\}, \]

and the transversal bundle \( tr(TM) \) is given by \( tr(TM) = \text{Span}\{N, u\} \).
Denote by $\nabla$ the Levi-Civita connection on $\mathbb{R}^3$. By straightforward calculations, we obtain

$$
\nabla_U U = \xi + 2x_2^2 N, \quad \nabla_U V = 0, \quad \nabla_U \xi = 2U + \frac{x_1 x_4}{2x_2^2} \xi - x_1 x_4 N,
$$

$$
\nabla_U N = \frac{1}{2x_2^2} U - \frac{x_1 x_4}{x_2^2} N + \frac{x_1 x_4}{x_2^2} \xi, \quad \nabla_U u = 0,
$$

$$
\nabla_V U = 0, \quad \nabla_V V = -2u, \quad \nabla_V \xi = 0, \quad \nabla_V N = 0, \quad \nabla_V u = 2V,
$$

$$
\nabla_U U = U + \frac{x_4}{2x_1} \xi + \frac{x_4^2 x_4}{x_1} N, \quad \nabla_U \xi = \frac{x_4}{x_1} U + \left(\frac{3}{2} + \frac{x_4^2}{2x_2^2}\right) \xi - x_4^2 N,
$$

$$
\nabla_U V = 0, \quad \nabla_U N = -N, \quad \nabla_U u = 0.
$$

Then taking into account of Gauss and Weingarten formulas infer

$$
A^2 U = -U, \quad A^2 V = 0, \quad A_\xi U = -\frac{1}{2x_2^4} U, \quad A_N V = 0, \quad A_N \xi = 0,
$$

$$
\tau(U) = \tau(V) = \tau(\xi) = 0, \quad \rho(U) = \rho(V) = \rho(\xi) = 0.
$$

Thus $A_\xi X = (1/2x_2^2) A^2 X$ for any $X \in \Gamma(TM)$ and $M$ is a screen conformal half-lightlike submanifold of $\mathbb{R}^3$ with a conformal factor $\varphi = 1/2x_2^2$.

**Definition.** A vector field $X$ on $\hat{M}$ is said to be a *conformal Killing* [15] if $\hat{L}_X g = -2\delta \hat{g}$, where $\delta$ is a non-vanishing smooth function on $\hat{M}$ and $\hat{L}_X$ denotes the Lie derivative with respect to $X$. In particular, if $\delta = 0$, then $X$ is called a *Killing*. A distribution $\mathcal{G}$ on $\hat{M}$ is said to be a *conformal Killing* (Killing) if each vector field belonging to $\mathcal{G}$ is a conformal Killing (Killing).

**Theorem 2.1.** Let $(M, g, S(TM))$ be a half lightlike submanifold of a semi-Riemannian manifold $(\hat{M}, \hat{g})$. Then the coscreen distribution is a conformal Killing if and only if $D(X, Y) = \epsilon \delta g(X, Y)$ for any $X, Y \in \Gamma(TM)$.

**Proof.** By straightforward calculations and use (1.6) and (1.15), we have

$$
(\hat{L}_X \hat{g})(X, Y) = \hat{g}(\nabla_X u, Y) + \hat{g}(X, \nabla_Y u),
$$

$$
\hat{g}(\nabla_X u, Y) = -g(A_\xi X, Y) + \phi(X) \eta(Y) = -\epsilon D(X, Y)
$$

for any $X, Y \in \Gamma(TM)$. Therefore, we obtain $(\hat{L}_X \hat{g})(X, Y) = -2\epsilon D(X, Y)$. □

Let $(M, g, S(TM))$ be a screen conformal half lightlike submanifold of a semi-Riemannian space form $(\hat{M}(c), \hat{g})$ with a conformal Killing coscreen. For all $X, Y, Z, W \in \Gamma(TM)$, by (1.9), (1.14) and (1.15), we have

$$
D(X, Y) = \epsilon \delta g(X, Y), \quad \phi(X) = 0, \quad A_\xi X = \delta P X + \epsilon \rho(X) \xi.
$$

Using (2.1) and (2.2), the Gauss equations (1.17) and (1.22) reduce to

$$
g(R(X, Y) Z, PW) = (c + \epsilon \delta^2) \{g(Y, Z) g(X, PW) - g(X, Z) g(Y, PW)\}
$$

$$
+ \varphi \{B(Y, Z) B(X, PW) - B(X, Z) B(Y, PW)\},
$$
Using (1.19), (1.23), (2.1) and (2.5), we obtain

\[ \{ Y \} \]

Replacing \( \phi \) by virtue of (2.10). Thus, from (1.9), (2.1) and (2.10), we have

\[ \{ \delta \} g(Y, Z) - Y[\delta] g(X, Z). \]

Replacing \( Y \) by \( \xi \) in the last equation and using (1.9), we obtain

\[ \{ \delta - \epsilon \rho(\xi) \} B(X, Z) = \xi[\delta] g(X, Z). \]

Using (1.19), (1.23), (2.1) and (2.5), we obtain

\[ \{ X[\varphi] - 2\varphi \tau(X) \} B(Y, PZ) - \{ Y[\varphi] - 2\varphi \tau(Y) \} B(X, PZ) \]

\[ = \{ \epsilon \eta(X) + \delta \rho(X) \} g(Y, PZ) - \{ \epsilon \eta(Y) + \delta \rho(Y) \} g(X, PZ). \]

Replacing \( Y \) by \( \xi \) in the last equation and using (1.9), we obtain

\[ \{ \xi[\varphi] - 2\varphi \tau(\xi) \} B(X, PZ) = (c + \delta \rho(\xi)) g(X, PZ). \]

**Theorem 2.2.** Let \((M, g, S(TM))\) be a screen conformal half lightlike submanifold of a semi-Riemannian space form \((M^{m+3}(c), \bar{g}), m > 2, \) with a conformal Killing coscreen distribution. Then we have \( c + \delta \rho(\xi) = 0. \)

**Proof.** Assume that \( c + \delta \rho(\xi) \neq 0. \) Then we have \( \xi[\varphi] - 2\varphi \tau(\xi) \neq 0 \) and \( B \neq 0 \) by virtue of (2.10). Thus, from (1.9), (2.1) and (2.10), we have

\[ B(X, Y) = \sigma g(X, Y), \ C(X, PY) = \varphi \sigma g(X, Y), \forall X, Y \in \Gamma(TM), \]

where \( \sigma = (c + \delta \rho(\xi)) \langle \xi[\varphi] - 2\varphi \tau(\xi) \rangle^{-1} \neq 0. \) From the first equation of (2.2) and (2.11), \( M \) is totally umbilical in \( M \) and \( S(TM) \) is also totally umbilical in \( M \) and \( \bar{M}. \) As \( \bar{M} \) has a constant curvature \( c, \) from (2.4) and (2.11), we have

\[ R^*(X, Y) Z = (c + 2\varphi \sigma^2 + \epsilon \delta^2) \{ g(Y, Z) X - g(X, Z) Y \} \]

for all \( X, Y, Z \in \Gamma(S(TM)). \) Let \( M^* \) be the leaf of \( S(TM) \) and \( \text{Ric}^* \) be the Ricci tensor of \( M^*. \) Then, from the last equation, we have

\[ \text{Ric}^*(X, Y) = (c + 2\varphi \sigma^2 + \epsilon \delta^2)(m - 1) g(X, Y), \forall X, Y \in \Gamma(S(TM)). \]

Thus \( M^* \) is Einstein. As \( m > 2, \) \( (c + 2\varphi \sigma^2 + \epsilon \delta^2) \) is a constant and \( M^* \) is a space of constant curvature \( (c + 2\varphi \sigma^2 + \epsilon \delta^2). \) Differentiating the first equation of (2.11) and using (1.10) and (2.5), we have

\[ \{ X[\sigma] + \sigma \tau(X) - \sigma^2 \eta(X) \} g(Y, Z) = \{ Y[\sigma] + \sigma \tau(Y) - \sigma^2 \eta(Y) \} g(X, Z) \]

for all \( X, Y, Z \in \Gamma(TM). \) Replacing \( Y \) by \( \xi \) in this equation, we have \( \xi[\sigma] = \sigma^2 - \sigma \tau(\xi). \) From (2.8) and (2.11), we have \( \xi[\delta] = \sigma \delta - \epsilon \sigma \rho(\xi). \) Since \( (c + 2\varphi \sigma^2 + \epsilon \delta^2) \) is a
$\epsilon\delta^2$ is a constant, we have $\xi[c+2\varphi^2+\epsilon\delta^2] = 2\sigma(c+2\varphi^2+\epsilon\delta^2) = 0$. Therefore, as $\sigma \neq 0$, we have $c+2\varphi^2+\epsilon\delta^2 = 0$ and consequently we get $R^* = 0$. Thus $M^*$ is a semi-Euclidean space. As the second fundamental form of the totally umbilical semi-Euclidean space $M^*$ as a submanifold of the semi-Riemannian space form $M$ vanishes [3, Section 2.3], we get $C = 0$. Consequently, from (2.1), we get $B = 0$ and $c + \delta\rho(\xi) = 0$ due to (2.10). It is a contradiction to $c + \delta\rho(\xi) \neq 0$. Thus we have $c + \delta\rho(\xi) = 0$. \[\Box\]

**Corollary 2.3** ([10]). Let $(M, g, S(TM))$ be a screen conformal half lightlike submanifold of a semi-Riemannian space form $(M^{m+3}(c), \bar{g}), m > 2$, with a Killing coscreen distribution. Then we have $c = 0$ and $\delta = 0$.

**Theorem 2.4.** Let $(M, g, S(TM))$ be a screen conformal Einstein half lightlike submanifold of a semi-Riemannian space form $(M^{m+3}(c), \bar{g}), m > 2$, with a conformal Killing coscreen distribution of conformal factor $\delta$. Then the leaf $M^*$ of $S(TM)$ is an Einstein manifold and $\delta$ is a constant.

**Proof.** From (2.3) and (2.4), we show that

$$2g(R(X, Y)PZ, PW) = g(R^*(X, Y)PZ, PW) + (c + \epsilon\delta^2)\{g(Y, PZ)g(X, PW) - g(X, PZ)g(Y, PW)\}$$

for all $X, Y, Z, W \in \Gamma(TM)$. Using the equations (1.27), (2.12) and the fact that $\bar{g}(R(\xi, X)Y, N) = (c + \delta\rho(\xi))(X, Y) = 0$, we get

$$2R^{(0,2)}(X, Y) = Ric^*(X, Y) + (m - 1)(c + \epsilon\delta^2)g(X, Y).$$

This shows that the induced tensor $R^{(0,2)}$ on $M$ is symmetric. Thus $M$ admits a symmetric Ricci tensor and $R^{(0,2)} = Ric$. Since $M$ is Einstein, i.e., $Ric = \gamma g$, where $\gamma$ is a constant if $m > 2$, the last equation reduces to

$$Ric^*(X, Y) = \{2\gamma - (m - 1)(c + \epsilon\delta^2)\}g(X, Y), \quad \forall X, Y \in \Gamma(TM).$$

Thus $M^*$ is also Einstein. Since $m > 2$, the function $\{2\gamma - (m - 1)(c + \epsilon\delta^2)\}$ is a constant. Therefore, the conformal factor $\delta$ is a constant. \[\Box\]

**Theorem 2.5.** Let $(M, g, S(TM))$ be a screen conformal Einstein half lightlike submanifold of a semi-Riemannian space form $(M^{m+3}(c), \bar{g}), m > 2$, with a conformal Killing coscreen distribution of conformal factor $\delta$. If either $\gamma \neq (m - 1)(c + \epsilon\delta^2)$ or rank $A^*_{\xi} > 0$, then we have $c + \epsilon\delta^2 = 0$.

**Proof.** Since $M$ is Einstein, the conformal factor $\delta$ is a constant by Theorem 2.4. From (2.8) with $c + \delta\rho(\xi) = 0$, we get $\{c + \epsilon\delta^2\}B(Y, Z) = 0$, or equivalently, $\{c + \epsilon\delta^2\}A^*_{\xi}X = 0$ for any $X, Y \in \Gamma(TM)$. First, if rank $A^*_{\xi} > 0$, we get $c + \epsilon\delta^2 = 0$. Next, if $c + \epsilon\delta^2 \neq 0$, then, since $(c + \epsilon\delta^2)$ is a constant, we have $B(X, Y) = 0$ for any $X, Y \in \Gamma(TM)$. Thus, from (1.27), (2.3) and the fact that $\bar{g}(R(\xi, X)Y, N) = (c + \delta\rho(\xi))(X, Y) = 0$, we have $\gamma = (m - 1)(c + \epsilon\delta^2)$. This implies that if $\gamma \neq (m - 1)(c + \epsilon\delta^2)$, then we get $c + \epsilon\delta^2 = 0$. \[\Box\]
Recall the following notion of null sectional curvature [2, 5, 6, 8]. Let $x \in M$ and $\xi$ be a null vector of $T_xM$. A plane $H$ of $T_xM$ is called a null plane directed by $\xi$ if it contains $\xi$, $g_x(\xi, W) = 0$ for any $W \in H$ and there exists $W_0 \in H$ such that $g_x(W_0, W_0) \neq 0$. Then, the null sectional curvature of $H$, with respect to $\xi$ and $\nabla$, is defined as a real number

$$K_\xi(H) = \frac{g_x(R(\xi, W)W, \xi)}{g_x(W, W)},$$

where $W \neq 0$ is any vector in $H$ independent with $\xi$. It is easy to see that $K_\xi(H)$ is independent of $W$ but depends in a quadratic fashion on $\xi$. An $n(\geq 3)$-dimensional Lorentzian manifold is of constant curvature if and only if its null sectional curvatures are everywhere zero [12].

**Theorem 2.6.** Let $(M, g, S(TM))$ be a screen conformal half lightlike submanifold of a semi-Riemannian space form $(M^{m+3}(c), g)$, $m > 2$, with a conformal Killing coscreen distribution. Then every null plane $H$ of $T_xM$ directed by $\xi$ has everywhere zero null sectional curvatures.

**Proof.** From (1.9), (1.19) and (2.3), we show that $g(R(\xi, X)Y, PW) = 0$ and $g(R(\xi, X)Y, N) = (c + \delta g(\xi))g(X, Y) = 0$ for any $X, Y \in \Gamma(TM)$. Thus the curvature tensor $R$ of $M$ satisfies $R(\xi, X)Y = 0$ for any $X, Y \in \Gamma(TM)$. Thus $K_\xi(H) = \frac{g_x(R(\xi, W)W, \xi)}{g_x(W, W)} = 0$ for any null plane $H$ of $T_xM$ directed by $\xi$. \(\square\)

### 3. Einstein submanifolds

In this section, let $(M, g, S(TM))$ be a screen conformal half lightlike submanifold of a Lorentzian space form $(M, g)$ with a conformal Killing coscreen distribution. Then $\xi = 1, \phi = 0$ and $S(TM)$ is a Riemannian and integrable vector bundle. As $M$ is a Lorentzian space form, then $R(\xi, Y)X = c\bar{g}(X, Y)\xi, R(u, X)Y = c\bar{g}(X, Y)\xi$ and $\text{Ric}(X, Y) = (m + 2)c\bar{g}(X, Y)$. Thus the equation (1.28) reduces to

(3.1) \[\text{Ric}(X, Y) = mc\bar{g}(X, Y) + B(X, Y)\text{tr}A_X + D(X, Y)\text{tr}A_u - \varphi g(A_\xi^*X, A_\xi^*Y) - g(A_uX, A_uY), \quad \forall X, Y \in \Gamma(TM).\]

From (1.16), $\xi$ is an eigenvector field of $A_\xi^*$ corresponding to the eigenvalue 0. Since $A_\xi^*$ is $\Gamma(S(TM))$-valued real self-adjoint operator on $\Gamma(TM)$ with respect to $\bar{g}$, $A_\xi^*$ have $m$ real orthonormal eigenvector fields in $S(TM)$ and is diagonalizable. Consider a frame field of eigenvectors $\{\xi, E_1, \ldots, E_m\}$ of $A_\xi^*$ such that $\{E_1, \ldots, E_m\}$ is an orthonormal frame field of $S(TM)$. Then

$$A_\xi^*E_i = \lambda_i E_i, \quad 1 \leq i \leq m.$$  

Let $M$ be an Einstein manifold. Then $\text{Ric} = \gamma g$ and (3.1) reduces to

(3.2) \[g(A_\xi^*X, A_\xi^*Y) - sg(A_\xi^*X, Y) + Fg(X, Y) = 0,\]
where $s = \text{tr}A_\xi^2$ is the trace of $A_\xi^2$ and $F = \varphi^{-1}\{\gamma - mc - \delta\rho(\xi) + (1 - m)\delta^2\}$ is a smooth function. In case $m > 2$, we show that $F = \varphi^{-1}\{\gamma - (m - 1)(c + \delta^2)\}$.

Put $X = Y = E_i$ in (3.2), the eigenvalue $\lambda_i$ is a solution of

\[(3.3) \quad x^2 - sx + F = 0.\]

The equation (3.3) has at most two distinct solutions. Assume that there exists $p \in \{0, 1, \ldots, m\}$ such that $\lambda_1 = \cdots = \lambda_p = \alpha$ and $\lambda_{p+1} = \cdots = \lambda_m = \beta$, by renumbering if necessary. From (3.3), we have

\[(3.4) \quad s = \alpha + \beta = pa + (m - p)\beta, \quad \alpha\beta = F.\]

Although $S(TM)$ is not unique, it is canonically isomorphic to the factor vector bundle $S(TM)^2 = TM/RadTM$ considered by Kupeli [11]. Thus all $S(TM)$ are isomorphic. For this reason, let $(M, g, S(TM))$ be a screen conformal Einstein half lightlike submanifold equipped with the canonical null pair $\{\xi, \eta\}$ of a Lorentzian space form $(M^{m+1}(c), \bar{g})$, $m > 2$, with a conformal Killing coscreen distribution.

**Theorem 3.1.** Let $(M, g, S(TM))$ be a screen conformal Einstein half lightlike submanifold of a Lorentzian space form $(M^{m+1}(c), \bar{g})$, $m > 2$, with a conformal Killing coscreen distribution. Then $M$ is locally a product manifold $L \times M_\alpha \times M_\beta$, where $L$ is a null curve and $M_\alpha$ and $M_\beta$ are totally umbilical leaves of some distributions of $M$.

**Proof.** If (3.3) has only one solution $\alpha$, then, since $M$ is screen conformal, $M = L \times M^* \cong L \times M^* \times \{x\}$ for any $x \in M$, where $M_\alpha = M^*$ is a leaf of $S(TM)$ and $M_\beta = \{x\}$ is a leaf of the trivial vector bundle $\{0\}$. Since $B(X, Y) = g(A_\xi^2X, Y) = \alpha g(X, Y)$ for all $X, Y \in \Gamma(TM)$, we get $C(X, Y) = \varphi \alpha g(X, Y)$ for all $X, Y \in \Gamma(TM)$ by (2.1). Thus $M^*$ is totally umbilical and $\{x\}$ is also totally umbilical. In this case, our assertion is true.

Assume that (3.3) has exactly two distinct solutions $\alpha$ and $\beta$. If $p = 0$ or $p = m$, then we also show that $M = L \times M^* \cong L \times M^* \times \{x\}$ for any $x \in M$, and $M^* = M_\alpha$ and $M_\beta = \{x\}$ (if $p = m$) or $M_\alpha$ and $M_\beta = \{x\}$ (if $p = 0$). In these cases, $M^*$ is totally umbilical. If $0 < p < m$. Consider the following four distributions $D_\alpha, D_\beta, D_\alpha^*\beta$ and $D_\beta^*\alpha$ on $M$:

\[
\Gamma(D_\alpha) = \{X \in \Gamma(TM) \mid A_\xi^2X = \alpha PX\}, \quad D_\alpha^* = PD_\alpha;
\]

\[
\Gamma(D_\beta) = \{U \in \Gamma(TM) \mid A_\xi^2U = \beta PU\}, \quad D_\beta^* = PD_\beta.
\]

Then $D_\alpha \cap D_\beta = \text{Rad}(TM)$ and $D_\alpha^* \cap D_\beta^* = \{0\}$.

Since $A_\xi^2PX = A_\xi^2X = \alpha PX$ for all $X \in \Gamma(D_\alpha)$ and $A_\xi^2PU = A_\xi^2U = \beta PU$ for all $U \in \Gamma(D_\beta)$, $PX$ and $PU$ are eigenvector fields of the real symmetric operator $A_\xi^2$ corresponding to the different eigenvalues $\alpha$ and $\beta$ respectively. Thus $PX \perp PU$ and $g(X, U) = g(PX, PU) = 0$, that is, $D_\alpha \perp D_\beta$. Also, since $B(X, U) = g(A_\xi^2X, U) = \alpha g(PX, PU) = 0$, we show that $D_\alpha \perp D_\beta$. 


For any $x \in M$, since $\{E_i\}_{1 \leq i \leq p}$ and $\{E_a\}_{p+1 \leq a \leq m}$ are $p$ and $(m - p)$ smooth linearly independent vector fields of $D_\alpha^s$ and $D_\beta^s$ respectively, $D_\alpha^s$ and $D_\beta^s$ are smooth distributions. Also, as $\{\xi, E_i\}_{1 \leq i \leq p}$ and $\{\xi, E_a\}_{p+1 \leq a \leq m}$ are $(p + 1)$ and $(m - p + 1)$ smooth linearly independent vector fields of $D_\alpha$ and $D_\beta$ respectively, $D_\alpha$ and $D_\beta$ are also smooth distributions on $M$. Thus $D_\alpha^s$ and $D_\beta^s$ are orthogonal vector subbundle of $S(TM)$, $D_\alpha^s$ and $D_\beta^s$ are non-degenerate distributions of rank $p$ and rank $(m - p)$ respectively. Thus $S(TM) = D_\alpha^s \oplus_{\text{orth}} D_\beta^s$. Consequently, $TM = \text{Rad}(TM) \oplus_{\text{orth}} D_\alpha^s \oplus_{\text{orth}} D_\beta^s$.

From (3.2), we show that $(A_\xi^s)^2 - (\alpha + \beta)A_\xi^s + \alpha \beta P = 0$. Let $Y \in \text{Im}(A_\xi^s - \alpha P)$, then there exists $X \in \Gamma(TM)$ such that $Y = (A_\xi^s - \alpha P)X$. Then $(A_\xi^s - \beta P)Y = 0$ and $Y \in \Gamma(D_\beta^s)$. Thus $\text{Im}(A_\xi^s - \alpha P) \subset \Gamma(D_\beta^s)$. Since the morphism $A_\xi^s - \alpha P$ maps $\Gamma(TM)$ onto $\Gamma(S(TM))$, we have $\text{Im}(A_\xi^s - \alpha P) \subset \Gamma(D_\beta^s)$. By duality, we also have $\text{Im}(A_\xi^s - \beta P) \subset \Gamma(D_\alpha^s)$.

For $X, Y \in \Gamma(D_\alpha)$ and $U \in \Gamma(D_\beta)$, we have

$$
(\nabla_X B)(Y, U) = -g((A_\xi^s - \alpha P)\nabla_X Y, U) + \alpha^2 g(X, Y)\eta(U)
$$

and $(\nabla_X B)(Y, U) = (\nabla_Y B)(X, U)$ due to (2.5). Thus $g((A_\xi^s - \alpha P)[X, Y], U) = 0$. As $D_\beta^s$ is non-degenerate and $\text{Im}(A_\xi^s - \alpha P) \subset \Gamma(D_\beta^s)$, we have $(A_\xi^s - \alpha P)[X, Y] = 0$. Thus $[X, Y] \in \Gamma(D_\alpha)$ and $D_\alpha$ is integrable. By duality, $D_\beta$ is also integrable. Since $S(TM)$ is integrable, for any $X, Y \in \Gamma(D_\alpha^s)$, we have $[X, Y] \in \Gamma(D_\alpha^s)$ and $[X, Y] \in \Gamma(S(TM))$. Thus $[X, Y] \in \Gamma(D_\alpha^s)$ and $D_\alpha^s$ is integrable. So is $D_\beta^s$.

For $X, Y \in \Gamma(D_\alpha)$ and $Z \in \Gamma(TM)$, we show that

$$
(\nabla_X B)(Y, Z) = -g((A_\xi^s - \alpha P)\nabla_X Y, Z) + \alpha^2 g(X, Y)\eta(Z)
\quad + (X \alpha) g(Y, Z) + \alpha^2 \eta(Y) g(X, Z).
$$

Using this equation and the facts that $(\nabla_X B)(Y, Z) = (\nabla_Y B)(X, Z)$ due to (2.5) and $(A_\xi^s - \alpha P)[X, Y] = 0$ for any $X, Y \in \Gamma(D_\alpha)$, we have

$$
[X - \alpha^2 \eta(X)] g(Y, Z) = (Y - \alpha^2 \eta(Y)) g(X, Z), \quad \forall X, Y \in \Gamma(D_\alpha).
$$

Therefore, for $X, Y \in \Gamma(D_\alpha^s)$ and $Z \in \Gamma(S(TM))$, we obtain $(X \alpha) g(Y, Z) = (Y \alpha) g(X, Z)$. Since $S(TM)$ is non-degenerate, we have $d\alpha(X)Y = d\alpha(Y)X$. Suppose there exists a vector field $X_0 \in \Gamma(D_\alpha^s)$ such that $d\alpha(X_0)_{x} \neq 0$ at each point $x \in M$, then $Y = fX_0$ for any $Y \in \Gamma(D_\alpha^s)$, where $f$ is a smooth function. It follows that all vectors from the fiber $(D_\alpha^s)_x$ are colinear with $(X_0)_x$. It is a contradiction as dim $((D_\alpha^s)_x) = p > 1$. Thus we have $d\alpha|_{D_\alpha^s} = 0$. By duality, we also have $d\beta|_{D_\beta^s} = 0$. Thus $\alpha$ is a constant along $D_\alpha^s$ and $\beta$ is a constant along $D_\beta^s$. From the first equation of (3.4), we have $(p - 1)\alpha = -(m - p - 1)\beta$. Thus both $\alpha$ and $\beta$ are constants along $S(TM)$.

Using (2.9) with $c + \rho(\xi) = 0$ and $\tau = 0$, we have

$$
(3.5) \quad (X \varphi)B(Y, Z) - (Y \varphi)B(X, Z) = \delta(\rho(PX)g(Y, Z) - \rho(PY)g(X, Z))
$$

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for any $X, Y, Z \in \Gamma(TM)$. Take $X, Y, Z \in \Gamma(D_\alpha^\ast)$, then (3.5) reduces to
\[
\{\alpha(X\varphi) - \delta\rho(X)\}Y = \{\alpha(Y\varphi) - \delta\rho(Y)\}X.
\]
Since $\dim(D_\alpha^\ast) > 1$, we have $(X\varphi)\alpha = \delta\rho(X)$ for all $X \in \Gamma(D_\alpha^\ast)$. While, take $X \in \Gamma(D_\beta^\ast)$ and $Y, Z \in \Gamma(D_\alpha^\ast)$ in (3.5), we have $(X\varphi)\alpha = \delta\rho(X)$ for all $X \in \Gamma(D_\beta^\ast)$. Consequently, we obtain $(X\varphi)\alpha = \delta\rho(X)$ for all $X \in \Gamma(S(TM))$.

By duality, we get $(X\varphi)\beta = \delta\rho(X)$ for all $X \in \Gamma(S(TM))$. Thus we have $(X\varphi)\alpha = (X\varphi)\beta$ for all $X \in \Gamma(S(TM))$. Since $\alpha \neq \beta$, we have $X\varphi = 0$ for all $X \in \Gamma(S(TM))$, that is, $\varphi$ is a constant along $S(TM)$. Take $X, Y \in \Gamma(D_\alpha^\ast)$ in (2.10), we have $\xi[\varphi]\alpha = 0$. Also, take $X, Y \in \Gamma(D_\beta^\ast)$ in (2.10), we have $\xi[\varphi]\beta = 0$. Since $(\alpha, \beta) \neq (0, 0)$, we have $\xi[\varphi] = 0$. Thus we have $X\varphi = 0$ for all $X \in \Gamma(TM)$, i.e., $\varphi$ is a constant on $M$.

For all $X \in \Gamma(D_\alpha^\ast)$ and $U \in \Gamma(D_\beta^\ast)$, since $(\nabla_X B)(U, Z) = (\nabla_U B)(X, Z),\nabla((A_\xi^\ast - \beta P)\nabla X - (A_\xi^\ast - \alpha P)\nabla_U X, Z) = 0, \forall Z \in \Gamma(S(TM))$.

As $S(TM)$ is non-degenerate, we get $(A_\xi^\ast - \beta P)\nabla X U = (A_\xi^\ast - \alpha P)\nabla U X$. Since the left term of the last equation is in $\Gamma(D_\alpha^\ast)$ and the right term is in $\Gamma(D_\beta^\ast)$ and $D_\alpha^\ast \cap D_\beta^\ast = \{0\}$, we have $(A_\xi^\ast - \beta P)\nabla X U = 0$ and $(A_\xi^\ast - \alpha P)\nabla U X = 0$.

This imply that $\nabla_X U \in \Gamma(D_\beta^\ast)$ and $\nabla_U X \in \Gamma(D_\alpha^\ast)$. On the other hand, $\nabla_X U = \nabla^* U$ and $\nabla_U X = \nabla^* X$ due to $D_\alpha^\ast \perp \beta D_\beta$, we have
\[
\nabla X U \in \Gamma(D_\alpha^\ast), \quad \nabla_U X \in \Gamma(D_\beta^\ast), \quad \forall X \in \Gamma(D_\alpha^\ast), U \in \Gamma(D_\beta^\ast).
\]

For $X, Y \in \Gamma(D_\alpha^\ast)$ and $U, V \in \Gamma(D_\beta^\ast)$, since $g(X, U) = 0$, we have
\[
g(\nabla_Y X, U) + g(X, \nabla_Y U) = 0, \quad g(\nabla_U X, V) + g(U, \nabla_V X) = 0.
\]

Using (3.6), we have $g(X, \nabla_Y U) = g(U, \nabla_V X) = 0$. Thus we show that
\[
g(\nabla_X U, Y) = 0, \quad g(X, \nabla_U V) = 0.
\]

Since the leaf $M^\ast$ of $S(TM)$ is a semi-Riemannian manifold and $S(TM) = D_\alpha^\ast \oplus_{\text{orth}} D_\beta^\ast$, where $D_\alpha^\ast$ and $D_\beta^\ast$ are integrable and parallel distributions with respect to the induced connection $\nabla^\ast$ on $M^\ast$ due to (3.7), by the decomposition theorem of de Rham [13], we have $M^\ast = M_\alpha \times M_\beta$, where $M_\alpha$ and $M_\beta$ are some leaves of $D_\alpha^\ast$ and $D_\beta^\ast$ respectively. Thus we have our theorem. \qed

**Theorem 3.2.** Let $(M, g, S(TM))$ be a screen conformal half lightlike submanifold of a Lorentzian space form $(M^{m+1}(c), \bar{g})$, $m > 2$, with a conformal Killing coscreen distribution. If $M$ is Einstein, i.e., $\text{Ric} = \gamma g$, then $M$ is locally a product manifold $L \times M_\alpha \times M_\beta$, where $L$ is a null curve and $M_\alpha$ and $M_\beta$ are totally umbilical leaves of some distributions of $M$:

1. If $\gamma \neq (m - 1)(c + \delta^2)$, then either $M_\alpha$ or $M_\beta$ is an $m$-dimensional Einstein Riemannian space form which is isometric to a sphere ($\gamma > 0$) or a hyperbolic space ($\gamma < 0$) and the other is a point on $M$. \label{thm:3.2}
(2) If \( \gamma = (m-1)(c+\delta^2) \), then \( M_\alpha \) is an \((m-1)\) or \(m\)-dimensional Einstein Riemannian space form which is isometric to a sphere \((\gamma > 0)\) or a hyperbolic space \((\gamma < 0)\) or a Euclidean space \((\gamma = 0)\) and \( M_\beta \) is a spacelike curve or a point on \( M \).

Proof. First, we prove that \( \gamma = 0 \) and \( \alpha \beta = 0 \) if \( 0 < p < m \). If \( 0 < p < m \), then, since \( \text{rank} A_\xi^\alpha > 0 \), we have \( c + \delta^2 = 0 \) by Theorem 2.5. If \( p = 1 \) or \( p = m - 1 \), then, from the facts that \((p-1)\alpha + (m-p-1)\beta = 0 \) and \( m > 2 \), we show that if \( p = 1 \), then \( \beta = 0 \) and if \( p = m - 1 \), then \( \alpha = 0 \). Thus \( \gamma = \varphi \alpha \beta = 0 \). If \( 1 < p < m - 1 \), then, from (3.7), we know that \( \nabla_U U \) has no component of \( A_\alpha \). Since the projection morphism \( P \) maps \( \Gamma(D_\beta) \) onto \( \Gamma(D^\beta_\alpha) \) and \( S(TM) = D^\alpha_\gamma \oplusorth D^\beta_\gamma \),

\[
\nabla_U U = P(\nabla_U U) + \eta(\nabla_U U)\xi, \quad P(\nabla_U U) \in \Gamma(D^\alpha_\gamma).
\]

It follows that

\[
g(\nabla_X \nabla_U U, X) = g(\nabla_X P(\nabla_U U), X) + \eta(\nabla_U U)g(\nabla_X \xi, X)
\]

\[
= -\alpha \eta(\nabla_U U)g(X, X).
\]

As \( \eta(\nabla_U U) = -\bar{g}(U, \nabla_U N) = g(U, A_\alpha U) = \varphi g(U, A^\alpha_\gamma U) = \varphi \varphi g(U, U) \), we get

\[
g(R(X, U)U, X) = -\varphi \alpha \varphi g(X, X)g(U, U).
\]

While, from the Gauss equation (2.3), we have

\[
g(R(X, U)U, X) = \varphi \alpha \varphi g(X, X)g(U, U),
\]

due to \( c + \delta^2 = 0 \). From the last two equations, we get \( \gamma = \varphi \alpha \beta = 0 \).

(1) Let \( \gamma \neq (m-1)(c+\delta^2) \): In this case, we have \( c + \delta^2 = 0 \). The equation (3.3) has two non-vanishing distinct solutions \( \alpha \) and \( \beta \). If \( 0 < p < m \), then \( \gamma = 0 \). This implies that \( \gamma = (m-1)(c+\delta^2) \). Therefore, we have \( p = 0 \) or \( p = m \). If \( p = 0 \), then \( M = L \times M^* = L \times \{ x \} \times M^* \) and \( B(X, Y) = g(A^\alpha_\gamma X, Y) = \beta g(X, Y) \) for any \( X, Y \in \Gamma(TM) \). From this and (2.1), we show that \( C(X, Y) = \varphi \beta g(X, Y) \) for all \( X, Y \in \Gamma(TM) \). Thus \( M^* \) is totally umbilical. From (2.4) and (2.13), we have

\[
R^*(X, Y)Z = 2\varphi \beta^2 \{ g(Y, Z)X - g(X, Z)Y \},
\]

\[
\text{Ric}^*(X, Y) = 2\varphi \beta^2 (m-1) g(X, Y), \quad \forall X, Y, Z \in \Gamma(S(TM)).
\]

Thus \( M^* \) is Einstein and \( 2\varphi \beta^2 \) is a constant due to \( m > 2 \). By (2.13), we have

\[
2\gamma = 2\varphi \beta^2.
\]

Therefore, \( M^* \) is an Einstein space of constant curvature \( 2\gamma \). By duality, if \( p = m \), then \( M = L \times M^* = L \times M^* \times \{ x \} \) and \( B(X, Y) = \alpha g(X, Y) \) for any \( X, Y \in \Gamma(TM) \). Thus \( M \) is totally umbilical and \( M^* \) is a totally umbilical Einstein space of constant curvature \( 2\gamma = 2\varphi \alpha^2 \). In case \( s^2 = 4F \), the equation (3.3) has only one non-vanishing solution, named by \( \alpha \) and \( \alpha \) is a unique eigenvalue of \( A_\xi^\alpha \). In this case, the first equation of (3.4) reduces to \( 2\alpha = ma \). This implies \( m = 2 \). Thus this case is an impossible one.
(2) Let $\gamma = (m - 1)(c + \delta^2)$: The equation (3.3) reduces to $x(x - s) = 0$. In case $s \neq 0$. Let $\alpha = 0$ and $\beta = s$. Then we have $s = \beta = (m - p)\beta$, i.e., $(m - p - 1)\beta = 0$. So $p = m - 1$. Thus $M_\alpha$ is a totally geodesic $(m - 1)$-dimensional Riemannian manifold and $M_\beta$ is a spacelike curve in $M$. In the sequel, let $X, Y, Z \in \Gamma(D^\alpha_\alpha)$ and $U \in \Gamma(D^\beta_\beta)$. From (2.4), we have 

$$R^*(X, Y)Z = (c + \delta^2)\{g(Y, Z)X - g(X, Z)Y\},$$

$$Ric^*(X, Y) = (c + \delta^2)(m - 1)g(X, Y).$$

Thus $g(R^*(X, Y)Z, U) = 0$. This implies $\pi_\alpha R^*(X, Y)Z = R^*(X, Y)Z$, where $\pi_\alpha$ is the projection morphism of $\Gamma(S(TM))$ on $\Gamma(D^\alpha_\alpha)$ and $\pi_\alpha R^*$ is the curvature tensor of $D^\alpha_\alpha$. Thus $M_\alpha$ is an Einstein manifold of a constant curvature $(c + \delta^2)$. Therefore, $M$ is locally a product $L \times M_\alpha \times M_\beta$, where $M_\alpha$ is an $(m - 1)$-dimensional Einstein Riemannian space form of a constant curvature $(c + \delta^2)$ and $M_\beta$ is a spacelike curve in $M$. In case $s = 0$, we get $\alpha = \beta = 0$, $A^\alpha_\alpha = B = 0$ and $D^\alpha_\alpha = D^\beta_\beta = S(TM)$. Since $M$ is screen conformal, we also have $C = A_\lambda = 0$. Thus $M^*$ is totally geodesic. Using (2.4), we have 

$$R^*(X, Y)Z = (c + \delta^2)\{g(Y, Z)X - g(X, Z)Y\}$$

for all $X, Y, Z \in \Gamma(S(TM))$. Thus $M$ is locally a product $L \times M^* \times \{x\}$, where $M^*$ is an $m$-dimensional Einstein Riemannian space form of a constant curvature $(c + \delta^2)$ and $\{x\}$ is a point. In these cases, since $(c + \delta^2) = \frac{2}{m - 1}$, we have $\text{sgn}(c + \delta^2) = \text{sgn} \gamma$. Thus $M_\alpha$ and $M^*$ are isometric to spheres (if $\gamma > 0$) or hyperbolic spaces (if $\gamma = 0$) or Euclidean spaces (if $\gamma < 0$).

**Corollary 3.3.** Let $(M, g, S(TM))$ be a screen conformal Einstein half lightlike submanifold of a Lorentzian space form $(M^{m+3}(c), \bar{g})$, $m > 2$, with a Killing coscreen distribution. Then $M$ is locally a product manifold $L \times M_\alpha \times M_\beta$, where $L$ is a null curve and $M_\alpha$ and $M_\beta$ are totally umbilical leaves of some distributions of $M$:

1. If $\gamma \neq 0$, either $M_\alpha$ or $M_\beta$ is an $m$-dimensional Riemannian space form which is isometric to a sphere ($\gamma > 0$) or a hyperbolic space ($\gamma < 0$) and the other is a point in $M$.
2. If $\gamma = 0$, $M_\alpha$ is an $(m - 1)$ or $m$-dimensional Euclidean space and $M_\beta$ is a spacelike curve or a point in $M$.

**Proof.** (1) Let $\gamma \neq 0$: In case $s^2 \neq 4F$. If $0 < p < m$, then $\gamma = 0$. Thus $p = 0$ or $p = m$. Either $M_\alpha$ or $M_\beta$ is a totally umbilical Riemannian manifold $M^*$ of constant curvature $2\varphi\alpha^2$ or $2\varphi\alpha^2$ respectively due to $\delta = c = 0$. Thus $M$ is locally a product manifold $L \times M^* \times \{x\}$ or $L \times \{x\} \times M^*$, where $M^*$ is an $m$-dimensional totally umbilical Riemannian manifold of constant curvature $2\gamma = 2\varphi\alpha^2$ or $2\gamma = 2\varphi\alpha^2$ which is isometric to a sphere or a hyperbolic space according to the sign of $\gamma$ and $\{x\}$ is a point. The case $s^2 = 4F$ is not appear because $m > 2$. 


(2) Let \( \gamma = 0 \): In case \( s \neq 0 \). Then \( \alpha = 0 \) and \( \beta = s \). Since \( p = m - 1 \), \( M_\alpha \) is an \((m - 1)\)-dimensional Riemannian manifold of curvature \( c + \delta^2 = 0 \) and \( M_\beta \) is a spacelike curve. Thus \( M \) is locally a product manifold \( L \times M_\alpha \times M_\beta \), where \( M_\alpha \) is an \((m - 1)\)-dimensional Euclidean space and \( M_\beta \) is a spacelike curve in \( M \). In case \( s = 0 \). Then \( \alpha = \beta = 0 \) and \( D_\alpha^* = D_\beta^* = S(TM) \). Thus \( M^* \) is an \( m \)-dimensional Riemannian manifold of curvature \( c + \delta^2 = 0 \). Thus \( M \) is locally a product \( L \times M^* \times \{x\} \) where \( M^* \) is an \( m \)-dimensional Euclidean space, \( L \) is a null curve and \( \{x\} \) is a point. 

\[ n \]

Example 3. Consider a surface \( M \) in \( R^4_2 \) given by the equations

\[ x_3 = \frac{1}{\sqrt{2}}(x_1 + x_2), \quad x_4 = \frac{1}{2}\ln(1 + (x_1 - x_2)^2). \]

Then \( TM = \text{Span}\{U, V\} \) and \( TM^\perp = \text{Span}\{\xi, u\} \), where we set

\[ U = \sqrt{2}(1 + (x_1 - x_2)^2)\partial_1 + (1 + (x_1 - x_2)^2)\partial_3 + \sqrt{2}(x_1 - x_2)\partial_4, \]
\[ V = \sqrt{2}(1 + (x_1 - x_2)^2)\partial_2 + (1 + (x_1 - x_2)^2)\partial_3 - \sqrt{2}(x_1 - x_2)\partial_4, \]
\[ \xi = \partial_1 + \partial_2 + \sqrt{2}\partial_3, \]
\[ u = 2(x_2 - x_1)\partial_2 + \sqrt{2}(x_2 - x_1)\partial_3 + (1 + (x_1 - x_2))\partial_4. \]

By direct calculations we check that \( \text{Rad}(TM) \) is a distribution on \( M \) of rank 1 spanned by \( \xi \). Hence \( M \) is a half-lightlike submanifold of \( R^4_2 \). Choose \( S(TM) \) and \( S(TM^\perp) \) spanned by \( V \) and \( u \) which are timelike and spacelike respectively.

We obtain the lightlike transversal vector bundle

\[ \text{ltr}(TM) = \text{Span}\left\{ N = -\frac{1}{2}\partial_1 + \frac{1}{2}\partial_2 + \frac{1}{\sqrt{2}}\partial_3 \right\}, \]

and the transversal bundle \( tr(TM) = \text{Span}\{N, u\} \). Denote by \( \bar{\nabla} \) the Levi-Civita connection on \( R^4_2 \) and by straightforward calculations we obtain

\[ \bar{\nabla}_V V = 2(1 + (x_1 - x_2)^2) \left\{ 2(x_2 - x_1)\partial_2 + \sqrt{2}(x_2 - x_1)\partial_3 + \partial_4 \right\}, \]
\[ \bar{\nabla}_\xi V = 0, \quad \bar{\nabla}_X \xi = \bar{\nabla}_X N = 0, \quad \forall X \in \Gamma(TM). \]

Taking into account of Gauss and Weingarten formulae, we infer

\[ B = 0, \quad A_\xi = 0, \quad A_x = 0, \quad \nabla X \xi = 0, \quad \tau(X) = \rho(X) = 0, \]
\[ [D(X, \xi) = 0, \quad D(V, V) = 2, \quad \nabla X V = \frac{2\sqrt{2}(x_2 - x_1)^3}{1 + (x_1 - x_2)^2} X^2 V \]

for any \( X = X^1 \xi + X^2 V \) tangent to \( M \). As \( A_\xi X = A_x X = 0 \) for any \( X \in \Gamma(TM) \), \( M \) is a trivial screen conformal half lightlike submanifold of \( R^4_2 \). Since \( g(V, V) = -1 + (x_1 - x_2)^2 \) we have

\[ D(V, V) = \delta g(V, V), \quad \text{where} \quad \delta = -\frac{2}{1 + (x_1 - x_2)^2}. \]
Therefore $M$ is a screen conformal half lightlike submanifold of $\mathbb{R}^4_2$ with a conformal Killing coscreen distribution $S(TM^\perp)$. Thus $M$ is locally a product manifold $M = L_1 \times L_2$, where $L_1$ is a null curve tangent to $\text{Rad}(TM)$ and $L_2$ is a timelike curve tangent to $S(TM)$.

References


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